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On linear anharmonic oscillators and self-consistent harmonic approximation

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Through the years, anharmonic oscillators generated a great deal of technical work, both classically and quantally. In fact, they are reducible to harmonic oscillators, with a good approximation.

Let $T = m\dot{u}^2/2$ be the kinetic energy of a linear oscillator of mass m, and $U = (m\omega^2/2)(u^2 + 2au^3/3)$ its potential energy with cubic anharmonicities, where a is a parameter. Let u^3 be approximated by

$$u^3 = \frac{3}{2}(Au + Bu^2) \ , \tag{1}$$

where $A = \bar{u^2}$, $B = \bar{u}$, the averages being taken over the motion and the coefficients 3/2 in (1) being chosen such as $\bar{u^3} = 3\bar{u}\bar{u^2}$. It is easy to see that the oscillator becomes then a displaced one, with the frequency $\Omega = \omega(1 + aB)^{1/2}$; the solution is $u = u_0 \cos \Omega t - C$, where u_0 is an amplitude and $C = aA/2(1 + aB)$. The condition $\bar{u^3} = 3\bar{u}\bar{u^2}$ is fulfilled only for small values of C, as expected ($\bar{u} = -C$, $\bar{u^2} = u_0^2/2 + C^2$, $\bar{u^3} = -3u_0^2C/2$). It follows $C \cong au_0^2/4$ and $A \cong u_0^2/2$, $B = -C \cong -au_0^2/4$. The frequency shift is then given by

$$\Omega = \omega(1 + aB)^{1/2} \cong \omega(1 - a^2u_0^2/8) \ , \tag{2}$$

which compares rather satisfactorily with the exact result[1] $\Omega = \omega(1 - 5a^2u_0^2/12)$.

A similar decomposition $u^4 = 3Au^2/2$ holds for the quartic anharmonicity in the potential energy $U = (m\omega^2/2)(u^2 + bu^4/2)$, where $A = \bar{u^2}$ and b is the anharmonic parameter. The condition $\bar{u^4} = 3(\bar{u^2})^2/2$ is then fulfilled exactly ($\bar{u^2} = A = u_0^2/2$, $\bar{u^4} = 3u_0^4/8$ for solution $u = u_0 \cos \Omega t$ and frequency $\Omega = \omega(1 + 3Ab/4)^{1/2}$). It follows the frequency shift given by

$$\Omega = \omega(1 + 3Ab/4)^{1/2} \cong \omega(1 + 3bu_0^2/16) \ , \tag{3}$$

for small b, which again compares well with the exact result[1] $\Omega = \omega(1 + 3bu_0^2/8)$. Is is worth noting that the frequency shift is quadratic in amplitude for cubic anharmonicities, and linear for quartic anharmonicities.

Similar approximations can be used approximately for higher-order anharmonicities, without any loss of qualitative behaviour, and a satisfactory representation of the quantitative results. They are called generically the self-consistent harmonic approximation.

References

[1] L. Landau and E. Lifshitz, Mecanique, Mir, Moscow (1965).