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Classical Mechanics and Quantal Mechanics *versus* Statistical Phyiscs. The rest of the Universe.

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Abstract

A few common remarks are made upon the above subject.

A recent tuitional article[1] aims at introducing a "quantum mechanical" temperature; such an attempt is probably non-singular, and even older, often-quoted textbooks[2] seem to contain remarks which might be misread in this respect. This offers me an occasion to make here a few comments on the subject. As a general remark I would say that such an enterprise is unreasonable, because we already have a satisfactory picture of both classical and quantal mechanics, on the one hand, and of statistical physics on the other; as we also have a clear understanding of their mutual connections.

One may agree that the motion of a body is represented by the time dependence of its coordinates; this dependence is given by the Principle of Inertia and Newton's Law of classical mechanics; for any given initial conditions the entire subsequent evolution of a mechanical system is determined, and the solution of the motion may be expressed as prime integrals, which are constant in time; some prime integrals are additive, like, for instance, the energy, corresponding to the symmetries of the system; the energy is conserved when the force does not depend on time, as, for instance, for an isolated system. Two isolated systems make an isolated system, and the additivity of some of the prime integrals is the definition of the multiplicity of the independent physical systems.

The coordinates may not be determined by necessity, in which case a wave moves, and a wavefunction would describe the corresponding quantal system, or the behaviour of a mechanical system on the quantal scale; but other mechanical quantities may have determined values, and the wavefunction gives these values. The time evolution of the wavefunctions proceeds by phase factors of well-determined energies, according to the energy quantization, so that wavefunctions of determined energy exist, and they are the eigenfunctions, or eigenstates, of the energy operator, *i.e.* of the hamiltonian; and the time evolution of the wavefunctions is given by Schrodinger's equation through the hamiltonian operator. Similarly, the momenta may be well-determined, for eigenfunctions of the space-translation operator, according again to the quantization, of the momentum this time. For given initial conditions the wavefunction is perfectly determined at any subsequent moment of time by Schrodinger's equation, and all the knowledge of a quantal system is contained in this wavefunction; only that this knowledge is given, essentially, in terms of probabilities, and average quantities, in accordance with the general framework of the quantal mechanics, *i.e.* in accordance with the principle of complementarity, that not all of the mechanical quantities have determined values; possibly none of them; the quantal mechanics has a statistical character in this respect, on the scale of the Planck's quanta $\hbar$ of action; which recovers the entirely determined character of the classical mechanics in the classical limit $\hbar \rightarrow 0$, according to the principle of correspondence. The principle of superposition of the wavefunctions and the measurement concept (*i.e.* the structure of linear space of the wavefunctions) lead to a bilinear appearance of the physical quantities, and to the probabilistic, or statistical, nature of the quantal systems; which is expressed in general terms by the uncertainty principle. The probability is given by the square modulus of the wavefunction, and the statistical character of the quantal mechanics is intrinsic to the natural objects; as a matter of fact, the very existence of the natural objects is not absolutely determined.

Speaking of mechanical systems, or of general features of the motion of the physical systems, or natural objects, it is perhaps worth mentioning here the high degree of generality to which these objects are determined, or, perhaps more appropriately, the great extent to which they are not determined. For instance, it is worth remarking that a quantal object may be ubiquos, as a consequence of its probabilistic nature; dynamical systems may not be integrable, *i.e.* the solutions to their motion are not representable by, at least a finite number of, functions; mechanical systems, like the non-linear ones, may move "chaotically", in the sense that their motion may change much for only a slight change in the initial conditions, or parameters; or their global motion may not even be unique, may be multifarious, though determined; etc. This lack of contents of the natural world reveals its reasonability, and the ultimate essence of the natural world might very well be the logical consistency of the pure thought, as expressed in undetermined subjective representations, like numbers.

