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Fermi Liquid Theory

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Abstract

Landau's theory of the Fermi liquid is reviewed.

He^3 liquefies at cca 3.2K under normal pressure, where its mean inter-particle separation of a few angstroms is comparable with the range of the interaction potential (and with the mean inter-particle separation in the corresponding ideal gas); its thermal wavelength is about 8Å, so that, under these conditions, He^3 is a quantum liquid of fermions, or a Fermi liquid (sometimes called a normal Fermi liquid too). The motion of the He^3 atoms in the (repulsive) self-consistent, mean-field potential is affected by inertia effects, *i.e.* the particles possess an effective mass, and consequently they obey the Fermi distribution, like an ideal Fermi gas. The elementary excitations are therefore quasi-particles, lying near the Fermi level; indeed, the states lying deeply into, or highly above, the Fermi sea require, on one hand, high excitation energies, according to the Fermi character of the distribution, and, consequently, they do not contribute much to the liquid properties; on the other hand, they are neither well-defined excitations, because the interaction may redistribute their energy, and momentum, in many ways, and, consequently, it may affect them drastically. It follows that the interaction does not affect the step-wise shape of the Fermi distribution, and the relevant quasi-particles may be viewed as slight distortions of the Fermi sea near its surface; this is especially true for high values of particles concentration, where the Fermi energy is high and the treatment of the interaction effects has a perturbative character. Similarly, the temperature T smears out the step-wise Fermi distribution over a range of the order of T, much smaller than the Fermi energy. This is Landau's theory of the Fermi liquid.

He^3 preserves its nature of a Fermi liquid over a certain, limited, range of parameters, of course; at very low temperatures, for instance, weak attractive interactions of magnetic origin pair up the He^3 atoms into bosons, which exhibit superfluidity; under pressure He^3 may even solidify at low temperatures, in a rather disordered structure which requires additional entropy; as such, solid He^3 may serve as a natural tool for reaching significantly low temperatures; etc. Another example of a certain Fermi liquid can be provided by the particular case of the electrons in solids, which interact through long-range Coulomb potential, in contrast to He^3 where the potential is short-range. However, the long-range part of the Coulomb potential is screened out by the positively-charged, neutralizing ionic background, so that the system is stable. In addition, the electronic liquid in solids is degenerate at practically any usual temperature, due to the light mass of the electrons, and the electronic thermal wavelength is very large.

The Fermi sea and the Fermi surface are defined by minimizing the ground-state energy with respect to the fermion occupancy under the constraint of a fixed number of particles N; the

chemical potential μ is thereby obtained, and the Fermi momentum $\mathbf{p}_F$. The volume of the Fermi sea is determined by the concentration N/V of the particles, and for an isotropic liquid $N/V = gp_F^3/6\pi^2\hbar^3$, where g is the spin weight. The shape of the Fermi surface may, however, be different from a sphere, depending on the external fields which act upon the particles, as, for instance, the crystal field acting upon the electrons in a solid. The interaction does not change, however, the volume of the Fermi sea, since the number of particles is conserved; a common, isotropic interaction preserves also the shape of the Fermi sea, and, therefore, the Fermi momentum $\mathbf{p}_F$ is preserved, too. The quasi-particle energy is a function $\varepsilon(\mathbf{p})$ of the momentum $\mathbf{p}$, and it can be written as

$$\varepsilon(\mathbf{p}) = \mu + \mathbf{v}_F(\mathbf{p} - \mathbf{p}_F) \quad , \tag{1}$$

