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On a non-linear wave equation in elasticity

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Abstract

It is shown that anharmonic corrections to the elastic energy may lead to unphysical solutions for the elastic movement. The equation of motion for longitudinal deformations with third-order anharmonic terms is the continuum limit of the Fermi-Pasta-Ulam equation. This equation is solved exactly by elementary quadratures, and the corresponding time-dependence is shown to exhibit singularities at finite times. The first terms in the asymptotic series of the plane-waves solution are also computed. It is shown that resonances may appear in the elastic waves, as a consequence of their mutual coupling through non-linearities.

1 Introduction

In spite of the great deal of work on non-linear phenomena, the wave equation with anharmonic corrections has still received little attention in the continuum limit. Cubic and quartic anharmonicities have been considered in a one-dimensional discrete lattice,[1] and exact solutions have been identified as sinusoidal waves of finite amplitudes and amplitude-dependent frequency, in general, for certain wavevectors (see also Ref.2). Non-linear structures arising from modulated strain in ferroelectrics have also been studied within a semi-discrete approach to Ginsburg-Landau equations.[3] However, the continuum limit is usually quite different from the discrete lattice models. A connection has also been discussed of non-linear wave equations with the well-known anharmonic oscillators.[4] A breakthrough has been recorded recently[5] by applying Lie algebras of the equation symmetry group to the exact solutions of a class of non-linear wave equations, which includes the well-known Fermi-Pasta-Ulam equation in the continuum limit. We present here cubic anharmonic corrections to the elastic waves equation, and show that the corresponding equation of motion for longitudinal deformations is the continuum limit of the Fermi-Pasta-Ulam equation. Its exact solution is obtained herein by elementary quadratures, and shown to be unphysical, in the sense that it is boundless for finite times and space boundaries placed at infinite. However, the non-linear term in this equation may act as a perturbation on plane waves, and the corresponding asymptotic series is relevant for waves propagating over finite space regions and time intervals. The first terms in this asymptotic series are explicitly computed. The transverse waves with cubic anharmonic corrections are also analyzed, and a resonance is shown to appear as a consequence of the non-linear coupling of these waves to the longitudinal deformation waves.

It is well-known that terms of order higher than second in the strain tensor must be considered in the elastic energy for large values of the elastic deformations. These higher-order terms generate non-linear equations of motion, and they are usually called anharmonic corrections. The anharmonic corrections to the elastic energy and equations of motion may change drastically the character of the elastic movement. Indeed, the superposition of the solutions does not hold anymore for anharmonic corrections, in general, and the elastic waves exhibit the combined-frequency phenomenon and temporal resonances. The third-order anharmonic corrections are considered here for an isotropic elastic body, and the equation of motion is solved exactly for longitudinal deformations. It is shown that this equation is the continuum limit of the well-known Fermi-Pasta-Ulam equation. The solution exhibits a singular time-dependence at finite times, being therefore unphysical. In addition, it is boundless at the space boundaries placed at infinite. The contribution of the non-linear terms in this equation is also treated as a small perturbation to the plane waves, and the first terms in the asymptotic series are computed explicitly. The coupling of the transverse waves to the longitudinal waves is also considered, and a resonance is shown to occur for a frequency which depends on the ratio of the waves velocities.

Linear elasticity[6] assumes a linear strain (or deformation) tensor

$$u_{ij} = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \ , \tag{1}$$

which has a weak spatial variation, where u_i is the i-th (Cartesian) coordinate of the displacement vector $\mathbf{u}$ at position $\mathbf{r}$ of coordinates x_i ($i = 1, 2, 3$). The elastic energy for an isotropic body is then written as

$$E = \int d\mathbf{r}(\frac{\lambda}{2}u_{ii}^2 + \mu u_{ij}^2) \tag{2}$$

in the linear approximation, where λ and μ are (constant) Lame's coefficients ($\lambda > -2\mu/3$, $\mu > 0$) and u_{ii}^2 and u_{ij}^2 are the two second-order scalars under rotations ($u_{ij}^2 = u_{ij}u_{ji}$). Equations of motion