Another step toward the voidness of the natural objects is taken by statistical physics. The statistical character of quantal systems, which manifests itself on the $\hbar$-scale is one thing, while the statistical character of the statistical physics is a further, distinct thing. The time evolution of a mechanical object, either classical or quantal, is perfectly determined in its own terms, either with perfectly determined values for all the mechanical quantities, as for classical systems, or with probabilities, in the case of the quantal systems; but this time dependence is not absolute: it depends on the initial conditions, and this conditions must be known in order to know the subsequent behaviour of the system. If the initial conditions are not known then they are chaotical. Various unknown external interactions may also be represented as temporal sequences of unknown initial conditions; under such circumstances the system performs a chaotical motion, *i.e.* an undefined motion; it is therefore described by a statistical probability, or distribution, of its various mechanical states in the phase space of coordinates, or in the space of the quantal states. A perfectly chaotical motion has, however, positive determinations; it is identical with itself at any moment of time, everywhere and for any other circumstances; therefore, there should exist an equilibrium state for which the statistical probability is constant in time, and various subsystems must also be independent of each other; these are the principles of statistical equilibrium and statistical independence, and they determine the logarithm of the statistical probability as a linear combination of additive prime integrals, like the energy. So, we are led to the canonical statistical distribution, and to temperature, which is an undefined parameter for the moment. However, since the number of distinct mechanical states taken by a system in its chaotical motion is multiplicative, it is natural to use its logarithm, which is additive, and is the entropy; then it is easy to see that the temperature is the increase in energy caused by a unit increase in entropy. Motion being chaotical the system tends to maximize the number of distinct states it visits, and so we are led to the Second Law of Thermodynamics, according to which the entropy increases in time, and is maximal at equilibrium; and hence, knowing that, by their definitions, the entropy is the average of the logarithm of the statistical distribution, one arrives again at the canonical form of the statistical distribution. And one can check out that for very large systems, capable of a pure chaotical motion, the fluctuations are vanishing, and a thermodynamic state of equlibrium is possible; since, many independent, small sub-systems contribute to finite, large quantities in this case; which is no longer so, of course, when the general framework of statistical physics, as expressed by its principles, is not fulfilled anymore.

The difference, therefore, between statistical physics and classical or quantal mechanics consists in that the statistical physics refers to the chaotical motion of a mechanical system as described by the statistical probability, while mechanics refers to a well determined time evolution of a motion, which may be expressed in terms of a probability, as in quantal mechanics, or even in terms of a statistical probability which obeys however Liouville's equation, like in kinetical physics, but in either case the mechanical evolution is determined, in the sense that it depends on the original conditions, *i.e.* on the original mechanical states, or the original probabilities; which is not so for the chaotical motion of the statistical systems. In a sense, everything is mechanical, *i.e.* it is motion, and the reductionism has no object in fact; only that every mechanical object is so in a different context, under different circumstances, in a hierarchy of generality, of physical emptiness, which reveals the underlying universal reason of the pure thinking.

One is prone perhaps to mistake statistical physics for quantal mechanics, or viceversa, because both employ probabilities, and have a statistical nature; at the core of such a confusion, like in Ref.1, lies the density matrix.

Let $\psi(x)$ be the (normalized) wavefunction; the density matrix ρ is

$$\rho(x,x') = \psi^*(x)\psi(x') \ ; \qquad (1)$$

obviously, this is the matrix element (x,x') of the projector $\rho = \mid \psi >< \psi \mid$ in the coordinates representation; as such, it is positive definite and idempotent, $\rho^2 = \rho$, and has a unity trace, $tr\rho = 1$; its only non-vanishing eigenvalue is 1, of course, and, for some expansion $\psi = \sum c_n \varphi_n$ in orthogonal eigenfunctions, it reads

$$\rho(x,x') = \sum_{nm} c_n^* c_m \varphi_n^*(x)\varphi_m(x') \quad , \qquad (2)$$

where the $c_n^* c_m$ matrix is indeed diagonalizable with one eigenvalue unity, and all the rest vanishing; the diagonal representation being (1); time dependence being irrelevant. Its usefulness consists in its expressing the bilinear appearance of physical quantities, any average $\overline{f}$ being written as

$$\overline{f} = \int dxdx' \cdot \psi^*(x) f(x,x')\psi(x') = tr(\rho f) \ ; \qquad (3)$$

obviously the density matrux ρ is the operator of the quantal distribution of probability, the fundamental concept of the quantal mechanics, equivalent practically with the concept of wavefunction. The density matrix (1) may be considered a particular case, called the pure case. Because, one may generalize the above definition of the density matrix to a mixed case, where the (normalized) wavefunction $\psi(x,q)$ describes two genuinely entangled systems, one with the coordinates x, which we are interested in, and some "environment", described by the coordinates q, which is not of interest; such that the density matrix is

