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Amplification factors in oscillatory motion

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Abstract

The forced oscillatory motion with damping is reviewed and amplification factors are defined and computed at resonance for displacement, velocity and acceleration. The amplification for external shocks is also estimated.

1 Forced oscillatory motion with attenuation

Let a particle of mass m and coordinate $x(t)$ be subjected to an elastic force $-kx$, a friction force $-\alpha\dot{x}$ and an external force $f(t)$; k denotes the elastic force constant and α is the friction coeficient. The corresponding equation of motion reads

$$m\ddot{x}+kx+\alpha\dot{x}=f\ . \tag{1}$$

We introduce the frequency ω given by $\omega^2=k/m$ and look for a solution $x=\xi e^{-\lambda\omega t}$. For $\alpha/2m\omega=\lambda$ we obtain

$$\ddot{\xi}+\omega'^2\xi=(f/m)e^{\lambda\omega t}\ , \tag{2}$$

where $\omega'=\omega(1-\lambda^2)^{1/2}$. We introduce $\eta=\dot{\xi}+i\omega'\xi$ which obeys

$$\dot{\eta}-i\omega'\eta=(f/m)e^{\lambda\omega t}\ , \tag{3}$$

and look for a solution $\eta=ue^{i\omega' t}$. We find the new equation

$$\dot{u}=(f/m)e^{-i\omega' t+\lambda\omega t}\ , \tag{4}$$

whose solution is

$$u=\int_0^t d\tau(f/m)e^{-i\omega'\tau+\lambda\omega\tau}+u_0\ , \tag{5}$$

where u_0 is the initial condition. It follows

$$\eta=\int_0^t d\tau(f/m)e^{i\omega'(t-\tau)+\lambda\omega\tau}+u_0e^{i\omega' t}\ , \tag{6}$$

and $\xi=(1/\omega')Im\eta$, *i.e.*

$$\xi=\frac{1}{\omega'}\int_0^t d\tau(f/m)e^{\lambda\omega\tau}\sin\omega'(t-\tau)+\frac{1}{\omega'}\left|u_0\right|\sin(\omega' t+\varphi)\ , \tag{7}$$

where φ is an initial phase. Finally we obtain the coordinate

$$x=\frac{1}{\omega'}\int_0^t d\tau(f/m)e^{-\lambda\omega(t-\tau)}\sin\omega'(t-\tau)+\frac{1}{\omega'}\left|u_0\right|e^{-\lambda\omega t}\sin(\omega't+\varphi)\ . \tag{8}$$

We choose $u_0=0$ and get the forced oscillations with attenuation

$$x=\frac{1}{\omega'}\int_0^t d\tau(f/m)e^{-\lambda\omega(t-\tau)}\sin\omega'(t-\tau) \tag{9}$$

and

$$\dot{x}=\frac{1}{\omega'}\int_0^t d\tau(f/m)e^{-\lambda\omega(t-\tau)}[-\lambda\omega\sin\omega'(t-\tau)+\omega'\cos\omega'(t-\tau)]\ , \tag{10}$$

which both satisfy the initial conditions $x(0)=0$ and $\dot{x}(0)=0$. We get also

$$\begin{gathered}\ddot{x}=f/m-\tfrac{1}{\omega'}\int_0^t d\tau(f/m)e^{-\lambda\omega(t-\tau)}[(1-2\lambda^2)\omega^2\sin\omega'(t-\tau)+\\ +2\lambda\omega\omega'\cos\omega'(t-\tau)]\ ,\end{gathered} \tag{11}$$

for acceleration, which originally equals the external acceleration $\ddot{x}(0)=f(0)/m$. Usually, the damping parameter λ is small ($\lambda\ll 1$), so that we may replace ω' in the above equations by ω.

