Journal of Theoretical Physics

Founded and Edited by M. Apostol **4** (1995)

ISSN 1453-4428

On the critical condition of fission of a liquid droplet

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Abstract

The critical condition of fission of a liquid droplet is reexamined, by estimating the surface energy and the electrostatic energy in the presence of deformations.

The fissionability criterion of a liquid droplet is recently brought again in discussion,[1] with the advent of metallic clusters.[2],[3] Since a computational error in the original paper on the theory of nuclear fission[4] is sometimes perpetuated, it is worth examining again the basic features of the elementary theory of the liquid drop model. This may even be more timely, as the model might bear some relevance on another problem, namely that of the charge separation in a thunder cloud.

A spherical liquid droplet of radius R_0 has a surface energy $E_S = 4\pi\sigma R_0^2$, where σ is the surface tension, and, when uniformly electrified to a charge q, it has also an electrostatic (Coulomb) energy $E_C = \frac{2}{5}q^2/R_0$. The simplest deformation the droplet may acquire is an axially symmetric one, in which case its shape is given by $\varphi(x^2+y^2, z^2) = const$, where φ is the potential energy at the (x,y,z)-point of the surface; and the simplest form of φ is $\varphi = const(x^2+y^2) + const\cdot z^2$, which means that the droplet is an ellipsoid of equation

$$\gamma^2(x^2+y^2) + \delta^2 z^2 = R_0^2 \quad . \tag{1}$$

The parameters γ and δ are chosen such as to keep constant the volume of the droplet (as for an incompressible one), *i.e.* $\gamma^2\delta = 1$, which allows the parametrization

$$\begin{gathered} \gamma = 1+\beta \quad , \\ \delta = 1 - 2\beta + 3\beta^2 \quad , \end{gathered} \tag{2}$$

up to the second order in the deformation parameter β. The elongated (prolate) ellipsoids correspond to $\beta > 0$, while the oblate ones correspond to $\beta < 0$. We may introduce the spherical coordinates $\xi = \gamma x = R_0 \sin\theta\cos\varphi$, $\eta = \gamma y = R_0 \sin\theta \sin\varphi$, $\zeta = \delta z = R_0\cos\theta$, and get the square radius

$$R^2 = x^2+y^2+z^2 = R_0^2\left[1 + 2\beta\left(3\cos^2\theta - 1\right) + 3\beta^2\left(\cos^2\theta+1\right)\right] \quad , \tag{3}$$

which corresponds to

$$R = R_0\left[1 + \beta\left(3\cos^2\theta - 1\right) + \frac{1}{2}\beta^2\left(2 + 9\cos^2\theta - 9\cos^4\theta\right)\right] \quad , \tag{4}$$

within the second-order approximation in powers of β. The area of the ellipsoidal surface is given by

$$S = \int R\sin\theta\cdot d\varphi\sqrt{R^2 d\theta^2 + R'^2 d\theta^2} \quad , \tag{5}$$

whence, by using (3) and (4), we get

$$S = 4\pi R_0^2 \left(1 + \frac{32}{5}\beta^2\right) \quad . \tag{6}$$

The surface energy is therefore

$$E_S = 4\pi\sigma R_0^2 \left(1 + \frac{32}{5}\beta^2\right) \quad . \tag{7}$$

This result is wrong in *Ref.*4 where a factor of 8 is missing in the β^2-term. The exact area reads

$$S = 2\pi R_0^2 \frac{1}{\gamma^2} \left[\varepsilon + \frac{\ln\left(\varepsilon + \sqrt{\varepsilon^2 - 1}\right)}{\sqrt{\varepsilon^2 - 1}}\right] \tag{8}$$

for $\varepsilon = \gamma/\delta$ (> 1), whose second-order expansion is (6).

The Coulomb energy of the deformed droplet is given by

$$E_C = \frac{1}{2}\rho^2 \int dv_1 dv \left\{\frac{1}{\gamma^2}\left[(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2\right] + \frac{1}{\delta^2}(\zeta_1 - \zeta)^2\right\}^{-1/2} \quad , \tag{9}$$

where ρ is the (constant) density of charge, and a second-order expansion in powers of β reads

$$E_C = \frac{1}{2}\rho^2 \left[(1 + \beta) E_0 - 3\beta\left(1 + \frac{7}{2}\beta\right) E_1 + \frac{27}{2}\beta^2 E_2\right] \quad , \tag{10}$$

where

$$\begin{aligned} E_0 &= \textstyle\int dv_1 dv \frac{1}{d} \quad , \\ E_1 &= \textstyle\int dv_1 dv \frac{(\zeta_1 - \zeta)^2}{d^3} \quad , \\ E_2 &= \textstyle\int dv_1 dv \frac{(\zeta_1 - \zeta)^4}{d^5} \quad , \end{aligned} \tag{11}$$

and $d = \left[(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2\right]^{1/2}$. The integral over $\mathbf{r}= (\xi, \eta, \zeta)$ ($dv = d\mathbf{r}$) in E_0 does not depend on the direction of $\mathbf{r}_1 = (\xi_1, \eta_1, \zeta_1)$, so that we get easily $E_0 = \frac{2}{15}(4\pi)^2 R_0^5$. For $E_{1,2}$ we perform first a rotation of angle $\varphi_1 - \pi/2$ around the third axis, and thereafter another rotation of angle $-\theta_1$ around the first axis, in order to bring the third axis along the direction of $\mathbf{r}_1$. With the new coordinates we have