where $\mathbf{v}_F = \partial\varepsilon/\partial\mathbf{p}_F$ is the Fermi velocity, and $\mathbf{p}$ is close to $\mathbf{p}_F$. These quasi-particles lie in a narrow region Δp around the Fermi surface, determined by the interaction effects, and their uncertainty in momenta is $\sim (\Delta p)^2/p_F$; this uncertainty comes from the quasi-particles interaction, *i.e.* from the quasi-particles scattering, and one can see that it is negligible as long as the quasi-particles are located around the Fermi surface, *i.e.* as long as $\Delta p \ll p_F$. The corresponding lifetime of the quasi-particles is $\tau_{int} \sim \hbar\mu/(\varepsilon - \mu)^2$, and it is long enough under the same condition, which means that the range over which the quasi-particles are well-defined around the Fermi surface increases with increasing the particles concentration. Similarly, the quasi-particles are spread out around the Fermi surface over a Δp range due to the thermal effects, such that $\Delta p \sim T/v_F$, and the corresponding uncertainty in momenta is $(\Delta p)^2/p_F \sim T^2/v_F^2 p_F \ll \mu T/v_F^2 p_F \sim \Delta p \ll \mu^2/v_F^2 p_F \sim p_F$, which means again that the quasi-particles momenta are pretty well determined; similarly, their uncertainty in energy is $\sim T^2/\mu$, so that their lifetime is $\tau_T \sim \hbar\mu/T^2$, much longer that the characteristic time of the Fermi level μ. The two uncertainties, one due to the interaction and another due to the temperature, are competitive, and the net lifetime τ is given by $\tau^{-1} = \left(\tau_{int}^{-1} + \tau_T^{-1}\right) n$, where n is the Fermi distribution, according to the probabilistic nature of the quasi-particles. All these are consistent with the linear series expansion (1) of the quasi-particle spectrum.

The lowest energy levels of a Fermi liquid consists of particle-hole excitations around the Fermi surface, and they are given by

$$\varepsilon(\mathbf{p}) = \mathbf{v}_F\mathbf{p} \quad , \tag{2}$$

where $\mathbf{p}$ denotes here the small variation in the momentum; noteworthy, this energy depends on the relative orientation of the Fermi velocity with respect to the momentum, and, incidentally, the "superfluidity" criterion $v < \mathbf{v}_F\mathbf{p}/p$ is never satisfied; one says that the excitations (2) form a particle-hole continuum, and this is called a Fermi-type spectrum. In principle, the quasi-particle energy (1) may also depend on the particle spin $\mathbf{s}$; however, this dependence may only include contributions of the s^2- or $(\mathbf{sp})^2$-type; the former is irrelevant, while the latter splits the $2s + 1$ degenerate levels into $(1/2)(2s+1)$ levels, each twofold degenerate; so, one can say that the quasi-particles have a one-half spin. The spin weight g accounts for the corresponding density of states in the subsequent calculations.

Both the quasi-particles and the temperature distort the Fermi distribution by a certain, small, amount $\delta n(\varepsilon)$; consequently, one may define the quasi-particle energy $\varepsilon(\mathbf{p})$ by the corresponding small change in the energy

$$\delta E = \frac{gV}{(2\pi\hbar)^3} \int d\mathbf{p} \cdot \varepsilon\delta n(\varepsilon) \quad , \tag{3}$$

while the entropy is the Fermi entropy

$$S = -\frac{gV}{(2\pi\hbar)^3} \int d\mathbf{p} \left[n\ln n + (1-n)\ln(1-n)\right] \quad ; \tag{4}$$

of course, this corresponds to the Fermi distribution

$$n = \frac{1}{\exp{[(\varepsilon - \mu)/T]} + 1} \tag{5}$$

for the quasi-particles at thermal equilibrium and for a given number N of particles. The distribution (5) is smeared out over a small T-range around the chemical potential μ, and the thermal effects can be computed in the same way as those for the ideal Fermi gas (including the change in the chemical potential due to the temperature). It remains, of course, to account for the interaction effects, especially in the density of states at the Fermi surface, *i.e.* the dependence $\varepsilon(\mathbf{p})$, which amounts, in principle, to determining the two parameters μ and v_F in (1). Likewise (3) small changes in the distribution determine small changes in the quasi-particle energies, which may be represented as

$$\varepsilon(\mathbf{p}) = p^2/2m + \frac{g}{(2\pi\hbar)^3} \int d\mathbf{p}' \cdot f(\mathbf{p}, \mathbf{p}') \delta n(\varepsilon') \ ; \tag{6}$$