2 Periodic external force

With the notations introduced above we may rewrite equation (1) as

$$\ddot{x}+\omega^2x+2\lambda\omega\dot{x}=f/m\ , \tag{12}$$

and assume a periodic external force as given by

$$f=f_0\cos\omega_0t\ . \tag{13}$$

The solution $x(t)$ of the equation (12) is obtained from (9) by introducing there this force as given by (13) (for vanishing initial conditions). The same solution is also obtained as $x=x_0+x_1$, where x_0 is the solution of the homogeneous equation and x_1 is a particular solution of the inhomogeneous equation. It is easy to check that x_0 is given by $x_0=ae^{-\lambda\omega t}\cos(\omega't+\alpha)$, where the amplitude a and the phase α are not yet determined. The particular solution of the inhomogeneous equation reads $x_1=b\cos(\omega_0t+\beta)$, where

$$\begin{gathered}b=\frac{f_0/m}{\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2}}\ ,\\ \tan\beta=\frac{2\lambda\omega\omega_0}{\omega_0^2-\omega^2}\ .\end{gathered} \tag{14}$$

The phase β is always negative, $-\pi<\beta<0$, *i.e.*

$$\begin{gathered}\sin\beta=-2\lambda\omega\omega_0/\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2}\ ,\\ \cos\beta=(\omega^2-\omega_0^2)/\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2}\ ,\end{gathered} \tag{15}$$

so that the particle laggs always behind the external force. The amplitude b is maximal for $\omega_0 = \omega(1-2\lambda^2)^{1/2}$. For $\lambda \ll 1$ the resonance occurs for $\omega_0 = \omega$. Let $\omega_0 = \omega + \varepsilon$; then

$$b = \frac{f_0/2m\omega}{\sqrt{\varepsilon^2+\lambda^2\omega^2}} \ , \qquad \tan\beta = \lambda\omega/\varepsilon \ . \tag{16}$$

In the absence of the damping the phase of the oscillation undergoes a jump at resonance (b changes the sign), while the damping smooths this jump out ($\beta = -\pi/2$ at resonance).

The friction force $-\alpha\dot{x}$ can be derived from $-\partial F/\partial\dot{x}$ where $F = (1/2)\alpha\dot{x}^2$ is the dissipation function. It follows that the Euler-Lagrange equations reads

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = \frac{\partial L}{\partial x} - \partial F/\partial\dot{x} \ , \tag{17}$$

where L is the Lagrange's function. The energy E changes in time according to

$$dE/dt = \frac{d}{dt}(\dot{x}\partial L/\partial\dot{x} - L) = \dot{x}[\frac{d}{dt}(\partial L/\partial\dot{x}) - \partial L/\partial x] = -\dot{x}\partial F/\partial\dot{x} = -2F \ . \tag{18}$$

For a time long enough the motion is stabilized, *i.e.* $x \cong x_1$and the energy is constant. The particle absorbs continuously energy from the external force and dissipates it through friction. The dissipated average energy per unit time is given by

$$I(\omega_0) = 2\overline{F} = 2m\lambda\omega\omega_0^2 b^2\overline{\sin^2(\omega_0 t+\beta)} = m\lambda\omega\omega_0^2 b^2 \ , \tag{19}$$

and close to the resonance

$$I(\omega_0) = \frac{f_0^2}{4m}\frac{\lambda\omega}{\varepsilon^2+\lambda^2\omega^2} \ , \tag{20}$$

which is a dispersive function of the frequency ε. Its integral does not depend on frequency, $\int_0^\infty d\omega_0 I(\omega_0) = \pi f_0^2/4m$.

We now turn back to the general solution

$$x = ae^{-\lambda\omega t}\cos(\omega t+\alpha) + b\cos(\omega_0 t+\beta) \ , \tag{21}$$

where we neglected the small effect of the damping on the frequency, *i.e.* $\omega' \cong \omega$, and impose the vanishing initial conditions $x(0) = \dot{x}(0) = 0$. Making use of (14) and (15) we obtain

$$a = \frac{f_0}{m}\frac{\sqrt{(\omega^2-\omega_0^2)^2+\lambda^2(\omega^2+\omega_0^2)^2}}{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2} \tag{22}$$

and

$$\tan\alpha = -\frac{\lambda(\omega^2+\omega_0^2)}{\omega^2-\omega_0^2} \ . \tag{23}$$

At resonance the phase α passes through $\pi/2$, *i.e.*

$$\sin\alpha = \lambda(\omega^2+\omega_0^2)/\sqrt{(\omega^2-\omega_0^2)^2+\lambda^2(\omega^2+\omega_0^2)^2} \ , \qquad \cos\alpha = -(\omega^2-\omega_0^2)/\sqrt{(\omega^2-\omega_0^2)^2+\lambda^2(\omega^2+\omega_0^2)^2} \ . \tag{24}$$

Also at resonance the amplitude

$$a = f_0/2m\lambda\omega^2 \tag{25}$$

equals the amplitude $b = f_0/2m\lambda\omega^2$ as given by (16).