$$\rho\partial^2 u_i/\partial t^2 = (\lambda + \mu)\partial(divu)/\partial x_i + \mu\Delta u_i \ , \tag{3}$$

follow, where ρ is the density and Δ denotes the laplacean, which describe longitudinal and transversal plane waves with velocities $v_l = \sqrt{(\lambda + 2\mu)/\rho}$ and, respectively, $v_t = \sqrt{\mu/\rho}$.

Non-linear contributions to elasticity appear first through the full expression

$$u_{ij} = \frac{1}{2}[\partial u_i/\partial x_j + \partial u_j/\partial x_i + (\partial u_k/\partial x_i)(\partial u_k/\partial x_j)] \tag{4}$$

of the strain tensor, and secondly through higher-order terms in the elastic energy. There are three scalars of the third order that must be added to the elastic energy (2), which now reads

$$E = \int d\mathbf{r}(\frac{\lambda}{2}u_{ii}^2 + \mu u_{ij}^2 + \frac{1}{3}Au_{ij}u_{jk}u_{ki} + Bu_{ij}^2 u_{kk} + \frac{1}{3}Cu_{ii}^3) \ , \tag{5}$$

where A, B, C are constant coefficients. The fourth-order contributions which appears inadvertently in (5) must be removed, in order to keep the contributions up to the third order at most. It is worth noting that, in general, the elastic energy given by (5) has not an absolute minimum value for $\partial u_i/\partial x_j = 0$, so that the motion associated with deviations of the strain tensor around vanishing values may lead to a non-equilibrium motion and to the unstability of the elastic body. Therefore, additional restrictions are necessary to be imposed upon the values of the deformation tensor, in order to describe a physically meaningful motion.

First, we note that the third-order non-linear contributions to the elastic enegy (5) do not affect a transverse wave of the form, say, $u_2(x_1) = u(x)$, which obeys the same equation of motion (3) as for linear elasticity.

For a longitudinal displacement $u_1(x_1) = u(x)$ the strain tensor has only one component $u_{11} = \partial u/\partial x + (1/2)(\partial u/\partial x)^2 = u' + u'^2/2$, and the energy reads

$$E = \int d\mathbf{r}\rho(\frac{1}{2}v_l^2u'^2 + \frac{1}{6}v^2u'^3) \ , \tag{6}$$

where $v^2 = [3(\lambda + 2\mu) + 2(A + 3B + C)]/\rho$. The density of energy has a minimum value for $u' = 0$ and a maximum value $2v_l^6/3v^4$ for $u' = -2v_l^2/v^2$. For $|u'|$ larger than $2v_l^2/|v|^2$ the elastic deformation becomes unstable. We assume therefore that the initial strain tensor u' is much smaller than this limiting value, everywhere in the body. On the other hand, we also assume that u' is sufficiently large ($u' \sim 1$) so that non-linear terms be considered in the elastic energy. It is interesting to note that even if the explicit third-order contributions to the energy are absent, *i.e.* $A = B = C = 0$, the third-order non-linearities do occur in the energy, coming from the non-linear terms in the strain tensor. The coefficient v^2 becomes in this case $v^2 = 3v_l^2$ (corresponding to $A + 3B + C = 0$). For the elastic energy given by (6) the equation of motion reads

$$\partial^2u/\partial t^2 = (\partial^2u/\partial x^2)[v_l^2 + v^2(\partial u/\partial x)] \ . \tag{7}$$