the momenta $\mathbf{p}$ and $\mathbf{p}'$ are close to the Fermi surface, and the function $f(\mathbf{p}, \mathbf{p}')$, which is symmetrical in its variables, (and may also depend on spin) depends on the particle interaction. The energy $\varepsilon(\mathbf{p})$ is determined in (6) to the same accuracy as in (1); it is the hamiltonian of the quasi-particles, and it may depend on the position, too, for a slightly perturbed liquid, as in a quasi-classical description. It is worth noting here the self-consistent character of the equation above, together with the quasi-particles Fermi distribution (5) (and, as such, the quasi-particle energy depends also on temperature, though to a higher-order approximation). The f-term in (6) accounts for the quasi-particles scattering, and one says that it dresses the free particles with interaction, the free particles being, therefore, bare particles. Solving for the interaction effects in terms of quasi-particles is also called the renormalization of the interaction, the quasi-particles being renormalized particles. The f-term in (6) represents also the reaction of the liquid to a particle excitation, and, in this respect, the theory of the Fermi liquid amounts to a linear response theory. It is important to notice in this case that the interaction integrals of the f-type in (6) include the density of the quasi-particles states at the Fermi surface, and the latter is large for high values of the particle concentration (*i.e.* for a well-defined quasi-particle picture); indeed, the density of states $\mathcal{D}_F$ is given by

$$d\tau = g\frac{V}{(2\pi\hbar)^3}d\mathbf{p} = g\frac{V}{(2\pi\hbar)^3}\frac{dsd\varepsilon}{v_F} = g\frac{V}{(2\pi\hbar)^3}\frac{p_F^2}{v_F}dod\varepsilon = \mathcal{D}_F dod\varepsilon \ , \tag{7}$$

where $d\tau$ is the infinitesimal number of states, ds is the surface element, and $do = \sin\theta d\theta d\varphi$ is the infinitesimal solid angle, and one can see that $\mathcal{D}_F$ increases with increasing p_F. Consequently, the f-function, and the interaction effects, must be weak, according to the perturbational character of the theory. Noteworthy, the small variations of the Fermi distribution at zero temperature, located at the Fermi surface, are $\delta n = -\delta(\varepsilon - \mu)\delta\varepsilon$, and

$$\delta(\varepsilon - \mu) = \int \frac{ds}{v_F}\delta(\mathbf{p} - \mathbf{p}_F) \ , \tag{8}$$

which, together with (6), determines, in principle, the form of the f-function and the corresponding small changes δn and $\delta\varepsilon$. From (6) one may infere the form

$$\mu = p_F^2/2m + \frac{g}{(2\pi\hbar)^3} \int d\mathbf{p}' \cdot f(\mathbf{p}_F, \mathbf{p}')n(\varepsilon') \tag{9}$$

for the chemical potential μ, to the first approximation.

By Galilei's principle of relativity the momentum of the liquid is given by

$$\int d\tau \cdot \mathbf{p}n = m \int d\tau \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}}n \quad ; \tag{10}$$

for a quasi-particle distortion δn one obtains

$$\int d\tau \cdot \mathbf{p}\delta n = m \int d\tau \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}}\delta n + \frac{m}{V} \int d\tau d\tau' \cdot \frac{\partial f}{\partial \mathbf{p}}n\delta n' = $$

$$= m \int d\tau \cdot \frac{\partial \varepsilon}{\partial \mathbf{p}}\delta n - \frac{m}{V} \int d\tau d\tau' \cdot f\frac{\partial n'}{\partial \mathbf{p}'}\delta n \quad , \tag{11}$$

or

$$\frac{\mathbf{p}}{m^*} = \frac{\mathbf{p}}{m} - \frac{g}{(2\pi\hbar)^3} \int ds' \cdot \frac{\mathbf{v}'_F}{v'_F}f(\mathbf{p}, \mathbf{p}'_F) \quad , \tag{12}$$

where m^*, defined by $\mathbf{v} = \partial\varepsilon/\partial\mathbf{p} = \mathbf{p}/m^*$, is the quasi-particle mass. For a spherical Fermi surface

$$\frac{1}{m^*} = \frac{1}{m} - g\frac{p_F}{(2\pi\hbar)^3} \int do \cdot \cos\theta \cdot f \quad , \tag{13}$$

where θ is the angle between $\mathbf{p}_F$ and $\mathbf{p}'_F$, $do = \sin\theta d\theta d\varphi$, and f depends on p_F and $\cos\theta$.

