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Replacing sums by integrals

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Abstract

A few examples are given of replacing sums by integrals.

With $x_0 = a, x_N = b, x_n = a + n$ and $a + N = b$ we may write

$$\begin{aligned} \int_{a-1/2}^{b+1/2} dx \cdot f(x) &= \sum_{n=0}^{N} \int_{x_x-1/2}^{x_n+1/2} dx \cdot f(x) = \\ &= \sum_{n=0}^{N} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \frac{1}{(2k+1)!} f^{(2k)}(x_n) \quad , \end{aligned} \tag{1}$$

where $f^{(2k)}$ are the $2k$-th derivative of f, with $f^{(0)} = f$. Similarly, we have

$$\int_{a-1/2}^{b+1/2} dx \cdot f^{(2m)}(x) = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \frac{1}{(2k+1)!} \sum_{n=0}^{N} f^{(2k+2m)}(x_n) \quad , \tag{2}$$

and denoting

$$A_m = \int_{a-1/2}^{b+1/2} dx \cdot f^{(2m)}(x) \ \ , \ \ X_m = \sum_{n=0}^{N} f^{(2m)}(x_n) \ \ , \ \ C_k = \frac{1}{2^{2k}} \frac{1}{(2k+1)!} \quad , \tag{3}$$

(2) can be transcribed as

$$A_m = \sum_{k=0}^{\infty} C_k X_{m+k} \quad . \tag{4}$$

By iterating (4) we obtain

$$X_m = A_m - \sum_{k=1}^{\infty} C_k A_{m+k} + \sum_{k,l=1}^{\infty} C_k C_l A_{m+k+l} - \sum_{k,l,p=1}^{\infty} C_k C_l C_p A_{m+k+l+p} + ... \quad , \tag{5}$$

whence

$$X_0 = A_0 - \sum_{k=1}^{\infty} C_k A_k + \sum_{k=2}^{\infty} \left(\sum_{l=1}^{k-1} C_k C_{k-l} \right) A_l - \sum_{k=3}^{\infty} \left(\sum_{l,p=1}^{l+p \leq k-1} C_l C_p C_{k-l-p} \right) A_k + ... \quad , \tag{6}$$

i.e.

$$X_0 = A_0 - C_1 A_1 + \left(C_1^2 - C_2\right) A_2 \left(C_1^3 - 2C_1C_2 + C_3\right) A_3 + ... \quad . \tag{7}$$

It follows from (7)

$$\sum_a^b f(x_n) = \int_{a-1/2}^{b+1/2} dx \cdot f(x) - \frac{1}{24} f' \mid_{a-1/2}^{b+1/2} + \frac{7}{24.240} f''' \mid_{a-1/2}^{b+1/2} - ... \quad . \tag{8}$$

The series in (8) is a fast convergent one providing the function f is a smooth function.

Let us apply (8) to

$$S_1 = \sum_{n=1}^{N} \frac{1}{n^2} \tag{9}$$

for large N. Since $1/n^2$ is rather abrupt between 1 and 2 we separate the term $n = 1$ and get

$$S_1 = 1 + \frac{1}{3/2} + O(1/N) \approx 1.66 \tag{10}$$

which compares well with $\pi^2/6$, the well-known value of S_1 for $N \to \infty$. Let us compute by the same method

$$f_1(z) = \sum_k \frac{1}{k^2 + z} \quad , \tag{11}$$

where $k = n/N, n = 1, 2, ...N$ and $z > 0$. We get easily

$$f_1(z) = \frac{N}{\sqrt{z}} \arctan \frac{1}{\sqrt{z}} + O(1/N) \tag{12}$$

and

$$F_1(z) = \sum_k^{\infty} \frac{1}{k^2 + z} = \frac{\pi N}{2\sqrt{z}} \quad , \tag{13}$$

in both cases the singular behaviour of S_1 being reflected in the singular behaviour at $z = 0$.

By the same procedure we can compute

$$S_2 = \sum_{n_1, n_2 = 1}^{N} \frac{1}{n_1^2 + n_2^2} \quad . \tag{14}$$

Denoting $n_2^2 = z$ and following the computations above we get straightforwardly

$$S_2 = c \sum_{n=1}^{N} \frac{1}{n} \tag{15}$$

as the leading term ($\sum_{n=1}^{N}(1/n) \approx \ln N$+Euler's constant $C \approx 0.577$) where c is somewhere between $\arctan 1$ and $\arctan 2$. We may ask of summations in (14) up to distinct upper limits $N_{1,2}$, and ambiguities will appear in the order of taking the limits. By applying (8) successively we obtain also

$$f_2(z) = \sum_{\mathbf{k}} \frac{1}{k^2 + z} = N^2 c \ln \left(\frac{1 + \sqrt{1+z}}{\sqrt{z}} \right) \quad , \tag{16}$$

where $\mathbf{k} = (n_1/N, n_2/N), n_{1,2} = 1, 2, ...N$ in the limit of large N and for $z > 0$, c being somewhere between $\pi/4$ and $\pi/2$. Integrating directly over $\mathbf{k}$ we get

$$f_2(z) = \frac{\pi}{4} N^2 \ln \left(\frac{1+z}{z} \right) \quad , \tag{17}$$

which is a pretty good approximation.

Similarly,

$$S_3 = \sum_{n_1,n_2,n_3=1}^{N} \frac{1}{n_1^2 + n_2^2 + n_3^2} \tag{18}$$

is

$$S_3 \approx 1.77 \cdot c \cdot N \quad , \tag{19}$$

where $c \in \left(\arctan 2\sqrt{2}, \arctan 1/\sqrt{2}\right)$. Replacing the sum in (18) by a three-dimensional integral we get $(\pi/2)\,(6/\pi)^{1/3}\,N$, which is a very good approximation to (19). A similar treatment supports $f_3(z) = \sum_{\mathbf{k}} 1/\left(k^2 + z\right)$, for $z.0$, where $\mathbf{k} = (n_1/N, n_2/N, n_3/N), , n_{1,2,3} = 1, 2, ...N$, or n_i are let to go to infinity, for any $i = 1, 2, 3$, as for $f_{1,2}$ as well.