This is the continuum limit of the Fermi-Pasta-Ulam equation.[5, 7] It ensures the conservation of energy (continuity equation) $\partial w/\partial t + divj = 0$, where $w = \rho(\dot{u}^2/2 + v_l^2u'^2/2 + v^2u'^3/6)$ is the energy density (both kinetic and elastic) and $j = -\rho(v_l^2u' + v^2u'^2/2)\dot{u}$ is the energy flow. It is easy to see that (7) can also be written as

$$\partial^2u'/\partial t^2 = \frac{\partial^2}{\partial x^2}(v_l^2u' + \frac{1}{2}v^2u'^2) \ , \tag{8}$$

or

$$\frac{\partial^2}{\partial t^2}U = \frac{1}{2}v^2\frac{\partial^2}{\partial x^2}U^2 \ , \tag{9}$$

where $U = u' + v_l^2/v^2$ ($U > -v_l^2/|v|^2$). This equation has been analyzed in Ref. 5 by making use of the symmetry approach and the prolongation technique. The solution of this equation can be written as $U(t, x) = g(t)f(x)$ by the separation of the variables, leading to

$$\ddot{g} - \frac{3}{2}\omega^2g^2 = 0 \ , \tag{10}$$

and

$$(f^2)'' - 3(\omega/v)^2f = 0 \ , \tag{11}$$

where ω^2 is a (real) constant of integration. We show herein that these equations can be integrated by elementary quadratures.

2 Time dependence

Indeed, (10) leads straightforwardly to

$$\int_{-s}\frac{dg}{\sqrt{g^3 + s^3}} = \omega t \ , \tag{12}$$

where s^3 is another (real) constant of integration, $\omega^2s^3 = \dot{g}^2(0) - \omega^2g^3(0)$ (the origin of time has been put equal to zero, and t can assume both positive and negative sign). First, we assume $s > 0$ and $\omega t > 0$, so that (12) becomes

$$\int_{-1}^{\xi}\frac{dx}{\sqrt{x^3 + 1}} = \sqrt{s}\omega t \ , \tag{13}$$

where $\xi = g/s > 0$. It corresponds to initial conditions $g(0) = -s$, $\dot{g}(0) = 0$ (for more general conditions we may change the origin of time). The integration in (13) can be performed straightforwardly by the substitution $x + 1 = \sqrt{3}\tan^2(\alpha/2)$, leading to the elliptic integral of the first kind

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}} = \omega\sqrt{\sqrt{3}st} \ , \tag{14}$$

where $k^2 = (2 + \sqrt{3})/4\,(< 1)$ and $\xi + 1 = \sqrt{3}\tan^2(\varphi/2)$. Introducing the notation $\tau = \omega\sqrt{\sqrt{3}st}$ we obtain immediately the Jacobi elliptic sine-amplitude[8] (p.910) $\sin\varphi = sn\tau$, or

$$\xi = \sqrt{3}\frac{1 - cn\tau}{1 + cn\tau} - 1 \ , \tag{15}$$

where $cn\tau$ is the cosine-amplitude; the function ξ can also be written as $\xi = \sqrt{3}[sn(\tau/2)dn(\tau/2)/cn(\tau/2)]^2 - 1$, where $dn(\tau/2)$ is the delta-amplitude of $\tau/2$. It can also be expressed in terms of the elliptic Weierstrass function.[5, 8] A similar substitution allows the integration for $s < 0$ (as well as for $\omega^2 < 0$; we note that $sgn(\omega^2) = sgn(g + s) = sgn(s)$). Noting also that $g(t)$ is an even function of time, we can finally write down the solution of (10) as

$$g(t) = |s|\,\sqrt{3}\frac{1 - cn(\sqrt{\sqrt{3}\,|s|}\,|\omega t|)}{1 + cn(\sqrt{\sqrt{3}\,|s|}\,|\omega t|)} - 1sgn(\omega^2) \ . \tag{16}$$

