

On the Fermi sea-displacement operators as exact Bose operators

L. C. Cune and M. Apostol
 Department of Theoretical Physics,
 Institute of Atomic Physics,
 Magurele-Bucharest MG-6,
 POBox MG-35, Romania
 email: cune@theor1.ifa.ro

Abstract

The generalized sea-displacement operators introduced recently by Setlur and Chang for Fermi systems (see, for instance, Phys. Rev. **B57** 15 144 (1998)) are discussed with regard to their Bose character. It is shown that, apart from some technical details, the bosonic nature of these operators calls for certain precautions in direct computations, whose implications remain to be further explored by the general theory.

In a series of cca 6 recent preprints[1] and one printed paper[2] Setlur and Chang engaged themselves in delineating a general theory, aimed at solving exactly, or, at least, "exceedingly plausibly", [2] the interaction problem in both Fermi and Bose systems, irrespective of the interaction strength (or its sign), and in any spatial dimensions. Drawing largely from the illuminating work of Castro-Neto and Fradkin,[3] and generalizing the inspiring concepts of Haldane,[4] Setlur and Chang rework the whole body of the many-body theories in an original manner, based on the central concept of bosonization.[5] In particular, the single-particle propagator is claimed to be computed "exactly for all wavelengths and energies", including "short-wavelength behaviour". [2] While the general, programmatic framework of the theory is well-established by these authors, there remain still a few technical details to be elucidated, in order to complete this powerful approach in all its aspects. In this context, we offer here a few comments on such questions.

For Bose systems Setlur and Chang[2] introduce condensate-displacement operators which satisfy Bose commutation relations. Similarly, sea-displacement operators are postulated for Fermi systems, satisfying Bose commutation relations, and it is assumed that products of Fermi operators have the same functional dependence on these operators as for the case of the Bose systems. Making use of the analogy with the Bose systems, the following relations are proposed, and defended, for the Fermi systems:

$$\begin{aligned}
 c_{k+q/2}^+ c_{k-q/2} &= \left(\frac{N}{\langle N \rangle} \right)^{\frac{1}{2}} [\Lambda_k(q) a_k(-q) + a_k^+(q) \Lambda_k(-q)] + \\
 &+ T_1(k, q) \sum_{q_1} a_{k+q/2-q_1/2}^+(q_1) a_{k-q_1/2}(q_1 - q) - \\
 &- T_2(k, q) \sum_{q_1} a_{k-q/2+q_1/2}^+(q_1) a_{k+q_1/2}(q_1 - q) \quad ,
 \end{aligned} \tag{1}$$

where c_k are Fermi operators (spin label is irrelevant here and, therefore, it is omitted), $a_k(q)$ are sea-displacement operators,

$$[a_k(q), a_{k'}^{\dagger}(q')] = \delta_{kk'}\delta_{qq'} \quad , \quad [a_k(q), a_{k'}(q')] = 0 \quad , \quad (2)$$

$$a_k(0) = 0 \quad , \quad (3)$$

and the coefficients T_1, T_2 and Λ are given by:

$$\begin{aligned} T_1(k, q) &= \sqrt{1 - \bar{n}_{k+q/2}}\sqrt{1 - \bar{n}_{k-q/2}} \quad , \\ T_2(k, q) &= \sqrt{\bar{n}_{k+q/2}\bar{n}_{k-q/2}} \quad , \\ \Lambda_k(q) &= \sqrt{\bar{n}_{k+q/2}(1 - \bar{n}_{k-q/2})} \quad ; \end{aligned} \quad (4)$$

n_k in the above formulas represents the Fermi occupation number (occupation number operator), \bar{n}_k is its expectation value on the ground-state, N stands for the operator of the total number of particles and $\langle N \rangle$ denotes the average number of particles. As one can see, eqs. (1) and (2) provide a bosonic representation of the particle-density operators for Fermi systems. The occupation number itself has a bosonic representation in this theory, given by

$$n_k = n^{\beta}(k) \frac{N}{\langle N \rangle} + \sum_q a_{k-q/2}^{\dagger}(q) a_{k-q/2}(q) - \sum_q a_{k+q/2}^{\dagger}(q) a_{k+q/2}(q) \quad , \quad (5)$$

where

$$n^{\beta}(k) = \frac{1}{\exp(\beta(\epsilon_k - \mu)) + 1} \quad (6)$$

is the Fermi distribution.

The exact bosonic character of the Fermi sea-displacement operators is embodied in the ansatz[2]

$$a_k(q) = \frac{1}{\sqrt{\bar{n}_{k-q/2}}} c_{k-q/2}^{\dagger} M(k, q) c_{k+q/2} \quad , \quad (7)$$

where the operator $M(k, q)$ has to be determined in such a way as to ensure the Bose commutation relations required by (2). In the limit of the random-phase approximation (RPA) eq. (7) is also written as

$$a_k(q) = \frac{1}{\sqrt{\bar{n}_{k-q/2}}} c_{k-q/2}^{\dagger} \left(\frac{n^{\beta}(k - q/2)}{\langle N \rangle} \right)^{1/2} e^{i\theta(k, q)} c_{k+q/2} \quad , \quad (8)$$

where the phase $\theta(k, q)$ is a functional of the number operator.

