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On an inverse problem in elastic wave propagation

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Abstract

We describe a procedure of determining the seismic-moment tensor from the measurements of the far-field elastic waves (known in Seismology as P and S-waves). These elastic waves are produced by a tensorial point-like force distribution with a short temporal duration (pulse-like time dependence), which corresponds to a double couple representation of the seismic forces. The inverse problem involves more unknowns than (algebraic) equations, and the Kostrov representation is used for the seismic-moment tensor of a faulting source, together with the energy conservation and a covariance condition, in order to reduce the number of unknowns and to determine the system of equations. The explicit solution of this system of equations is obtained and the components of the seismic-moment tensor are given in terms of the far-field elastic waves. Also, the fault geometry and force mechanism are determined.

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Introduction. Recently, a tensorial point force distribution has been introduced,[1, 2] which corresponds to the double-couple representation of the forces which may act in a localized seismic focus.[3, 4] This force distribution is governed by the seismic-moment (symmetrical) tensor with components M_{ij} $(i,\, j = 1, 2, 3)$. The static deformations of a homogeneous, isotropic half-space[1] and the elastic waves generated by this force distribution[2] have been computed (the latter for a pulse-like time dependence). We present here the solution of an inverse problem which determines the tensorial components M_{ij} from measurements of the elastic waves in the far-field region. The problem is described by algebraic equations which relate the components M_{ij} to the amplitudes of the elastic waves. Apart from a more general interest in such inverse problems (*e.g.*, in Applied Mathematics,[5] Acoustics and Electromagnetism,[6, 7] Geophysics,[8] Seismology[9], etc), the problem presented here exhibits a special interest because it raises certain difficulties related to its definition (determination). Indeed, the equations relating the components M_{ij} to the wave displacement include seven unknowns (six components M_{ij} and the (short) duration T of the pulse), while the wave displacement provides only three parameters (input data): the magnitude v_l of the longitudinal-wave displacement $\mathbf{v}_l$ and two components of the transverse-wave displacement $\mathbf{v}_t$ (we assume that the observation radius $\mathbf{r}$ from the force distribution to the local reference frame is known).

The problem is solved by appealing to the Kostrov representation of a tensorial point force distribution in a fault (dyadic representation).[10, 11] This representation reduces the number of the unknowns from seven to four, but, among these four unknowns, it introduces a new one, the (small) volume of the region occupied by the force distribution (focal volume). This volume is related to the duration of the pulse by the equation of the energy conservation, derived in the present paper. Finally, the fourth equation needed for solving the system of equations is obtained from the covariance condition, which requires the equations to be invariant to rotations and translations. The problem is determined providing the equations are covariant, as expected. Apart from the tensorial components M_{ij}, other parameters of the force source are determined, like the volume of the region occupied by the force distribution, the duration of the force, the released energy and the geometry of the fault, *i.e.* the normal to the fault and the fault slip. These two vectors determine the asymptotes of a hyperbola which characterizes the fault mechanism.

From the practical standpoint, the problem described above is currently solved by various seismological agencies by fitting synthetic seismograms to data measured at various locations and times.[12]-[25] This procedure requires a special care for the covariance of the fitting equations; in addition, semi-empirical fitting parameters are introduced. The procedure presented here makes use of the waves measured at one location only, without additional parameters, which allows a direct implementation of the covariance condition. On the other hand, the numerical results presented here are determined up to a factor of the order of unity, which arises from the lack of accurate knowledge of the model of localized fault.