$$\begin{aligned} \xi &= \xi' \sin\varphi_1 + (\eta' \cos\theta_1 + \zeta' \sin\theta_1)\cos\varphi_1 \quad , \\ \eta &= -\xi' \cos\varphi_1 + (\eta' \cos\theta_1 + \zeta' \sin\theta_1)\sin\varphi_1 \quad , \\ \zeta &= \zeta' \cos\theta_1 \quad , \end{aligned} \tag{12}$$

and

$$E_1 = \int dv_1 dv' \frac{1}{d^3}\left[\left(r_1 - \zeta'\right)^2 \cos^2\theta_1 + \eta'^2 \sin^2\theta_1 + 2\left(r_1 - \zeta'\right)\eta' \sin\theta_1 \cos\theta_1\right] \quad , \tag{13}$$

where $d = (r_1^2 + r^2 - 2r_1 r\cos\Theta)^{1/2}$, Θ being the angle between $\mathbf{r}$ and $\mathbf{r}_1$. The integration in (13) is now easily performed, leading to $E_1 = \frac{1}{3}E_0$. Similarly, we get $E_2 = \frac{1}{5}E_0$, so trhat the Coulomb energy is

$$E_C = \frac{3}{5}\frac{q^2}{R_0}\left(1 - \frac{4}{5}\beta^2\right) \quad . \tag{14}$$

If one follows the fissionability of the droplet in the "deformed channel" one finds, from (7) and (14), the critical condition $q^2 > 40\sigma V$, where V is the volume of the droplet. However, a "spherical explosion" described by $R_0 \to R_0(1 + \alpha)$ (without conserving the volume) leads to the critical condition $q^2 > 10\sigma V$, which is easier to meet than the former; which suggests that the fission proceeds by a "spherical" blow-up. Of course, if the ground-state of the droplet, by various reasons, is deformed, the critical condition of fissionability should be obtained from that "deformation channel", and, for example, the former condition given above will hold ($q^2 > 40\sigma V$) for an "elliptical" ground-state of the droplet. It is also worth remarking that the elliptical shape given by (4) contains a certain superposition of the Legendre polynomials $P_2(\cos\theta)$ and $P_4(\cos\theta)$; a general expansion in Legendre polynomials would describe the motion of the surface of the droplet. For a more general deformation of the ground-state one may use the same method as that indicated above for estimating the surface and the electrostatic energies.

It is rather widely agreed [5] that a process of charge separation may occur in the thunder clouds, which would explain the thunders, the lightnings, and the charging of the earth's surface with negative electricity. It is likely that this separation is performed by supercooled water droplets, in the region of temperature $\sim -12\ ^\circ C$ (which corresponds typically to a height of $\sim 8\ Km$). In the electric field $\mathcal{E}$ of the earth ($\mathcal{E} \sim 100\ V/m \sim 3 \cdot 10^{-3}\ ues$) a water droplet of radius R_0 may acquire, due to its polarization, an electric charge $q \sim \varepsilon \mathcal{E} R_0^2$, at most, where $\varepsilon \sim 80$ is the dielectric constant of water. The origin of this charge, though controversial, may be that of negative ions, or electrons, produced by the ionization of the air. The estimation of the Coulomb energy of a droplet whose surface is uniformly charged to q proceeds in the same way as that indicated above; we get for a deformed droplet

$$E_C = \frac{q^2}{2R_O}\left(1 - \frac{4}{5}\beta^2\right) \quad , \tag{15}$$

whence, by the same line of reasoning as above, the critical condition of fissionability (which is again that of a "spherical channel") reads

$$q^2 > 12\sigma V \quad . \tag{16}$$

By using the estimation of q given above we get a critical radius $R_0 \sim \sigma/\varepsilon^2\mathcal{E}^2 \sim 10^4\ cm$ ($\sigma \sim 70\ dyn/cm$), which is too large, although the electrical field may acquire much higher values at the microscopic level of an ionized gas; or another mechanism may work for electrifying the droplets. We note that for falling down with a velocity of $\sim 1\ cm/s$ under the Stokes' force (air viscosity being $\eta \sim 10^{-4}\ g/cm \cdot s$ at $20\ ^\circ C$) a droplet must have a radius $\sim 1\ mm$.

If this mechanism would work, a negative charge is accumulated at the bottom of the cloud, which will sharply increase the electric field; which, in turn, will accelerate the falling of the droplets by the dipolar attraction, reducing thereby their chance of gaining charges. Consequently, larger droplets will be stable and they will precipitate. (Recall that for falling with higher velocities v the resistance force of the air is given by $k \cdot \frac{\rho v^2}{2} S$, where ρ is the air density, S is the cross section of the droplet, and the numerical coefficient $k \cong 0.24$ for a sphere; which would require a droplet radius $R_0 \sim 10^{-3}\ cm$ for free sustenance.). In such a scenario the precipitation, *i.e.* the rain, is a consequence of the electrification of the cloud.

References

[1] See, for example, W. A. Saunders, Phys. Rev. Lett. **64** 3046 (1990); W. A. Saunders and N. Dam, Z. Phys. D**20** 111 (1991).

[2] W. A. de Heer, Revs. Mod. Phys. **65** 611 (1993).

[3] M. Brack, Revs. Mod. Phys. **65** 677 (1993).

[4] N. Bohr and J. A. Wheeler, Phys. Rev. **56** 426 (1939).

[5] See, for example, J. A. Chalmers, Atmospheric Electricity, Pergamon, Oxford (1967).

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