A displacement field $\mathbf{u}$ generates a volume change $\delta V = V\,div\mathbf{u}$; for sound, whih is adiabatic, this implies a potential energy per unit volume $-(1/2V)(\partial p/\partial V)(\delta V)^2$, which obviously implies the (adiabatic) compressibility $\kappa = -(1/V)(\partial V/\partial p)$; this energy can further be written as $(\rho/2)(\partial p/\partial \rho)(div\mathbf{u})^2$, where $\rho = mN/V$ is the density; the corresponding kinetic energy per unit volume is $\rho \, \dot{\mathbf{u}}^2 /2$, so that the sound velocity u is given by

$$u^2 = \partial p/\partial \rho = -\frac{V^2}{mN}\frac{\partial p}{\partial V} = \frac{1}{\rho\kappa} \quad . \tag{14}$$

Since $Nd\mu = Vdp - SdT$, one gets for vanishing temperatures (where there is no need to distinguish between adiabatic and isothermal compressibility)

$$u^2 = -\frac{V}{m}\frac{\partial\mu}{\partial V} = \frac{N}{m}\frac{\partial\mu}{\partial N} \quad , \tag{15}$$

where use has been made of the fact that the chemical potential μ depends only on the concentration N/V. This equation may serve to determining the chemical potential. Now,

$$\delta\mu = \frac{\partial\varepsilon}{\partial\mathbf{p}_F}\delta\mathbf{p}_F + \frac{1}{V} \int d\tau' \cdot f\delta n' \quad , \tag{16}$$

which agrees with (9), and where the effective mass m^* is to be noted, as for a local change in the density; it is also worth remarking that the changes associated with the compressibility amount, naturally, to changes in the distribution at the Fermi surface. The first term in the right-hand side of (16) gives $[(2\pi\hbar)^3/4\pi gVm^*p_F]\,\delta N$ for an isotropic Fermi sea; the second term gives $[(1/4\pi V)\int do' \cdot f]\,\delta N$, so that

$$u^2 = \frac{p_F^2}{3mm^*} + \frac{N}{4\pi mV} \int do' \cdot f(\mathbf{p}_F, \mathbf{p}'_F) \quad , \tag{17}$$

or

$$u^2 = \frac{p_F^2}{3m^2} + \frac{gp_F^3}{3m(2\pi\hbar)^3} \int do' \cdot (1 - \cos\theta) f(\mathbf{p}_F, \mathbf{p}_F') \quad . \tag{18}$$

It is worth noting, however, that the sound does not propagate in a Fermi liquid at vanishing tempeartures, as a consequence of its total absorption by the particle-hole excitations; this is called the quasi-particle damping of the sound. Indeed, the absorption coefficient of the sound is defined as $\gamma \sim 1/\omega\tau_s \sim \lambda/l$, where ω is the sound frequency, λ is its wavelength, τ_s is the sound lifetime and l is the mean free-path of the sound; by definition $1/l$ is proportional to the viscosity; for $\omega\tau \ll 1$, where τ is the quasi-particles lifetime, the frequency is as low as only the temperature effects govern the quasi-particles lifetime, so that the absorption rate is $\gamma \sim \omega^2/T^2$ (by definition, $\tau_s \sim (1/\omega)\,(1/\omega\tau)\,(1/\omega)$, where a characteristic frequency due to the interaction processes appears in the last factor); hence, the sound can not be propagated at vanishing temperatures, where the viscosity of the quasi-particles is infinite. For $\omega\tau \gg 1$ the thermal effects are negligible, and the absorption rate goes, obviously, like $\gamma \sim \omega^2$. A minimum of γ appears evidently in-between, of the order of $\gamma \sim T^2$, which shows that a certain sound-type disturbance can be propagated even at vanishing temperature; obviously, this is nothing but a local perturbation in the quasi-particles density, *i.e.* a space-time perturbation of the Fermi distribution; this perturbation is called zero sound, in order to distinguish it from the ordinary, first, sound (and to distinguish it from the second sound which may propagate in superfluids). The zero sound (like the superfluid sound u too) is, in fact, ordinary sound propagating in the context of interaction and at low temperatures, *i.e.* the aspect the ordinary sound takes in quantum liquids.