$$\rho(x,x') = \int dq \cdot \psi^*(x,q)\psi(x',q) \ ; \qquad (4)$$

obviously, this density matrix is positive definite, has a unity trace, $tr\rho = 1$, though it is no longer idempotent; in fact, $tr\rho^2 < 1$; the averages of any quantities which refer to the x-system are again represented as $\overline{f} = tr(\rho f)$, and so we have a complete quantal knowledge of the x-system for a given wavefunction ψ of the entire system. An expansion of the form $\psi(x,q) = \sum c_n(q)\varphi_n(x)$ leads to the representation

$$\rho(x,x') = \sum_{nm} a_{nm}\varphi_n^*(x)\varphi_m(x') \quad , \tag{5}$$

where the matrix

$$a_{nm} = \int dq \cdot c_n^*(q)c_m(q) \tag{6}$$

is the density matrix in this representation. It can be diagonalized, and the interesting case is for the system being in a stationary state ψ, *i.e.* the system does not depend on time; the x-system does not depend on time too, in which case the density matrix (5) acquires a diagonal form

$$\rho(x,x') = \sum_n \rho_n\varphi_n^*(x)\varphi_n(x') \tag{7}$$

in terms of the energy eigenfunctions $\varphi_n(x)$, where ρ_n are the eigenvalues of the density matrix a_{nm}. Obviously, ρ_n is the probabilitiy distribution of the x-system on the energy eigenstates φ_n. Quite formally this is very similar with statistical physics; in fact, a density matrix of the form given by (7) is also used often with arbitrary ρ_n, not derived from a wavefunction ψ, but assumed to describing a certain "mixture" of quantal states; this latter form is sometimes called also the "statistical" case, which is enough misleading; it is also referred to as an incomplete description, as motivated by an incomplete knowledge, *i.e.* the absence of a complete system of observables; which, however, pertains also to the q-integration in (4), in fact; anyway, a "quantum mechanical" entropy is defined in theses cases by $S = -\sum \rho_n \ln \rho_n$ (which, however, does not increase in time!); the rationale behind being that $S = 0$ for the pure case, when all ρ_n vanishe except for one which is unity, while in the mixed, or "statistical" case, $S > 0$ for positive ρ_n such that $\sum \rho_n = 1$; which makes S a sort of measure of our ignorance of the x-system (as a consequence of integrating over q, for instance), its maximum value $\ln N$ being reached for a uniform distribution $\rho_n = 1/N$, where N is the total number of states; for two states only the minimum of information one may get is that corresponding to $\rho_n = 1/2$, which gives a maximum $S = \ln 2$ of entropy; which is taken as the unit of information, and is called bite; the information being therefore referred to as a sort of "negentropy", *i.e.* a "negative entropy". All this may have appeared in an attempt to get a formal representation of the quantal measurements, and to the natural dissipation of a signal through noise.[4]

However, all this has little to do with statistical physics; equoting, as in Ref.1, the probability ρ_n in (7), obtained from a wavefunction ψ, with a Gibbs canonical statistical distribution $\exp(-\varepsilon_n/T)/(\sum\exp(-\varepsilon_n/T))$ may lead to a "quantum mechanical" temperature T, which may either possibly be taken as another notation for the interaction between the x-system and the q-"environment"-without any enlightment unfortunately-, or as a simple mistake, because this T is not the temperature; indeed, it depends on the state ψ of the entire system, and the x-system does not move chaotically, and it is not statistically independent, and it is not independent of the q-"environment"; and for every state of the entire system one would have another T. The statistical systems have no wavefunctions, while the x-system here has one, or, at least, part of one: for any q its wavefunction is $\psi(x,q)$, and integrating over q removes q not the wavefunction. Of course, this does not mean that the operator $\rho = \exp(-H/T)$ of the (unrenormalized) statistical distribution of the x-system may not be represented as

$$\rho(x,x') = \sum_n e^{-\varepsilon_n/T}\varphi_n^*(x)\varphi_n(x') \quad , \tag{8}$$

like in (7), where H is the hamiltonian of the x-system, probably the main formal basis of confusion.

