Journal of Theoretical Physics

Founded and Edited by M. Apostol **37** (1999)

ISSN 1453-4428

On the "bosonization" in two and three dimensions

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Abstract

It is shown that the "bosonization" of the fermions in two and three dimensions proposed by Castro Neto and Fradkin (see, for instance, A. H. Castro Neto and Eduardo Fradkin, Phys. Rev. Lett. **72** 1393 (1994), Phys. Rev. **49** 10 877 (1994)) is inconsistent. It is also shown that the bosonic "coherent" state introduced by these authors is not a coherent state, and the corresponding classical action is chosen arbitrarily. In addition, the single-fermion results obtained by Castro Neto and Fradkin in Phys. Rev. **51** 4084 (1995) by using this method depend on the momentum-transfer cut-off, and, consequently, are unphysical.

Few years ago, Castro Neto and Fradkin[1]·[2] (see also Ref.3) noticed that the Fourier transforms

$$\rho_{\mathbf{q}}(\mathbf{k}) = c^{+}_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2} \tag{1}$$

of the fermion-density operator satisfy boson-like commutation relations

$$\langle FS \mid [\rho_{\mathbf{q}}(\mathbf{k}), \rho_{-\mathbf{q}'}(\mathbf{k}')] \mid FS\rangle = \mathbf{q}\mathbf{v}_{\mathbf{k}} \delta(\mu - \varepsilon_{\mathbf{k}}) \delta_{\mathbf{k}\mathbf{k}'} \delta_{\mathbf{q}\mathbf{q}'} \tag{2}$$

in the limit $\mathbf{q} \to 0$, when averaged over the Fermi sea $\mid FS\rangle$; here $\varepsilon_{\mathbf{k}}$ denotes the one-fermion energy, $\mathbf{v}_{\mathbf{k}} = \partial\varepsilon_{\mathbf{k}}/\partial\mathbf{k}$ is the corresponding velocity (Planck's constant is set equal to one), μ denotes the chemical potential, and the spin labels are omitted for simplicity. This is an old observation, and it was systematically exploited, probably for the first time, by Sawada[4] in 1957. Based on this observation the above authors[1]·[2] attempted to "bosonize" the fermions in two and three dimensions. To this end, normal-ordered operators

$$a_{\mathbf{q}}(\mathbf{k}) = \rho_{\mathbf{q}}(\mathbf{k}) \theta(\mathbf{q}\mathbf{v}_{\mathbf{k}}) + \rho_{-\mathbf{q}}(\mathbf{k}) \theta(-\mathbf{q}\mathbf{v}_{\mathbf{k}}) \tag{3}$$

are introduced, where θ is the step function, such as $a_{\mathbf{q}}(\mathbf{k}) \mid FS\rangle = 0$, and boson-like commutation relations

$$\left[a_{\mathbf{q}}(\mathbf{k}), a^{+}_{\mathbf{q}'}(\mathbf{k}')\right] = |\mathbf{q}\mathbf{v}_{\mathbf{k}}| \, \delta(\mu - \varepsilon_{\mathbf{k}}) \delta_{\mathbf{k}\mathbf{k}'} (\delta_{\mathbf{q}\mathbf{q}'} + \delta_{\mathbf{q},-\mathbf{q}'}) \tag{4}$$

are adopted for these operators, as suggested by (2). However, the boson-like commutation relations (4) are not consistent with the definition given in (1) and (3), since

$$a^{+2}_{\mathbf{q}}(\mathbf{k}) = 0 \tag{5}$$

for $\mathbf{q} \neq 0$. Indeed, one can check easily that

$$\rho^{2}_{\mathbf{q}}(\mathbf{k}) = c^{+}_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2} c^{+}_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2} = n(\mathbf{k}) \delta_{\mathbf{q},0} \quad , \tag{6}$$