This is a periodic function with period $\sqrt{\sqrt{3}\,|s|}\,|\omega t| = 4K$ ($\Delta\varphi = 2\pi$), where K is the complete elliptic integral $F(\pi/2, k)$ (~ 4). It has also singularities at $\sqrt{\sqrt{3}\,|s|}\,|\omega t| = 4K(n + 1/2)$, where n is an integer (corresponding to $\varphi = (2n+1)\pi$), as expected directly from (12). These singularities make unphysical the solution of (7). We note here that a similar treatment is applicable to the classical cubic anharmonic oscillator, leading to an exact (oscillatory) solution which can be expressed in terms of elliptic functions.[9]

3 Spatial dependence

First, we note that solution f of equation (11) changes sign as $(\omega/v)^2$ does and it is an even function of x. Making use of the substitution $(f/h)^2 = F$, where h is a (real) non-vanishing constant of integration, the spatial equation (11) becomes $F'^2 = (4\omega^2/v^2\,|h|)(F^{3/2} - 1)$, which leads to

$$\int_0^z dt \cdot t^{-1/2}(1 + t)^{-1/3} = 3\,|\omega x/v|\,/\sqrt{|h|} \ , \tag{17}$$

where $F^{3/2} - 1 = t = |f/h|^3 - 1 > 0$. It corresponds to the boundary condition $f(0) = h$ and $f'(0) = 0$ (the origin of space is set equal to 0). Using $t \to -t$ one can see that the integral in (17) is an analytic continuation of the incomplete beta function $B_{-z}(1/2, 2/3) = 2\sqrt{-z}\,{}_2F_1(1/2, 1/3, 3/2; -z)$, where ${}_2F_1$ is the Gauss hypergeometric function F.[8] (pp.950, 1039). Therefore, the spatial function f is given by the implicit equation

$$\sqrt{(|f/h|)^3 - 1}\,F(1/2, 1/3, 3/2; 1 - (|f/h|)^3) = 3\,|\omega x/v|\,/2\sqrt{|h|} \ , \tag{18}$$

up to a constant of integration (which can be chosen as the origin of space). Making use of the transformation formulas of the hypergeometric function,[8] (p. 1043), or using directly the integral representation (17), we find the solution of this equation

$$f \sim |h|\,sgn(\omega/v)^2 + (\omega/2v)^2 x^2 \ , \ x \sim 0 \tag{19}$$

near the origin, and

$$f \sim (\omega/2v)^2 x^2 \ , \ x \to \pm\infty \tag{20}$$

for large x. The remarkable particular case $h = 0$, corresponding to $f = (\omega/2v)^2 x^2$ has been pointed out in Ref. 5.