The situation is perhaps similar to some extent to the microcanonical distribution; a large (*i.e.* with many degrees of freedom) isolated system of energy E is decomposed there into two sub-systems, a small one, corresponding to, say, the x-system here, and the remaining "environment"; the difference is that the two sub-systems are independent, *i.e.* their energies can be written as $E = \varepsilon_n + \varepsilon'$, where ε' is the energy of the q-"environment"; which is not the case here, where the corresponding writing reads $E = \varepsilon_n + (E - \varepsilon_n)$, where $E - \varepsilon_n$ is not the energy of the q-"environment"; because the wavefunction $\psi(x, q)$ is entangled, and does not factorize into a wavefunction of x and another of q, to lead to a pure case; of course, though independent, the two sub-systems of the microcanonical distribution are not in given, stationary quantal states, because they are in chaotical motion; occupying such states with certain probabilities, which are the statistical probabilities; one may ask how is it possible the chaotical motion, which makes the systems independent, while the systems interact, *i.e.* how is it possible the statistical physics? Well, it might be that the interaction is so finely diffused by so many interaction processes, that ultimately it results into a chaotical motion; while for small, non-thermodynamic systems this may only happen much slower, everything being a matter of scale; and the independence is statistical, *i.e.* at the level of probabilities and averages, but there are still fluctuations; anyway, the chaotical motion is possible through itself; like the inertial motion of a mechanical body, and the motion of a wave; and it may also be possible by its physical emptiness, like quantal mechanics which is also possible by its lacking of the localization, and like classical mechanics-by the undefinedness of the space and time. Actually, they are not possible in fact, they are impossible on an absolute scale, all of them are possible as ideal cases only, and in themselves; and the natural world itself is impossible, rigurously speaking, it does not exist, there exist only the motion, as expressed by physics, and natural philosophy.

Let me add some final, miscellaneous remarks. The classical, completely determined, mechanics, with all its arbitrariness in the initial conditions, is not only unpractical for macroscopic bodies, like a gas sample, for instance, but not even desirable, because it is impossible on the atomic scale; the chaotical motion of these latter systems, with its positive determinations, is perfectly self-consistent; this having been most convincingly perceived perhaps when the brownian motion has been represented as a stochastic motion which called for temperature; on this occasion, the molecular chaos has been seen clearly, probably for the first time. Moreover, a classical mechanics representation has further proven itself as being inadequate for statistical physics, with the ergodic theorem, the principle of detailed balancing, and the irreversibility of the statistical motion ("statistical mechanics" being probably the most appropiate term for all of this); but certainly with the counting of the mechanical states, which called seriously for some universal action; this being perhaps one of the main calls for the quantal mechanics; which showed clearly enough not only the rigurous impossibility of the classical mechanics, but the rigurous impossibility of the contents of the natural world; while, however, macroscopic bodies may consistently exist and move in the vanishing limit of three universal physical constants, namely $\hbar$ and $1/c$ and G, where c is the light velocity and G is the gravitation constant, there is still no quantitative limitation to the chaotical motion which would allow the approximation of life and consience. In the natural world the pure thinking has only an approximate existence, and to the extent to which science is perfect it does not apply, and to the extent to which it is only an approximation it works. And the science limitations are natural limitations, they are precisely the non-existence of the natural world. The limits of the formal thinking themselves, as expressed in ancestral paradoxes and Godel's theorem, reinforce both the consistency of the natural thinking and its undefined nature; subjective representations like space, time, non-localization and non-existence, chaotical motion, all embodying numbers, might very well be last milestones of the scientific quest. But that limitation which would allow both life and conscience, as ordered, and ephemeral approximations to a chaotical motion, is still missing; it might well be an universal amount of a minimum of information, as inscribed in the human genetic code, whose origin, if it exists, could only be of divine nature. The "ontological" argument for the existence of God is the non-existence of the natural world, according to science. *Cogito, ergo* **est**.