where $n(\mathbf{k}) = c^+_{\mathbf{k}} c_{\mathbf{k}}$ is the fermion occupation number. Therefore, the "bosonic" operators $a_{\mathbf{q}}(\mathbf{k}), a^+_{\mathbf{q}}(\mathbf{k})$ are in fact "fermionic" operators, operating only on two states, $\mid FS\rangle$ and $\mid \mathbf{qk}\rangle = a^+_{\mathbf{q}}(\mathbf{k}) \mid FS\rangle$, and the "bosonization" is actually a "fermionization". Since $a_{\mathbf{q}}(\mathbf{k}) = a_{-\mathbf{q}}(\mathbf{k})$, one may restrict oneself to $\mathbf{qv_k} > 0$, and the boson-like commutation relations (4) may be used (approximately) for $\mathbf{k} \neq \mathbf{k}'$ or $\mathbf{q} \neq \mathbf{q}'$, but not for $\mathbf{k} = \mathbf{k}', \mathbf{q} = \mathbf{q}'$, where the a-operators satisfy fermion-like commutation relations $\left\{a_{\mathbf{q}}(\mathbf{k}), a^+_{\mathbf{q}}(\mathbf{k})\right\} = 1$ (and $a^{+2}_{\mathbf{q}}(\mathbf{k}) = 0$). This would suffice to say that the attempt made in Refs.1 and 2 at "bosonizing" the fermions in two and three dimensions is inconsistent.

Overlooking the contradiction implied by (4) and (5) Castro Neto and Fradkin[1],[2] proceed to constructing a "coherent" state defined by

$$\mid u_{\mathbf{q}}(\mathbf{k})\rangle = \exp\left[\frac{v_{\mathbf{k}}}{2\left|\mathbf{qv_k}\right|} u_{\mathbf{q}}(\mathbf{k}) a^+_{\mathbf{q}}(\mathbf{k})\right] \mid FS\rangle \quad ; \tag{7}$$

in view of (5) this can also be written in various other forms, as, for instance,

$$\mid u_{\mathbf{q}}(\mathbf{k})\rangle = \left[1 + \frac{v_{\mathbf{k}}}{2\left|\mathbf{qv_k}\right|} u_{\mathbf{q}}(\mathbf{k}) a^+_{\mathbf{q}}(\mathbf{k})\right] \mid FS\rangle \quad ; \tag{8}$$

obviously, the scalar products are different for these states. Restricting oneself to the low-energy states defined by the a, a^+-operators, one may establish the (over)completeness of both (7) and (8) with a gaussian measure; however, the path-integral lagrangean is different, and in fact it is not unique. Indeed, from (7) one obtains the lagrangean density

$$\begin{aligned} \mathcal{L} \;\; = \;\; & \sum_{\mathbf{kq}} \frac{v^2_{\mathbf{k}} \delta(\mu - \varepsilon_{\mathbf{k}})}{2\left|\mathbf{qv_k}\right|} \cdot i u^*_{\mathbf{q}}(\mathbf{k}) \frac{\partial}{\partial t} u_{\mathbf{q}}(\mathbf{k}) - \\ & - \langle \{u_{\mathbf{q}}(\mathbf{k})\} \left|H\right| \{u_{\mathbf{q}}(\mathbf{k})\} \rangle / \langle \{u_{\mathbf{q}}(\mathbf{k})\} \,||\, \{u_{\mathbf{q}}(\mathbf{k})\} \rangle \quad , \end{aligned} \tag{9}$$

where H denotes the hamiltonian of the system, while, using (8), general factors of the form

$$\left[1 + \frac{v^2_{\mathbf{k}}}{\left|\mathbf{qv_k}\right|} \delta(\mu - \varepsilon_{\mathbf{k}}) \left|u^+_{\mathbf{q}}(\mathbf{k})\right|^2\right]^{-1} \tag{10}$$

would affect the terms in the lagrangean density (9); they arise from the fact that commutators of the type $[e^{ua}, a^+] = uCe^{ua}$ differ from commutators like $[1 + ua, a^+] = uC$, where C stands for the commutator $[a, a^+]$, $[a, a^+] = C$. Of course, such inconsistencies originate in the fact that "bosonic" coherent states of the form given by (7) can not be constructed with "fermionic" operators. One may conclude, therefore, that (7) (or its equivalent form (8)) is not a coherent state, and, in this respect, the choice (9) made by Castro Neto and Fradkin[1],[2] for the classical lagrangean appears to be arbitrary.