Elastic waves. The inverse problem. The tensorial force distribution has the components[1, 2]

$$f_i = M_{ij} h(t) \partial_j \delta(\mathbf{r}) \quad , \tag{1}$$

where $h(t)$ is an even, positive function, localized at the moment $t = 0$ and ∂_j ($j = 1, 2, 3$) denote the derivatives with respect to the components x_j of the vector $\mathbf{r}$ (throughout this paper we understand summation over repeating indices). We assume $max[h(t)] = h(0) = 1$ and denote by T the (short) duration of the action of the force; the time T is much shorter than any time of interest, such that we may view the function $h(t)$ as being represented by $T\delta(t)$, where $\delta(t)$ is the Dirac delta function. Similarly, the force distribution given by equation (1) is localized over a small volume $V = l^3$, of dimension l, placed at the origin, as shown by the Dirac delta function $\delta(\mathbf{r})$ (and its derivatives). We may use the representation $g(\mathbf{r})$ for $\delta(\mathbf{r})/V$, where $max[g(\mathbf{r})] = g(0) = 1$.

The force distribution given by equation (1) generates in the far-field region ($r \gg l$) spherical-shell elastic waves $\mathbf{u}_{l,t}$ proportional to $h'(t - r/c_{l,t})$, propagating with velocities c_l (longitudinal wave) and c_t (transverse velocity). They are given by[2]

$$\mathbf{u}_l = -\frac{h'(t-r/c_l)}{4\pi\rho c_l^3 r} M_4 \mathbf{n} \ , \ \mathbf{u}_t = \frac{h'(t-r/c_t)}{4\pi\rho c_t^3 r} \left(M_4 \mathbf{n} - \mathbf{m}\right) \quad , \tag{2}$$

where ρ is the density of the body, $\mathbf{n} = \mathbf{r}/r$ is the unit vector along the observation radius, $\mathbf{m}$ is the vector with the components $m_i = M_{ij} n_j$ and $M_4 = M_{ij} n_i n_j$. We estimate the amplitudes of the displacement produced by these waves as

$$\mathbf{v}_l = -\frac{1}{4\pi\rho T c_l^3 r} M_4 \mathbf{n} \ , \ \mathbf{v}_t = \frac{1}{4\pi\rho T c_t^3 r} \left(M_4 \mathbf{n} - \mathbf{m}\right) \quad . \tag{3}$$

We assume that the parameters ρ, $c_{l,t}$ and $\mathbf{r}$ (*i.e.*, r and $\mathbf{n}$) are known. In Seismology the longitudinal displacement $\mathbf{v}_l$ and the transverse displacement $\mathbf{v}_t$ are known as corresponding to the P- and S-waves, respectively.[3, 4] The inverse problem presented in this paper is to derive the componenets M_{ij} (of the seismic-moment tensor) from the displacements $\mathbf{v}_{l,t}$ measured far away from the origin of the force (on Earth's surface). We can see from equations (3) that we have seven unknowns (six components M_{ij} and the duration T of the action of the force) and only three input (measured, known) parameters (three equations): the magnitude v_l of the longitudinal displacement and two components of the transverse displacement $\mathbf{v}_t$ (we can check immediately from equations (3) the transversality condition $\mathbf{v}_l \cdot \mathbf{v}_t = 0$). We may view the equations

$$\mathbf{m} = -4\pi\rho Tr\left(c_l^3\mathbf{v}_l + c_t^3\mathbf{v}_t\right) \quad , \tag{4}$$

derived from equations (3), as three independent equations; multipling by n_i and summing over i, we get the first equation (3),

$$M_4 = M_{ij}n_in_j = -4\pi\rho Trc_l^3(\mathbf{v}_l\mathbf{n}) = -4\pi\rho Trc_l^3 v_l \quad , \tag{5}$$

which is not independent of the three equations written above. We note from equations (4) and (5) the relation $m^2 > M_4^2$ ($v_t^2 > 0$).