The slight space-time changes in the particle distribution are described by the kinetic, or Boltzmann, equation

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\frac{\partial \varepsilon}{\partial \mathbf{r}} = 0 \quad , \tag{19}$$

where the collision rate of the quasi-particles is too low to be kept in the right-hand side; and obviously the motion must satisfy $\omega\tau \gg 1$, as being the motion of the quasi-particles. For $n = n_0 + \delta n$, where n_0 is the unperturbed Fermi distribution and δn is a $\omega, \mathbf{k}$-wave, one gets

$$(\omega - \mathbf{p}\mathbf{k}/m^*)\,\delta n = \mathbf{p}\mathbf{k}\frac{gp_F}{(2\pi\hbar)^3}\int do' \cdot f\delta n' \quad , \tag{20}$$

for a spherical Fermi surface. Noteworthy, the changes in the distribution are localized at the Fermi surface. This is a typical equation describing the self-consistent effects of interacting particles (and as such it is called, among others, the Bethe-Salpeter equation). In principle, the f-function may depend on spin, too, in which cases spin summations are included in the g factor. First, the equation is solved for the spin dependence of the changes δn, the solution describing thus both particle- and spin-density waves. Thereafter, it is easy to see that the main contribution to the integral comes from $\mathbf{p}_F$ close to $\mathbf{p}_F'$ in the f-function, so that one may denote this contribution of f by $f(0)$ (spin effects included). Under these circumstances equation (20) amounts to

$$\frac{2\pi^2\hbar^3}{gm^*p_F f(0)} = \frac{1}{2}\int d\theta \cdot \sin\theta \frac{\cos\theta}{s - \cos\theta} \quad , \tag{21}$$

where $s = \omega/v_F k$; the integral is $-1 - (s/2)\ln|(s-1)/(s+1)|$ (which, sometimes, is called the Lindhard's function), and the equation has solution only for $f(0) > 0$, which amounts to repulsive interaction, and for $s = \omega/v_F k > 1$, this latter condition ensuring the absence of the excitation processes for the quasi-particles, *i.e.* the existence of the quasi-particles that propagate the zero

sound. For small values of the left-hand side the solution is obtained for $s \to \infty$, where the integral goes like $1/3s^2$; one obtains

$$\omega = \left[\frac{2}{gm^*} \frac{N}{V} f(0) k^2 \right]^{1/2} \quad ; \tag{22}$$

this is particularly relevant for interacting electrons ($g = 2$), where one can see easily that $f(0) = 4\pi e^2/k^2$, *i.e.* the Fourier transform of the Coulomb potential; indeed, in the long wavelength limit $k \to 0$ this is a strong coupling, and the corresponding frequency

$$\omega = \left(\frac{4\pi N e^2}{m} \right)^{1/2} \tag{23}$$

is that of the particle-density waves called plasmons (due to the long-range correlations the bare mass of the electrons is practically left unchanged). On the contrary, in the opposite limit of high values of the left-hand side of (21) one obtains the genuine zero sound frequency

$$\omega = v_F \left\{ 1 + 2 \exp \left[-4\pi^2 \hbar^3 / gm^* p_F f(0) \right] \right\} k \quad , \tag{24}$$

where $f(0) = V(0)$, *i.e.* the long-wavelength limit of the (short-range) original particle interaction. Despite their appearance as affecting only one-particle states (and in contrast with the quasi-particles), the zero sound, as well as the sound in superfluids, are in fact generated by interaction with the other particles, and as such they are collective modes of the liquid motion, like ordinary sound; and like the plasmons, where the long-range character of the Coulomb interaction affects all the particles.

The f-function in (6), expressing the change in the energy of a quasi-particle as a result of its interaction with the other quasi-particles located at the Fermi surface, can be derived from the original particle interaction. Indeed, for high values of particles concentration, and for an original interaction of the type $V + \sigma\sigma'U$, where $\sigma, \sigma' = \pm 1$ are spin variables for a one-half spin one gets

$$f_{\sigma\sigma'} = V(0) + \sigma\sigma'U(0) - [V(\mathbf{p} - \mathbf{p}') + U(\mathbf{p} - \mathbf{p}')] \delta_{\sigma\sigma'} \tag{25}$$

for short-range interaction, where the right-hand side includes the Fourier transforms of the interaction, and the momenta are close to the Fermi surface. It is worth noting in (25) the direct interaction corresponding to a vanishing momentum transfer, as well as the exchange contribution corresponding to a finite momentum transfer. For long-range interactions the long-range terms in (25) are absent, except for the $k \to 0$ limit, while the short-range terms are those produced by screening. It is important to notice that the frequency vanishing and the vanishing of the momentum transfer are not intervertible limits, due to the particle-hole continuum of excitations. The Coulomb interacting electrons in metals form a special liquid, which may be termed an electron liquid; the long-range degrees of freedom are affected by plasmons, while the remaining short-range motion is that of a weakly-interacting Fermi liquid; the corresponding f-function of the latter is given by ($U = 0$)