Equation (8) raises some technical difficulties. First, we notice that the Fermi number operator n_k has the idempotency property $n_k^2 = n_k$, and, therefore, $\sqrt{\bar{n}_{k-q/2}}$ in (8) might be taken simply as being equal with $\bar{n}_{k-q/2}$. Indeed, a general operatorial functional $f(A)$ may be defined by the associated Taylor series

$$f(A) = \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!} (A - a)^m \quad ; \quad (9)$$

for $A = n_k$ and $a = 1$ one obtains from (9)

$$f(n_k) = f(1) + (1 - n_k)(f(0) - f(1)) \quad , \quad (10)$$

which, for the particular case $f = \sqrt{x}$, leads to $\sqrt{n_k} = n_k$. In addition, the factor $1/\sqrt{n_{k-q/2}}$ in (8) might, conceivably, suggest divergencies problems, in view of the fact that the number operator is not an inversable operator; a certain sense should, consequently, be assigned to this writing in (8), and one of the simplest is to replace $n_{k-q/2}$ by $n_{k-q/2} + \varepsilon I$, where I is the identity operator, and take the limit $\varepsilon \rightarrow 0$ at the end of the calculations. Doing so, we obtain

$$\frac{1}{\sqrt{n_{k-q/2}}} \rightarrow \frac{1}{n_{k-q/2} + \varepsilon I} = \frac{1}{\varepsilon} \left(I - \frac{1}{\varepsilon + 1} n_{k-q/2} \right) \quad , \quad (11)$$

and

$$\frac{1}{\sqrt{n_{k-q/2}}} c_{k-q/2}^+ \rightarrow \frac{1}{\varepsilon} \left(I - \frac{1}{\varepsilon + 1} n_{k-q/2} \right) c_{k-q/2}^+ = \frac{1}{\varepsilon + 1} c_{k-q/2}^+ \rightarrow c_{k-q/2}^+ \quad . \quad (12)$$

The same result is obtained working with the function $f = 1/\sqrt{x + \varepsilon}$, and using the expansion (9). Equation (8) becomes then

$$a_k(q) = c_{k-q/2}^+ \left(\frac{n^\beta(k - q/2)}{\langle N \rangle} \right)^{1/2} e^{i\theta(k,q)} c_{k+q/2} \quad , \quad (13)$$

and using the fact that $\theta(k, q)$ is a functional of the number operator one obtains straightforwardly

$$[a_k(q), a_k^+(q)] = \frac{n^\beta(k - q/2)}{\langle N \rangle} (n_{k-q/2} - n_{k+q/2}) \quad . \quad (14)$$

Comparing this result with (2), one can see that the bosonic character of the sea-displacement operators $a_k(q)$ should probably be understood in a special, and more general, sense, which remains to be further explored, in order to "capture what one is looking for".[2] A possible suggestion toward solving this question would be that of taking the expectation value of (14) over the Fermi sea, which would lead, however, to the desired, approximate, bosonic property at the Fermi surface only, in contrast with Setlur and Chang's declaration that the new Fermi-sea displacement operators are "no longer restricted to be close to the Fermi surface".[2] Moreover, the sea-displacement operators defined by (13) are only consistent with (5) for

$$\frac{n_k}{\langle N \rangle} \left[\sum_{k_1} n_{k_1} n^\beta(k_1) - N n^\beta(k) \right] = 0 \quad , \quad (15)$$

which requires $n^\beta(k_1) = n^\beta(k) = \text{constant}$, *i.e.* the absence of the Fermi surface. Apart from the interesting consequences of such a physical picture, one can emphasize here the apparent consistency of the present bosonic theory for the Fermi systems.

The above considerations are not restricted to the RPA limit. Indeed, making use of (12) the general ansatz expressed in (7) becomes

$$a_k(q) = c_{k-q/2}^+ M(k, q) c_{k+q/2} \quad ; \quad (16)$$

let $|v \rangle = |1, 1, 1, \dots, 1, 0_{k-q/2}, 1, \dots, 1, 0, 0, 0, \dots, 0, 1_{k' \neq k+q/2}, 0, 0, 0, \dots \rangle$ be a state vector in the space of the occupation numbers, *i.e.* an empty fermion state at $k - q/2$ below the Fermi surface, and an occupied fermion state at $k' \neq k + q/2$ above the Fermi surface; then, one obtains $\langle v | [a_k(q), a_k^+(q)] | v \rangle = 0$, which certainly is at variance with the bosonic character of the

sea-displacement operators. This suggests again that the physical states space where the sea-displacement Bose operators are defined should have a much subtler structure for the Fermi systems, in accordance with the distinct particularities of these systems.

In conclusion, one may say that the bosonic properties of the sea-displacement operators introduced by Setlur and Chang for Fermi systems[1][2] might be operational within certain, carefully determined, theoretical schemes, and that simple-minded, direct computations might be deceitful in the framework of a bosonization approach of such boldness and subtlety.

References

- [1] G. S. Setlur and Y. C. Chang, e-prints 9810043, 9809124, 9808264, 9706187, 9705219, 9701206 cond-mat@xxx.lanl.gov.
- [2] G. S. Setlur and Y. C. Chang, Phys. Rev. **B57** 15 144 (1998).
- [3] A. H. Castro-Neto and E. Fradkin, Phys. Rev. Lett. **72** 1393 (1994); Phys. Rev. **B49**, 10 877 (1994); Phys. Rev. **B51**, 4084 (1995).
- [4] F. D. M. Haldane, J. Phys. **C14** 2585 (1981); Helv. Phys. Acta **65** 152 (1992); *Perspectives in Many-Particle Physics*, Proceedings of the International School of Physics "Enrico Fermi", Course CXXI, Varenna, 1992, ed. by R. Schrieffer and R. A. Broglia (North-Holland, New York, 1994).
- [5] A relevant series of references are included in Refs. 1 and 2 above.