Kostrov representation. The Kostrov representation (or dyadic representation)[10, 11] establishes the tensor $\mathbf{M}$ (with components M_{ij}) for a fault which may be considered very small (localized) in comparison with the distances of interest (*i.e.* far-field distances). Such a fault may be viewed as the focal region (the focus) of typical tectonic seisms.[3, 4] We give here a simple derivation of this representation. Let us consider a fault consisting of two parallel surfaces with a small area S, of dimension l, separated by a small distance d, which may slide against each other. Let us denote by $\mathbf{s}$ the normal to such a surface and by $\mathbf{a}$ the unit vector along the slip direction (in the plane of the surfaces, $\mathbf{s} \cdot \mathbf{a} = 0$). We estimate the torque $t_{ij} = f_id_j$ of the force component f_i caused by the sliding of the two surfaces and the orthogonal distance component d_j. Obviously, the force component may be written as $f_i = 2\mu lu^0a_i$, where μ is the (shear) Lame coefficient and u^0 is the magnitude of the displacement at the point where the force acts (say, the centre of the surface S); the factor 2 arises from the relative displacement of the two surfaces. Similarly, we may write $d_j = ds_j$, such that we get the torque $t_{ij} = 2\mu lu^0da_is_j$. We may replace the product lu^0d by the small volume V of the region occupied by the fault, such that we get $t_{ij} = 2\mu Va_is_j$. We can see that the product u^0d is equated by this representation to the fault area $S = l^2$. The symmetrized torque t_{ij} is generalized to the components M_{ij} of the seismic-moment tensor (double couple representation), *i.e.* we write

$$M_{ij} = 2\mu V(s_ia_j + a_is_j) \ . \tag{6}$$

This is the Kostrov representation (dyadic representation) of the seismic-moment tensor (if we set $u^0 = d = l$, equation (6) gives also a representation for the focal strain $M_{ij}/4\mu V$). We note that the tensor given by equation (6) is traceless, $M_{ii} = 0$ ($s_ia_i = 0$). We can see easily that this representation involves four unknown parameters: for instance, two for the unit vector $\mathbf{s}$, one for the orthogonal unit vector $\mathbf{a}$ and the magnitude-related parameter $2\mu V$ (actually the volume V, since μ is a known parameter). Although the seven unknowns in equations (3) are reduced to four by equation (6), in order to make use of this representation in the problem formulated by equations (3) we need a relation between the duration T and the fault volume V. This relation is provided by the energy conservation.

Before passing to the energy conservation we note that the representation given by equation (6) has two symmetry operations. First, equation (6) is invariant under the simultaneous changes $\mathbf{s} \longrightarrow -\mathbf{s}$ and $\mathbf{a} \longrightarrow -\mathbf{a}$, which merely indicates a reflection of the orientation of the fault surfaces (or an interchange of the two oriented surfaces); second, equation (6) is invariant under the operation $\mathbf{s} \longleftrightarrow \mathbf{a}$, which indicates that we cannot distinguish between the fault orientation and the sliding direction. Indeed, matter conservation in the sliding process requires in fact that another fault, orthogonal to the former, is present.

Energy conservation. The equation of the elastic waves with force density given by equation (1) can be written as

$$\ddot{u}_i - c_t^2 \Delta u_i - (c_l^2 - c_t^2)\partial_i div\, \mathbf{u} \;\; = \;\; \frac{1}{\rho} M_{ij}(t)\partial_j \delta(\mathbf{r}) \quad , \tag{7}$$

where u_i are the components of the displacement vector $\mathbf{u}$ and the time dependence is incorporated in $M_{ij}(t)$. If we multiply equation (7) by $\dot{u}_i$ and sum over the suffix i, we get the law of energy conservation

$$\begin{gathered} \frac{\partial}{\partial t}\left[\frac{1}{2}\rho \dot{u}_i^2 + \frac{1}{2}\rho c_t^2 (\partial_j u_i)^2 + \frac{1}{2}\rho (c_l^2 - c_t^2)(\partial_i u_i)^2\right] \\ -\rho c_t^2 \partial_j (\dot{u}_i \partial_j u_i) - \rho (c_l^2 - c_t^2)\partial_j (\dot{u}_j \partial_i u_i) = \dot{u}_i M_{ij}(t)\partial_j \delta(\mathbf{r}) \ . \end{gathered} \tag{8}$$

According to this equation, a mechanical work $\dot{u}_i M_{ij}(t)\partial_j \delta(\mathbf{r})$ per unit volume and unit time is done by the external force at the origin (in the focal region). The corresponding energy is taken by the elastic waves (the square bracket in equation (8)) and carried through the space (the term including the *div* in equation (8)).