$$f_{\sigma\sigma'} = -\frac{4\pi e^2}{\left| \mathbf{p} - \mathbf{p}' \right|^2 / \hbar^2} \delta_{\sigma\sigma'} \quad , \quad \left| \mathbf{p} - \mathbf{p}' \right| > \hbar k_c \quad , \tag{26}$$

where k_c is a certain cut-off wavevector; it is reminiscent of the Thomas-Fermi screening wavevector

$$q_{TF} = (4 m p_F e^2 / \pi \hbar^3)^{1/2} \quad . \tag{27}$$

The effective mass of the electrons would be therefore

$$m^* = m \left\{ 1 + \frac{me^2}{\pi \hbar p_F} \left[1 - \beta^2/4 + \ln(\beta/2) \right] \right\} \quad , \tag{28}$$

where $\beta = k_c/k_F$ (where k_F denotes the Fermi wavevector); for electrons at metallic concentrations one obtains $\beta \sim 1$ from the minimum of the ground-state energy, so that the electron effective mass is practically left undressed (another contribution to the effective mass comes from the electron-plasmon coupling, which likewise is very small). The f-function above may also be used as a scattering amplitude, leading to small life-time effects for the short-range interacting electrons; the above short-range interaction which appears in (26) is a screened Coulomb interaction, similar to the Thomas-Fermi screening.

The change $\delta\mu$ in the chemical potential due to a slight spatial imbalance $\delta n(\mathbf{r})$ in the particle concentration is $\delta\mu = v_F \delta p_F = (\pi^2\hbar^3 v_F/p_F^2)\delta n$ (the total spatial variation of the Fermi distribution vanishes, so that the quasi-particle interaction does not contribute to this change); this change equals the potential energy $-e\varphi$, which must satisfy the Poisson equation $k^2\varphi = 4\pi e + 4\pi e\delta n = 4\pi e - (4e^2 p_F^2/\pi\hbar^3 v_F)\varphi$; hence, the Thomas-Fermi screening of the Coulomb potential in (27).

The magnetic momentum of an electron is $2\beta s = \beta\sigma$, where $\beta = e\hbar/2mc$ is the Bohr magneton (2 is the gyromagnetic factor, $s = 1/2$ is the spin and c is the light velocity). In a magnetic field H its energy changes by

$$\delta\varepsilon_\sigma = -\beta\sigma H + \frac{1}{(2\pi\hbar)^3}\sum_{\sigma'}\int d\mathbf{p}' \cdot f_{\sigma\sigma'}\delta n_{\sigma'} \;\;; \tag{29}$$

this change takes place at the Fermi surface, so that

$$\delta\varepsilon_\sigma = -\beta\sigma H - \frac{m^* p_F}{4\pi^2\hbar^3}\sum_{\sigma'}\int d\theta \cdot \sin\theta f_{\sigma\sigma'}\delta\varepsilon_{\sigma'} \;\;, \tag{30}$$

whose solution is

$$\delta\varepsilon_\sigma = -\frac{1}{2}g\beta\sigma H \;\;, \tag{31}$$

where

$$g^{-1} = \frac{1}{2}\left[1 + \frac{m^* p_F}{4\pi^2\hbar^3}\int d\theta \cdot \sin\theta(f_{\sigma\sigma} - f_{\sigma\bar\sigma})\right] \;\;, \tag{32}$$

with $\bar\sigma = -\sigma$. The magnetization is given by

$$\chi H = \frac{\beta}{(2\pi\hbar)^3}\sum_\sigma\int d\mathbf{p}\cdot\sigma\delta n_\sigma = -\frac{\beta m^* p_F}{2\pi^2\hbar^3}\sum_\sigma\sigma\delta\varepsilon_\sigma = \frac{\beta^2 m^* p_F}{2\pi^2\hbar^3}gH \;\;, \tag{33}$$

whence the spin susceptibility

$$\chi = \frac{\beta^2 m^* p_F}{2\pi^2\hbar^3}g \;\;. \tag{34}$$

Making use of (26) one obtains

$$\chi = \frac{\beta^2 m p_F}{\pi^2\hbar^3}\left[1 + \frac{me^2}{\pi\hbar p_F}\ln(\beta/2)\right] \tag{35}$$

for electrons. Similar computations hold for any other Fermi liquid. This is called the Pauli paramagnetism; the orbital motion of the electrons gives the Landau diamagnetism, which is minus one-third of the χ above.