We also establish now the coordinates for a motion of the particle proceeding solely under the action of the external force f . Since $f = f_0 \cos\omega_0 t$ it follows that acceleration is $ac = (f_0/m)\cos\omega_0 t$, which has a maximum value $ac_{max} = f_0/m$. The velocity is given by $v = (f_0/m\omega_0)\sin\omega_o t$, which initially vanishes and has a maximum value $v_{max} = f_0/m\omega_0$. Finally, the displacement d is given by

$$d = -(f_0/m\omega_0^2)\cos\omega_0 t + (f_0/m\omega_0^2) = (2f_0/m\omega_0^2)\sin^2(\omega_0 t/2) \ , \tag{26}$$

for a vanishing initial displacement; its maximum value is $d_{max} = 2f_0/m\omega_0^2$.

3 Amplification factors at resonance

According to the results derived above at resonance $a = b = f_0/2m\lambda\omega^2$, $\alpha = \pi/2$ and $\beta = -\pi/2$. Then, the general solution given by (21) becomes

$$x = \frac{f_0}{2m\lambda\omega^2}(1 - e^{-\lambda\omega t})\sin\omega t \ . \tag{27}$$

We look for the local minima of this function, *i.e.* the solutions of the equation

$$\tan\omega t = -\frac{1}{\lambda}(e^{\lambda\omega t} - 1) \ ; \tag{28}$$

they are close to

$$\omega t = (2k+1)\pi/2 \tag{29}$$

(and slightly above), where $k = 0, 1, 2....$ The maximum values of the coordinate modulus are given by

$$|x|_{max} \cong \frac{f_0}{2m\lambda\omega^2}(1 - e^{-\lambda(2k+1)\pi/2}) \ . \tag{30}$$

The amplification factor of the displacement is defined as the ratio

$$F_d = |x|_{max}/d_{max} \cong \frac{1}{4\lambda}(1 - e^{-\lambda(2k+1)\pi/2}) \ . \tag{31}$$

For small values of the damping coefficient λ the amplification factor may attain considerably higher-than-unity values. Indeed, for $\lambda(2k+1)\pi/2 \ll 1$ we get

$$F_d \cong (2k+1)\pi/8 \tag{32}$$

from (31). Typical values for λ allows the integer k go up to $k = 1, 2, 3, 4$, where the amplification factor reaches the values 1.18, 1.96, 2.75 and 3.53, respectively, for times $t = (2k+1)T/4$, where T is the period of the oscillations. For higher values of the damping ($\lambda > 0.25$, for instance) the amplification factor is less than unity.

A similar analysis holds for the velocity

$$\dot{x} = \frac{f_0}{2m\lambda\omega}[\lambda e^{-\lambda\omega t}\sin\omega t + (1 - e^{-\lambda\omega t})\cos\omega t] \ , \tag{33}$$

which reaches the maximum modulus values

$$|\dot{x}|_{max} \cong \frac{f_0}{2m\lambda\omega}(1 - e^{-\lambda k\pi}) \tag{34}$$

for $\omega t \cong k\pi$, $k = 1, 2, 3...$(the maximum placed between 0 and $\pi/2$ is left aside). The amplification factor for velocities is defined by

$$F_v = |\dot{x}|_{max} / v_{max} \cong \frac{1}{2\lambda}(1 - e^{-\lambda k\pi}) \ . \tag{35}$$

For small values of the damping coefficient the amplification factor is given by

$$F_v \cong k\pi/2 \ , \tag{36}$$

and it may attain higher values than the amplification factor for displacement (up to 2π for instance, corresponding to $k = 4$).

Within the approximation $\lambda\omega t \ll 1$ and $\lambda \ll 1$ employed here the acceleration can be written as

$$\ddot{x} = \frac{f_0}{2m}(\omega t \sin\omega t - 2\cos\omega t) \ ; \tag{37}$$

its modulus attains maximum values for ωt satisfying $\tan\omega t = -\omega t/3$. The approximate solutions of this equations, corresponding to higher values of the acceleration, are given by $\omega t \cong (2k+1)\pi/2$ for $k = 2, 3....$ The amplification factor for acceleration is defined by

$$F_a = |\ddot{x}|_{max} / ac_{max} = \frac{1}{2} |\omega t \sin\omega t - 2\cos\omega t|_{max} \ . \tag{38}$$

Its maximum values are (slightly less than) $F_a \cong (2k+1)\pi/4$.