In the far-field region the displacement $\mathbf{u}$ in equation (8) can be decomposed in longitudinal and transverse waves, *i.e.* we can write $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$, where $curl\, \mathbf{u}_l = 0$ and $div\, \mathbf{u}_t = 0$; this decomposition leads to

$$\frac{\partial e_{l,t}}{\partial t} + c_{l,t} div\, \mathbf{s}_{l,t} = 0 \quad , \tag{9}$$

where

$$e_{l,t} = \frac{1}{2}\rho \left(\dot{u}^f_{l,ti}\right)^2 + \frac{1}{2}\rho c_{l,t}^2 \left(\partial_i u^f_{l,tj}\right)^2 \tag{10}$$

' is the energy density and

$$s_{l,ti} = -\rho c_{l,t} \dot{u}^f_{l,tj} \partial_i u^f_{l,tj} \ ; \tag{11}$$

$c_{l,t} s_{l,ti}$ are energy flux densities per unit time (energy flow). From equation (9) we can see that the energy is transported with velocities $c_{l,t}$ (as it is well known). The volume energy $E = \int d\mathbf{r}(e_l + e_t)$ is equal to the total energy flux

$$\Phi = -\int dt d\mathbf{r} \left(c_l div\, \mathbf{s}_l + c_t div\, \mathbf{s}_t\right) = -\int dt \oint d\mathbf{f} \left(c_l \mathbf{s}_l + c_t \mathbf{s}_t\right) \quad , \tag{12}$$

where $d\mathbf{f}$ denotes the element of the volume-enclosing surface. Making use of equations (11) and the waves given by equations (2) for $M_{ij}(t) = M_{ij} h(t)$, we estimate the surface integral in equation (12) as

$$E = \Phi = \frac{4\pi\rho}{T} r^2 \left(c_l v_l^2 + c_t v_t^2\right) \ ; \tag{13}$$

this relation gives the energy released by the tensorial force distribution acting a short duration T in terms of the displacement magnitudes $v_{l,t}$ measured in the far-field region; it is equal to the mechanical work W. Making use of equations (3) we get an additional relation

$$E = \frac{1}{4\pi\rho c_t^5 T^3}\left[m^2 - \left(1 - c_t^5/c_l^5\right) M_4^2\right] \tag{14}$$

between energy and the seismic moment. We note that the lack of an accurate knowledge of the function $h(t)$ affects this result by a numerical factor, which, for a very short duration T, is of the order unity.

From equation (8) the mechanical work in the focal region is given by

$$W = \int dt \int d\mathbf{r} \dot{u}_i^0(t) M_{ij}(t)\partial_j \delta(\mathbf{r}) \quad , \tag{15}$$

where $\dot{u}_i^0(t)$ is the time derivative of the i-th component of the focal displacement $u_i^0(t) = u_i^0 h(t)$, $-T/2 < t < 0$; for $M_{ij}(t) = M_{ij}h(t)$, equation (15) becomes

$$W = \frac{1}{2}\int d\mathbf{r} u_i^0 M_{ij}\partial_j \delta(\mathbf{r}) \ . \tag{16}$$

For a fault, we use the localized function $g(\mathbf{r})$ for the spatial distribution and assume $u^0 = d$; we get immediately $W \simeq \frac{1}{2}M_{ij}a_i s_j$. Making use of equation (6), we get $W = \mu V$. We can see that the mechanical work done in the focal region is of the order of the elastic energy stored in that region, as expected. It is worth noting that (small) deviations from the equality $u^0 = d$ affect this result by a numerical factor (likely, slightly larger than unity), which remains undetermined. We adopt here the value unity for the ratio of this numerical factor and the (unknown) numerical factor in equation (13), such that, by equating W with energy E (and Φ) given by equation (13), we get

$$V = \frac{4\pi r^2}{c_t^2 T}\left(c_l v_l^2 + c_t v_t^2\right) \tag{17}$$

($c_t^2 = \mu/\rho$). This equation provides the relation between the volume V occupied by the force distribution and its duration of action T. The product VT is affected by the uncertainty in the numerical factors discussed above.