4 Exact solution

The general solution of (7) reads

$$u(t,x) = g(t - t_0) \int_0^x dx f(x - x_0) - (v_l/v)^2 x + c \ , \tag{21}$$

where time and space origins t_0 and, respectively, x_0 are introduced, and c is another constant of integration. These constants of integration, together with ω^2 , s and h introduced previously, are determined from initial and boundary conditions. The movement described by (21) looks rather like a vibration and not a propagation. The density of kinetic energy $\rho\dot{u}^2/2$ increases boundlessly in time, while the density of elastic energy $e = \rho(v_l^2 u'^2 + v^2 u'^3/3)/2 = \rho v^2 (f^3 g^3 - 3v_l^4 fg/v^4 + 2v_l^6/v^6)/6$ (which requires $fg > -v_l^2/|v|^2$ for avoiding the unstability of the body), decreases initially with increasing time and thereafter increases boundlessly. This boundless increase in both energies is performed at the expense of the energy flow $j = -\rho(v_l^2 u' + v^2 u'^2/2)\dot{u} = -\rho v^2 (f^2 g^2 - v_l^4/v^4) f\dot{g}$, which, although acquires the same value at symmetric boundaries $x = \pm L$ (due to the fact that f is an even function of x), increases itself boundlessly in time. Indeed, the main characteristic of the solution (21) is its singular behaviour near the periodical times $t = (4K/|\omega|\sqrt{\sqrt{3}\,|s|})(n + 1/2)$ as indicated by (16). These singularities are unphysical, they lead to ruptures in the elastic body, corresponding to jumps of the solution from one temporal oscillating branch to another, with corresponding singularities in the time derivative of the solution (angular points of solution) at the singularities times, and with corresponding loss of energy. This singular behaviour of the solutions of the non-linear elastic movement indicates a main mechanism of energy transfer and dissipation through ruptures. It is a general phenomenon exhibited by non-linear equations of elastic motion, because for large values of u' in (8) the *rhs* of this equation reduces to $(\partial^2/\partial x^2)u'^n$, where n ($n = 2, 3, ...$) is an integer corresponding to the non-linearity degree, and such an equation can be integrated by separation of variables, leading to singular solutions for finite times. The ruptures associated with such non-linear elastic movements are non-uniformly distributed in space and propagates (from the boundaries) with a non-uniform velocity given by $v = dx/dt = (\partial u/\partial t)/(\partial u/\partial x)$. For the third-order non-linearities discussed here the time dependence is given by $g \sim 1/\omega^2(t-T)^2$ near a singularity, where $T = 4K/|\omega|\sqrt{\sqrt{3}\,|s|}$ is the period, and the spatial dependence is given by $\int^x dx f \sim (\omega/v)^2 x^3$ for large values of $|x|$. It folows that ruptures appears during a time of the order of a period T propagating with a velocity of the order of v . For $A + 3B + C = 0$ this velocity is $v = \sqrt{3((\lambda + 2\mu)/\rho} = \sqrt{3}v_l$.

5 Asymptotic series

It is useful to compute the asymptotic series of the solution of (7) by viewing the non-linear term as a small perturbation. To this end, we introduce the parameter $\varepsilon = (v/v_l)^2$. Equation (7) reads now

$$\ddot{u} - v_l^2 u'' = \varepsilon v_l^2 u' u'' \ . \tag{22}$$

The solution of (22) can be written as an expansion $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ...$ in powers of ε, where $u_0 = a\cos(\omega t - kx)$ is a plane wave of amplitude a and frequency $\omega = v_l k$, k being the corresponding wavevector. The first-order approximation u_1 obeys the equation

$$\ddot{u}_1 - v_l^2 u_1 = -\frac{1}{2} v_l^2 a^2 k^3 \sin[2(\omega t - kx)] \ , \tag{23}$$

whose solution is of the form $u_1 = f\cos[2(\omega t - kx)]$, with f a linear function of time and space. Similarly, the second-order approximation includes a second-order f-function of space and time. Straightforward computations lead to

$$\begin{aligned} u = a\cos(\omega t - kx) + \tfrac{1}{16}\varepsilon a^2 k^2 (x + v_l t)\cos[2(\omega t - kx)] + \\ + \tfrac{1}{128}\varepsilon^2 a^3 k^4 (x + v_l t)^2 \cos[3(\omega t - kx)] - \cos(\omega t - kx) + ... \end{aligned} \tag{24}$$

which is, in fact, a triple expansion in powers of ε, ak and lk, where $l = x + v_l t$ is a characteristic length. It is worth noting the expansion parameter $ak = a/\lambda$ in (24), where λ is the wavelength, which shows indeed that the non-linear contributions are controlled by the ratio of the wave amplitude to the wavelength, as expected. The asymptotic character of the solution, however, makes boundless these contributions, over the characteristic length l. We also note the higher harmonics appearing in the asymtotic series (24), as well as various amplification factors of the order of $1 + \varepsilon al/4\lambda^2$ in the amplitude, velocity and acceleration of the asymptotic solution. A similar asymptotic series can be computed for higher-order anharmonic corrections to the elastic waves equation.