The number of particles at finite temperatures is given by

$$N = g\frac{V}{(2\pi\hbar)^3}\int d\mathbf{p}\cdot n \;\;, \tag{36}$$

where n is the Fermi distribution. Let F be given through f by

$$F = g\frac{V}{(2\pi\hbar)^3}\int d\mathbf{p}\cdot fn \quad ; \tag{37}$$

according to the properties of the Fermi distribution it may be written as

$$
\begin{aligned}
F &= g\frac{V}{(2\pi\hbar)^3}\int \frac{ds d\varepsilon}{v}\cdot fn = g\frac{V}{(2\pi\hbar)^3}\int do d\varepsilon\cdot \left(p^2 f/v\right)n = \\
&= g\frac{V}{(2\pi\hbar)^3}\int do\left[\int_0^\mu d\varepsilon\cdot\left(p^2 f/v\right) + \frac{\pi^2 T^2}{6}\left(p^2 f/v\right)'_\mu + ...\right] = \\
&= F_0 + g\frac{V}{(2\pi\hbar)^3}\int do\left[\left(p^2 f/v\right)_{\mu_0}\delta\mu + \frac{\pi^2 T^2}{6}\left(p^2 f/v\right)'_{\mu_0} + ...\right] \quad ,
\end{aligned}
\tag{38}
$$

where F_0 is that quantity at zero temperature and μ_0 is the chemical potential at zero temperature. For a spherical Fermi surface and the conserved number of particles one obtains

$$\delta\mu = -\pi^2 T^2 m^*/6p_F^2 \quad . \tag{39}$$

The change in energy can be written as

$$\delta E = g\frac{V}{(2\pi\hbar)^3}\int d\mathbf{p}\cdot\varepsilon(n - n_0) \quad , \tag{40}$$

where n_0 denotes the Fermi distribution at zero temperature; by using (38) for a spherical Fermi surface one obtains

$$\delta E = g\frac{V}{12\hbar^3}m^* p_F T^2 = \frac{\pi^2 T^2}{2p_F^2}m^* N \quad ; \tag{41}$$

hence the entropy

$$S = g\frac{V}{6\hbar^3}m^* p_F T \quad , \tag{42}$$

the free energy

$$\delta F = -g\frac{V}{12\hbar^3}m^* p_F T^2 \quad , \tag{43}$$

and the heat capacity

$$C = g\frac{V}{6\hbar^3}m^* p_F T = S \quad ; \tag{44}$$

as well as the change in pressure

$$\delta p = g\frac{1}{18\hbar^3}m^* p_F T^2 \quad . \tag{45}$$

The thermodynamic potentials Ω and Φ change by

$$\delta\Omega = \delta\Phi = \delta F = -g\frac{V}{12\hbar^3}m^* p_F T^2, \tag{46}$$

when expressed in proper variables, *i.e.* μ instead of N, and, respectively, p instead of V in δF. Noteworthy, $\delta\Omega_0 = -N\delta\mu$, so that the total change in Ω is $-(gV/18\hbar^3)m^* p_F T^2 = -2\delta E/3 = -\delta(pV) = -V\delta p$. Similarly, the change in Φ_0 is $V\delta p$, which, together with (45), leads to a total change $-(gV/36\hbar^3)m^* p_F T^2 = \delta(\mu N) = N\delta\mu$ in Φ. From (45) one can see easily that $C_p - C_v \sim T^3$ (as expected from $S \sim T$), and there is no need, therefore, to distinguish between the two heat capacities (in particular, the zero sound does contribute insignificantly to the thermodynamic

properties). By using again (45) one finds the coefficient of thermal expansion $(1/V)(\partial V/\partial T)_p = -(gT/6V\hbar^3)\partial(m^*p_F)/\partial p$, with $\partial\Phi/\partial p = V$; one can see that its ratio to the heat capacity is independent of temperature (Gruneisen's law).

Liquid He^3 under normal pressure has a Fermi wavevector $p_F/\hbar \sim 0.8\text{Å}^{-1}$ (from density), an effective mass $m^* \sim 2.4m$ (from the heat capacity), and a sound velocity $u \sim 190$m/s (from compressibility).

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