Covariance condition. Source parameters. Let us write the Kostrov representation (equation (6)) as

$$M_{ij} = M(s_i a_j + a_i s_j) \ , \ \ M = 2\mu V \tag{18}$$

and introduce the notations $\alpha = \mathbf{n}\cdot\mathbf{a}$, $\beta = \mathbf{n}\cdot\mathbf{s}$ and $\mathbf{p} = \mathbf{m}/M$; we get

$$\alpha\mathbf{s} + \beta\mathbf{a} = \mathbf{p} \ . \tag{19}$$

If the vector $\mathbf{n}$ has a component perpendicular to the plane made by the vectors $\mathbf{s}$ and $\mathbf{a}$, a similar equation for this vector would not be covariant (the out-of plane component would be an undetermined parameter). Therefore, we assume that the vector $\mathbf{n}$ is in the plane made by the two vectors $\mathbf{s}$ and $\mathbf{a}$ and write

$$\beta\mathbf{s} + \alpha\mathbf{a} = \mathbf{n} \ ; \tag{20}$$

we call this equation the covariance condition. We get immediately

$$\alpha^2 + \beta^2 = 1 \ , \ \ 2\alpha\beta = p_4 \ \ , \tag{21}$$

where $p_4 = M_4/M$ and

$$m^2 = M^2 \tag{22}$$

($p^2 = 1$). Making use of equations (4), (17) and (22) we get the duration

$$T = \sqrt{2r}\frac{(c_l v_l^2 + c_t v_t^2)^{1/2}}{(c_l^6 v_l^2 + c_t^6 v_t^2)^{1/4}} \ \ , \tag{23}$$

the volume

$$V = \frac{\pi(2r)^{3/2}}{c_t^2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \tag{24}$$

and the parameter

$$M = 2\mu V = 2\pi\rho(2r)^{3/2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \tag{25}$$

in terms of measurable quantities (displacements $v_{l,t}$).

The quantity $\left(M_{ij}{}^2\right)^{1/2}=\sqrt{2}M$ (equation (18)) is called the magnitude of the tensor M_{ij}; we call the parameter M the reduced magnitude of the tensor M_{ij}. From $E=W=\mu V$ (equation (16)) we have $E=M/2$. It follows that equation (25) allows the estimation of the magnitude of the seismic-moment tensor and the released energy E. These estimates can be used in the Gutenberg-Richter relation $\lg\left(M_{ij}{}^2\right)^{1/2}=1.5M_w+16.1$ which defines the (moment) magnitude M_w of the seism.[3, 4]

Derivation of the seismic-moment tensor. The solutions of the system of equations (21) are

$$\alpha=\sqrt{\frac{1+\sqrt{1-p_4^2}}{2}}\ ,\ \beta=sgn(p_4)\sqrt{\frac{1-\sqrt{1-p_4^2}}{2}} \tag{26}$$

and $\alpha\longleftrightarrow\pm\beta$, α, $\beta\longleftrightarrow-\alpha$, $-\beta$. Making use of equations (4), (23) and (24), the parameters p_i and p_4 are given by

$$p_i=-\frac{c_l^3v_{li}+c_t^3v_{ti}}{\left(c_l^6v_l^2+c_t^6v_t^2\right)^{1/2}}\ ,\ p_4=-\frac{c_l^3v_l}{\left(c_l^6v_l^2+c_t^6v_t^2\right)^{1/2}}\ . \tag{27}$$