6 Coupled equations

Let us assume both a longitudinal displacement $u_1(x_1) = u(x)$ and a transverse displacement $u_2(x_1) = v(x)$. The strain tensor has then the components $u_{11} = u' + u'^2/2 + v'^2/2$ and $u_{12} = u_{21} = v'/2$. Making use of (5) the equations of motion are obtained as

$$\begin{aligned} \ddot{u} - v_l^2 u'' = \varepsilon v_l^2 u' u'' + \zeta v_l^2 v' v'' \ , \\ \ddot{v} - v_t^2 v'' = \zeta v_l^2 u' v'' \ , \end{aligned} \tag{25}$$

where $\zeta = 1 + (A + 2B)/2\rho v_l^2$. The solutions can be written as a double expansion in powers of ε and ζ. The zeroth-order approximation are plane waves $u_0 = a\cos(\omega_1 t - k_1 x)$ and $v_0 = b\cos(\omega_2 t - k_2 x)$, where a and b are amplitudes and $\omega_{1,2} = v_{l,t} k_{1,2}$. Apart from the asymptotic character, the solution exhibits a new feature originating in the combined-frequency phenomenon. Indeed, the first-order approximation to the transverse wave obeys the equation

$$\ddot{v}_1 - v_t^2 v_1'' = -\frac{1}{2} v_l^2 abk_1 k_2^2 \sin(\Omega t - Kx) \ , \tag{26}$$

where $\Omega = \omega_1 \pm \omega_2$ and $K = k_1 \pm k_2$. The solution of this equation is $v_1 = B\sin(\Omega t - Kx)$, where

$$B = \frac{1}{2} abk_2 \frac{v_l^2 k_1 k_2}{\Omega^2 - v_t^2 K^2} = \frac{1}{2} abk_2 \frac{v_l}{v_l - v_t} \cdot \frac{v_l k_2}{(v_l + v_t)k_1 \pm 2v_t k_2} \ . \tag{27}$$

It is worth noting here that a resonance may appear for $(v_l + v_t)k_1 - 2v_t k_2 = 0$, which corresponds to $\omega_2 = \omega_1(1 + v_t/v_l)/2$, *i.e.* $\Omega = \omega_1(1 - v_t/v_l)/2$. Similar resonances may also appear in higher-order approximations both to longitudinal and transverse waves, as a consequence of the combined-frequency phenomenon originating in the non-linear contributions. The damping can be considered here, by introducing the term $\eta(\dot{u} - v_l u')$ in the original wave equation, where η is the damping coefficient (it leads to a damped plane wave of the form $u = ae^{-\eta t}\cos(\omega t - kx)$ for $t > 0$; a similar term holds also for the transverse waves). The resonance singularity is then smoothed out by the small damping coefficient η, while the asymptotic series (24) is not changed significantly.

7 Conclusion

The main conclusion of this paper is that anharmonic corrections to the elastic energy lead, in general, to unphysical solutions of the elastic movement, which involve singularities in the time-dependence at finite times and boundless movement at the space boundaries placed at infinite. This phenomenon is illustrated in the present paper by solving exactly the equation of motion for a longitudinal deformation for the third-order anharmonic corrections to the elastic energy. It is shown that this equation is the continuum limit of the Fermi-Pasta-Ulam equation, and a solution obtained by elementary quadratures is provided. This phenomenon is rather general, it appears also for higher-order non-linear equation of motion, which makes unphysical the solutions of these equations. It follows that, for a consistent physical picture, the elastic energy both for small and for large deformations is distributed among wave-like solutions, which obey linear equations of motion with a satisfactory approximation over finite spaces and times, while the non-linear contributions act as a small perturbation. The first terms in such a perturbation series are computed for the third-order anharmonic corrections to the longitudinal elastic wave, in order to illustrate the asymptotic character of the non-linear contributions. The transverse waves are not affected by the third-order non-linearities, though a superposition of transverse and longitudinal deformations propagating along the same direction exhibits resonances for certain frequencies that depend on the ratio of the waves velocities, as a consequence of their mutual coupling via non-linear terms. Such non-linear couplings betwen waves propagating along different directions is worth of a more detailed investigation.

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