The situation is different in one dimension, where the bosonization is a genuine one. Indeed, as it is well-known, boson-like operators can be defined there by

$$\rho_{1,2\mathbf{q}} = \sum_{\mathbf{k} \sim \pm \mathbf{k}_F} c^+_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2} \quad , \tag{11}$$

where the summation over $\mathbf{k}$ is restricted to a small range around the two $\pm\mathbf{k}_F$-Fermi "points", $\mathbf{k}_F$ being the Fermi momentum; it is easy to see that, due to the summation over $\mathbf{k}$,

$$(b^+_{1,2\mathbf{q}})^n \mid FS\rangle \neq 0 \tag{12}$$

for any n, where

$$b_{_{1,2}\mathbf{q}} = \rho_{_{1,2}\mathbf{q}}(\mathbf{k})\theta(\pm\mathbf{q}\mathbf{k}_F) + \rho_{_{1,2}-\mathbf{q}}(\mathbf{k})\theta(\mp\mathbf{q}\mathbf{k}_F) \quad . \tag{13}$$

One may use boson-like commutation relations for these operators, similar with those given by (2), by averaging their commutators over the Fermi sea, and (12) gives then boson states for any integer n. One may conclude that this is indeed a genuine bosonization. Moreover, the low-energy excitations of the one-dimensional fermions can be described entirely in terms of the $b_{_{1,2}\mathbf{q}}$-boson operators;[5] as often emphasized, this is the origin of the non-Fermi liquid behaviour of the fermions in one dimension. In contrast with the one dimensional case, a construction similar to (11) in two- or three-dimensions would imply the continuum of particle-hole excitations, as a consequence of the motion along the Fermi surface; such a construction requires certain restrictions which, however, lead to arbitrary, unphysical results.

Indeed, drawing on the one-dimensional case Castro Neto and Fradkin[3] employ also boson-like operators defined by

$$\sum_{\mathbf{k}} \Phi_\Lambda \left(|\mathbf{k} - \mathbf{k}_F|\right) a_{\mathbf{q}}(\mathbf{k}) \quad , \tag{14}$$

where $a_{\mathbf{q}}(\mathbf{k})$ is given by (3) and $\Phi_\Lambda \left(|\mathbf{k} - \mathbf{k}_F|\right)$ is a "smearing" function with the support within a narrow range Λ around the origin (eq.(2.5) in Ref.3); Λ plays here the role of a cut-off on the fermion states (which may be called a "bandwidth" cut-off), these states being thereby restricted to a narrow region around each value of the Fermi momentum in a given set of values. In other words, the Fermi surface is divided in a number of "boxes", "sectors", or "patches", of various shapes (spheres, pill boxes, etc), centered around a discrete set of $\mathbf{k}_F$-values, and it is assumed that the Fermi surface may be "rectified" within each of these boxes, *i.e.* it is assumed that it may be approximated by planes (or lines in two dimensions); this assumption is needed in order to express the kinetic energy of the fermions in terms of the operators given by (14), and it may be fulfilled providing $\Lambda \ll k_F$. Further on, the operators given by (14) are not, in general, disentangled from each other, due to the "proximity" effects between adjacent boxes; in order to achieve this decoupling, and transform (14) in boson operators, the momentum transfer $\mathbf{q}$ is required to be much smaller that the size of the boxes, $q \ll \Lambda$; obviously, this means another cut-off, say Λ', this time on the range of the interaction (which transfers the momentum $\mathbf{q}$), and the condition of validity for the above construction would be $q < \Lambda' \ll \Lambda \ll k_F$. Such a construction is used in a number of recent publications,[6] in slightly different versions, but the above assumption of the two cut-offs is a common element. It may be useful, to a somewhat extent, for describing the collective modes of the interacting fermions, but it leads to unphysical results for single-fermion properties, as expected. This unphysical behaviour originates in the dependence of the single-fermion propagator on the cut-offs; for instance, leaving aside the bandwidth cut-off Λ (which may be absorbed in the definition of the boson operators and the coupling constants), the momentum-transfer cut-off Λ' appears in the single-fermion propagator. Such a cut-off dependence is a well-known matter too, and it was pointed out long ago in connection with the relation between the bosonized fermions and the original fermions in one dimension.[7]