Finally, we get the vectors

$$\begin{aligned}\mathbf{s}&=\tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{p}-\tfrac{\beta}{\alpha^2-\beta^2}\mathbf{n}\ ,\\ \mathbf{a}&=-\tfrac{\beta}{\alpha^2-\beta^2}\mathbf{p}+\tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{n}\ ;\end{aligned} \tag{28}$$

these solutions are symmetric under the operations $\mathbf{s}\longleftrightarrow\mathbf{a}$ ($\alpha\longleftrightarrow-\beta$) and $\mathbf{s}\longleftrightarrow-\mathbf{a}$ ($\alpha\longleftrightarrow\beta$, or α, $\beta\longleftrightarrow-\alpha$, $-\beta$), as discussed before. The seismic moment given by equation (18) is determined up to these symmetry operations.

The eigenvalues of the seismic moment given by equation (18) are $\pm M$ (we leave aside the eigenvalue zero); the corresponding eigenvectors $\mathbf{w}$ are given by $\mathbf{a}\cdot\mathbf{w}=\pm\mathbf{s}\cdot\mathbf{w}$, which imply $\mathbf{p}\cdot\mathbf{w}=\pm\mathbf{n}\cdot\mathbf{w}$; the vectors $\mathbf{w}$ are directed along the bisectrices of the angles made by $\mathbf{s}$ and $\mathbf{a}$, or $\mathbf{p}$ and $\mathbf{n}$ ($\mathbf{w}\sim\mathbf{s}\pm\mathbf{a}$). The associated quadratic form $M_{ij}x_ix_j=const$ is a rectangular hyperbola in the reference frame defined by the vectors $\mathbf{s}$ and $\mathbf{a}$; by using the coordinates $u=\mathbf{s}\cdot\mathbf{x}$ and $v=\mathbf{a}\cdot\mathbf{x}$ in equation (18), the equation of this hyperbola is $uv=const/2M$. Actually, in the local frame (coordinates x_i), the quadratic form $M_{ij}x_ix_j=const$ is a degenerate hyperboloid, consisting of a family of parallel hyperbolas displaced along the third axis (perpendicular to the u- and v-axes). Making use of equations (28), this quadratic form can also be written as

$$2\xi\eta-p_4\left(\xi^2+\eta^2\right)=const\ \ , \tag{29}$$

where the coordinates $\xi=p_ix_i$ and $\eta=n_ix_i$ are directed along the vectors $\mathbf{p}$ and $\mathbf{n}$, respectively. The asymptotes of this hyperbola are $\xi=p_4\eta/\left(1+\sqrt{1-p_4^2}\right)$ and $\eta=p_4\xi/\left(1+\sqrt{1-p_4^2}\right)$ (corresponding to the asymptotes $u=(\alpha\xi-\beta\eta)/(\alpha^2-\beta^2)=0$ and $v=(-\beta\xi+\alpha\eta)/(\alpha^2-\beta^2)=0$). Equations (26)-(28) define the fault geometry and force mechanism, as derived from the far-field elastic waves.

Finally, by making use of equations (28) in equation (18) we get the solution for the seismic moment

$$M_{ij}=\frac{M}{1-p_4^2}\left[p_in_j+p_jn_i-p_4\left(p_ip_j+n_in_j\right)\right]\ \ , \tag{30}$$

where M is given by equation (25) and p_i, p_4 are given by equations (27). In equation (30) there are only three independent components of the seismic tensor, according to the equations

$M_{ij}n_j/M = p_i$: the vectors $\mathbf{n}$ and $\mathbf{p}$ are known (equation (27)) from experimental data, such that these equations can be viewed as three conditions imposed upon the six components M_{ij}. Also, we can see that there exist only three independent components of the seismic tensor M_{ij} from the conditions $M_{ii} = 0$, $M_{ij}s_js_i = 0$ (or $M_{ij}a_ia_j = 0$) and $p_i^2 = 1$. The later equality arises from the covariance condition, which, together with the energy conservation, determines the duration T, the (small) volume V occupied by the force distribution and the reduced magnitude M of the seismic moment.