Making use of the decomposition (8) for the statistical operator opens the way up to path integration, with an evolution in an imaginary time as the inverse of the temperature; this is not a representation of the chaotical motion, but of the wavelike nature of quantal objects, Planck's constant entering the partition function together with temperature; equilibrium, like the principle of least action, being representable both by decaying real exponentials and by rapidly, and destructively, interfering imaginary exponentials, like in a steepest descent of a wave; this singles out classical paths for small $\hbar$ and large T, but the quantal nature of the technique remains, even when working with coherent states, where the classical aspect of their coordinates is perhaps more apparent; and, by the way, awareness should be exercised when representing dynamics, either mechanical or statistical, in terms of certain other coordinates, like the ones introduced through coherent states, or Hubbard-Stratonovisch fields, etc, as to exhaust the motion modes.

It is perhaps interesting in this connection the classical limit, though it does not touch at all essentially upon the statistical character; in particular $\hbar$ is there to stay, for a consistent statistical physics, and while classical mechanics is an approximation to quantal mechanics, both are yet more particular than statistical physics; though, however, the latter requires them. As a matter of fact, there is perhaps as much a contamination between quantal mecanics and statistical physics, as a consequence of their statistical character, as between quantal mechanics and classical mechanics, due to the mechanical quantities they both use; yet, in either case there is no identity, and no derivative relation, not even the same fundamental outlook, except perhaps a quite general and undefinable attitude.

As for the classical limit, first, the expansion (5) in plane waves reads

$$\rho(x,x') = \frac{1}{V}\sum_{pp'}\rho(p,p')e^{-(i/\hbar)(px-p'x')} \quad , \tag{9}$$

where V denotes the volume; changing to $x = X + y/2, x' = X - y/2, p = P + q/2, p' = P - q/2$, one can rewrite (9) as

$$\rho(P+q/2, P-q/2) = \frac{1}{V}\int dXdy\cdot\rho(X+y/2, X-y/2)e^{(i/\hbar)qX}e^{(i/\hbar)Py} \quad ; \tag{10}$$

introducing Wigner's distribution

$$f(X,P) = \frac{1}{V}\int dy\cdot\rho(X+y/2, X-y/2)e^{(i/\hbar)Py} \quad , \tag{11}$$

one can write

$$\rho(P,P) = \int dX\cdot f(X,P) \tag{12}$$

from (10) and

$$\int dP\cdot f(X,P) = ((2\pi\hbar)^3/V)\rho(X,X) \tag{13}$$

from (11); since $\rho(X,X)$ and $\rho(P,P)$ are the localization probabilities of the coordinates and, respectively, momenta, Wigner's distribution has the disguise of a classical distribution; however, it may take negative values, as it can be seen easily for small P, which shows the genuine quantal behaviour.

Moreover, suppose an x-system as part of the Universe; the latter may be thought as being in its quasi-classical limit; the wavefunction is then

$$\psi(x,q) \sim e^{(i/\hbar)S(x,q)} \quad , \tag{14}$$

where $S(x,q)$ is the action; the coefficients of the plane waves expansion are given by

$$c_p(q) \sim \int dx \cdot e^{(i/\hbar)S(x,q)} e^{-(i/\hbar)px} \quad ; \tag{15}$$

by steepest descents

$$\partial S(x_0,q)/\partial x = p \tag{16}$$

determines x_0 as functions of p and q; one obtains

$$c_p(q) \sim \int dx \cdot e^{(i/\hbar)S''(x_0,q)(x-x_0)^2} \sim (\hbar/\left|S''(x_0,q)\right|)^{3/2}... \tag{17}$$

for a sui-generis representation of the second-order expansion; therefore the density matrix reads

$$\int dq \cdot c_p^*(q) c_{p'}(q) \sim \int dq \cdot (\hbar/\left|S''(x_0,q)\right|)^{3/2} (\hbar/\left|S''(x_0',q)\right|)^{3/2}...; \tag{18}$$

it is natural to assume that the only entanglement with the Universe is for $x \sim q$, *i.e.* $S''(x_0,q) \sim S''(x_0,x_0)\delta(x_0-q)$, which diagonalizes the density matrix as

$$\int dq \cdot c_p^*(q) c_{p'}(q) \sim (\hbar/\left|S''(x_0,x_0)\right|)^3 \delta(p-p')... \tag{19}$$

up to normalization factors; this may serve as a representation for the eigenvalues ρ_n of the density matrix.

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