Far from resonance the amplification factors decrease. It is worth noting that the amplification factors are higher than unity because of the large amplitudes of oscillations at resonance, and far from resonance these amplitudes decrease according to (14) and (22), and, consequently, the amplification factors decrease too. Indeed, it is worth analyzing the energy forced into the oscillating particle for zero damping. From (6) we obtain

$$\eta = e^{i\omega t}\int_{-\infty}^{+\infty} d\tau (f/m) e^{-i\omega\tau} \ , \tag{39}$$

in this case, for a very large duration t and vanishing initial conditions at $t \rightarrow -\infty$. On the other hand the energy of the particle is $E = m(\dot{x}^2 + \omega^2 x^2)/2 = m|\eta|^2/2$, *i.e.*

$$E = \frac{1}{2m}\left|\int_{-\infty}^{+\infty} d\tau f e^{-i\omega\tau}\right|^2 \ , \tag{40}$$

which shows that the pumped energy is associated with the Fourier component of the external force corresponding to the particle frequency. For $f = f_0\cos\omega_0 t$ we obtain $E = \pi^2 f_0^2 \delta^2(\Delta\omega)/2m$ close to resonance, where $\Delta\omega = \omega_0 - \omega$. The energy per unit time leads to $I = E/t = \pi f_0^2\delta(\Delta\omega)/4m$, since $t\Delta\omega = 2\pi$; it may also be written as

$$I = \frac{f_0^2}{4m}\frac{\lambda\omega}{(\Delta\omega)^2 + \lambda^2\omega^2} \ , \tag{41}$$

c where λ is a vanishing parameter, which coincides with the dissipated energy per unit time given by (20). The latter equation shows that the energy absorbed from the external force equals the dissipated energy and it has a maximum value at resonance.

4 Shocks

In more realistic cases the external force is distributed around a certain frequency ω_0, of the order of the frequency ω, as given by a gaussian

$$f = const \cdot f_0 \int d\omega_1 e^{-(\omega_1-\omega_0)^2/2\Delta^2} \cos\omega_1 t \ , \tag{42}$$

where Δ is the frequency extension of the external spectrum. If $\Delta \ll \omega_0, \omega$ then the external wavepacket is similar with a monochromatic wave, and the results for the amplification factors are similar with those given in the previous sections. More interesting is the opposite limit $\Delta \gg \omega_0, \omega$, which corresponds to a shock of a short duration as given by

$$f = -f_0 \Delta t e^{-\Delta^2 t^2/2} \ , \tag{43}$$

extending over a time interval $t \sim 1/\Delta$. Such a force is obtained as a gaussian distribution of the form

$$f = (f_0/\sqrt{2\pi}\Delta) \int d\omega_1 [-(\omega_1-\omega_0)/\Delta] e^{-(\omega_1-\omega_0)^2/2\Delta^2} \sin\omega_1 t \ . \tag{44}$$

Now, it is easy to compute the maximum value of the acceleration under the action of such an external force, $ac_{max} = f_0/m\sqrt{e}$, as well as the velocity $v_{max} = f_0/m\Delta$ and the displacement

$$d = (f_0/m\Delta) \int_{-\infty}^{t} d\tau e^{-\Delta^2\tau^2/2} \ , \tag{45}$$

which gives a maximum value $d_{max} = \sqrt{2\pi} f_0/m\Delta^2$.

The coordinate of the particle can easily be obtained by introducing the force f given by (43) into the general solution of the form given by (9). We may neglect the small contribution of the damping in this case, and make use of the inequalities $\Delta \gg \omega_0, \omega$ in estimating the integral (9. We obtain

$$x = \frac{f_0}{m\Delta} t e^{-\Delta^2 t^2/2} \ , \tag{46}$$

which has a maximum value $|x|_{max} = (f_0/m\Delta^2\sqrt{e})$. Therefore, the amplification factor for displacement is

$$F_d = 1/\sqrt{2\pi e} \ . \tag{47}$$

Similarly, we get the amplification factors $F_v = 1$ for velocity, and $F_a = 2.28$ for acceleration. As one can see, the amplification is less than unity for displacement, equal to unity for velocity and higher than unity for acceleration for shocks of very short duration.