For instance, leaving aside a rapidly oscillating term, it is claimed in Ref.3 that the single-fermion propagator in two dimensions requires the estimation of the integral

$$\int d\mathbf{q} \cdot \left(q_N / q_T^2\right) \quad , \tag{15}$$

where $q_{N,T}$ are the components of the wavector $\mathbf{q}$ normal, and, respectively, tangential, to the Fermi "circle" (and $d\mathbf{q} = dq_T dq_N$); Castro Neto and Fradkin[3] cut the q_N-integration by a $1/\Lambda$-exponential,

$$\int dq_N \cdot q_N \rightarrow \int dq_N \cdot q_N e^{-q_N/\Lambda} = \Lambda^2 \tag{16}$$

(eq. (8.10) in Ref.3), while the divergent q_T-integral is replaced by

$$\int dq_T \cdot \left(1/q_T^2\right) \to N(0) v_F/\Lambda^2 \tag{17}$$

(eq. (8.9) in Ref.3), where $N(0)$ is the density of states at the Fermi surface, and v_F is the Fermi velocity; these manipulations are done in order to enforce the idea that the single-fermion propagator, and the quasi-particle residue, would not depend on the cut-off choice (indeed, by using (16) and (17) one can see that (15) does not depend on Λ). Unfortunately, the above estimations are incorrect, and the single-fermion properties depend on the momentum-transfer cut-off Λ'. Indeed, first one should notice that this latter cut-off should be used in the q_N-integration, according to the construction of the "boxes" on the Fermi surface (however, if the Λ cut-off is absorbed in the boson operators and the coupling constants, then one may agree to denote the remaining interaction cut-off Λ' by Λ). Secondly, leaving aside the infrared divergence in the q_T-integral (17), the density of states $N(0)$ appearing in (17) must be replaced by a local density of states, *i.e.* $\int dq_T = \Lambda'$, for instance, and the result depends on the integration cut-off. A similar result is obtained by performing the integral in (17) with an infrared cut-off, *i.e.* $\int dq_T \cdot (1/q_T^2) = 1/\Lambda'$, and again the result depends on cut-off. In this connection, the statement made in Ref.3 that one may "substitute" the local density of states

$$N_\Lambda(\mathbf{k}_F) = \frac{1}{V} \sum_{\mathbf{k}} |\Phi_\Lambda(|\mathbf{k} - \mathbf{k}_F|)|^2 \, \delta(\mu - \varepsilon_{\mathbf{k}}) \tag{18}$$

"by its natural average which is the total density of states $N(0)$,

$$N(0) = \frac{1}{V} \sum_{\mathbf{k}} \delta(\mu - \varepsilon_{\mathbf{k}}) \tag{19}$$

divided by the solid angle of the Fermi surface $S_d = \int d\Omega$" (eqs. (2.10) and (2.11) on p. 4087 of Ref.3) is incorrect. According to the definitions used by these authors, a direct computation gives $N_\Lambda(\mathbf{k}_F) = (N(0)/S_d) \cdot (\Lambda/k_F)^{d-1}$ (for spherical Fermi surfaces), for instance, where d denotes the number of dimensions.

In conclusion, one may say that the fermion "bosonization" attempted in Refs.1-3 is inconsistent, and leads to arbitrary and unphysical results.

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