Discussion and conclusion. An isotropic tensor $M_{ij} = -M\delta_{ij}$ is an interesting particular case, since it can be associated with a force distribution caused by explosions. In this case the transverse displacement is vanishing ($\mathbf{v}_t = 0$), $\mathbf{m} = -M\mathbf{n}$ and $M_4 = -M$. From equations (4) and (13) we get

$$\mathbf{m} = -4\pi\rho T r c_l^3 \mathbf{v}_l \ , \ E = \frac{4\pi\rho r^2}{T} c_l v_l^2 \ . \tag{31}$$

we can see that $\mathbf{v}_l\mathbf{n} > 0$ corresponds to $M > 0$ (explosion), while the case $\mathbf{v}_l\mathbf{n} < 0$ corresponds to an implosion. The region occupied by the force distribution is a sphere with radius of the order l, and the vectors $\mathbf{s}$ and $\mathbf{a}$ are equal ($\mathbf{s} = \mathbf{a}$) and depend on the point on the focal sphere; the magnitude of the focal displacement is $u^0 = l$. The considerations made before for the geometry of the focal region (Kostrov representation) lead to

$$M_{ij} = -2V(2\mu + \lambda)\delta_{ij} = -2\rho c_l^2 V\delta_{ij} \ \ , \tag{32}$$

where the focal volume is written as $V = Sl$, S being the area of the spherical surface. Similarly, the energy is $E = W = \frac{1}{2}M$ ($M > 0$), such that, making use of equations (31), we get $c_lT = \sqrt{2rv_l}$,

$$M = 2\pi\rho c_l^2(2rv_l)^{3/2} = 2\rho c_l^2 V \ , \tag{33}$$

and the focal volume $V = \pi(2rv_l)^{3/2}$. These equations determine the magnitude M and the volume of the focal region from the displacement v_l measured at (large) distance r. We note that a superposition of shear faulting and isotropic focal mechanisms cannot be resolved, because the longitudinal displacement $\mathbf{v}_l$ includes indiscriminately contributions from both mechanisms.

In conclusion, an inverse problem is solved here for the elastic wave propagation in a homogeneous, isotropic body, which consists in determining the tensor of the force distribution from the measurements of the far-field elastic waves. The source of the waves is a tensorial point-like force distribution with a short duration (pulse-like time dependence), which corresponds to a double couple representation of the seismic forces. The static deformation produced by such a force distribution in a homogeneous, isotropic half-space have been computed previously.[1] The elastic waves produced by this pulse-lke force distribution in a homogeneous, isotropic body are spherical-shell waves,[2] which are known in Seismology as the P- and S-waves. The inverse problem involves three (algebraic) equations (the displacement amplitudes of the longitudinal and transverse elastic waves) and seven unknowns, which include six components of the force tensor and the duration of the temporal pulse. For a fault-like region occupied by the force distributution (focal region) the Kostrov representation holds, which reduces the number of unknowns to four, but, unfortunately, introduces the volume of the focal region as a new unknown. From the energy conservation we derive in this paper an equation which relates this volume to the duration of the pulse, such that we are left with three equations and four unknowns. The fourth equation needed for determining the system of equations is obtained from the covariance condition, which eliminates undetermined quantities. The determined system of equations is solved explicitly, and the components of the force tensor are given in terms of the far-field waves. In addition, other parameters of the wave source are determined, like the duration of the pulse, the focal volume, the released energy, the orientation of the fault and the slip in the fault, all in terms of the far-field waves. From the fault geometry and the force tensor, we derive a hyperbola which characterize the force source, in the sense that its asymptotes are directed along the normal to the fault and the fault slip. Also, the particular case of an isotropic source, which may correspond to explosions, is presented.

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