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On parametric excitations of magnetization under electric flows

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1 Spin waves

As it is well-known the Landau-Lifshitz equation for magnetization $\overrightarrow{M}$ is given by

$$\frac{\partial \overrightarrow{M}}{\partial t} = \gamma \overrightarrow{M} \times \overrightarrow{H} + D\overrightarrow{M} \times \nabla^2 \overrightarrow{M} \tag{1}$$

with usual notations. Let m_x and m_y be small deviations from an equilibrium uniform magnetization M_0 parallel to the z-axis. In the absence of an anisotropy field we have $\overrightarrow{H} = H_i \overrightarrow{e_z}$. Keeping only linear contributions, we get

$$\gamma H_i (m_x \overrightarrow{e_x} + m_y \overrightarrow{e_y}) \times \overrightarrow{e_z} = \gamma H_i (m_y \overrightarrow{e_x} - m_x \overrightarrow{e_y})$$

for the first term, and

$$DM_0 \overrightarrow{e_z} \times \left(\overrightarrow{e_x} \frac{\partial^2 m_x}{\partial z^2} + \overrightarrow{e_y} \frac{\partial^2 m_y}{\partial z^2} \right) = -DM_0 \left(\overrightarrow{e_x} \frac{\partial^2 m_y}{\partial z^2} - \overrightarrow{e_y} \frac{\partial^2 m_x}{\partial z^2} \right)$$

for the second one in the *rhs* of Eq.(1). Therefore, the equation (1) led to the following system of differential equations

$$\begin{aligned} \frac{\partial m_x}{\partial t} &= \gamma H_i m_y - DM_0 \frac{\partial^2 m_y}{\partial z^2} \\ \frac{\partial m_y}{\partial t} &= -\gamma H_i m_x + DM_0 \frac{\partial^2 m_x}{\partial z^2} \end{aligned} \tag{2}$$

Introducing $m_\pm = m_x \pm i m_y$ we get

$$\begin{aligned} \frac{\partial m_\pm}{\partial t} &= \frac{\partial m_x}{\partial t} \pm i \frac{\partial m_y}{\partial t} \\ &= \gamma H_i (m_y \mp i m_x) - DM_o \frac{\partial^2}{\partial z^2}(m_y \mp i m_x) \\ &= \mp i \gamma H_i (m_x \pm i m_y) \pm i DM_0 \frac{\partial^2}{\partial z^2}(m_x \pm i m_y) \\ &= \mp i \gamma H_i m_\pm \pm i DM_0 \frac{\partial^2 m_\pm}{\partial z^2} \end{aligned}$$

The eigenmodes are given by

$$m_+ = a_+(t)e^{\pm ikz}$$

$$m_- = a_-(t)e^{\mp ikz}$$

where $a_\pm$ are complex conjugate to one another and satisfy the equation

$$\frac{\partial a_\pm}{\partial t} = \mp i(\gamma H_i + DM_0 k^2)a_\pm$$

Therefore, we obtain the eigenmodes

$$m_+ = e^{-i\omega_k t}e^{\pm ikz}$$

$$m_- = e^{i\omega_k t}e^{\mp ikz}$$

where

$$\omega_k = \gamma H_i + DM_0 k^2 \tag{3}$$

Then, we have

$$\begin{aligned} m_x &= \cos\left(\pm kz - \omega_k t\right) \\ m_y &= \sin\left(\pm kz - \omega_k t\right) \end{aligned} \tag{4}$$

2 Spin waves under dc currents

The equation of motion is written as

$$\frac{\partial m_\pm}{\partial t} = \pm i\left(-\gamma H_i + DM_0 \frac{\partial^2}{\partial z^2}\right) m_\pm + X\frac{\partial m_\pm}{\partial z}$$

where X depends linearly of the *dc* current, parallel to the z-axis.
The solutions are of the form

$$m_+ = a(t)e^{\pm ikz} \quad , \quad m_- = m_+^* = a^*(t)e^{\mp ikz}$$

where

$$m_+ = e^{-i\omega_k t}e^{\pm ik(z+Xt)}$$

$$m_- = e^{i\omega_k t}e^{\mp ik(z+Xt)}$$

and

$$\begin{aligned} m_x &= \cos\left(\pm k(z+Xt) - \omega_k t\right) \\ m_y &= \sin\left(\pm k(z+Xt) - \omega_k t\right) \end{aligned}$$

where ω_k is given by (3). We can see that

$$m_{x,y}(z,t) = m^0_{x,y}(z+Xt,t)$$

where $m^0_{x,y}$ are the components of the magnetization in the absence of the *dc* current given by (4).

3 Time dependent currents

If X has a time dependence, $X = X(t)$, the equations are solved in the same manner. The only difference is that in the final result we have

$$\int X(t)dt$$

instead of Xt. Therefore, the solution is

$$m_x = \cos\left[\pm k\left(z + \int X(t)dt\right) - \omega_k t\right] = m_x^0(z + \int X(t)dt)$$

$$m_y = \sin\left[\pm k\left(z + \int X(t)dt\right) - \omega_k t\right] = m_y^0(z + \int X(t)dt)$$

The amplitude has no time dependence, therefore the conditions for a *parametric resonance* are not fulfilled. The parametric resonance occur when the amplitude of the oscillations behave like $\mu^{\omega t}$, with $\mu > 1$, giving instability for $t \gg 1$.

For instance, for $X = X_0 \cos\omega t$ we get

$$m_x = m_x^0(z + X_0 \frac{\sin\omega t}{\omega})$$

4 Parametric excitations

By means of the Fourier transform equation (2) becomes

$$\begin{aligned} \frac{\partial a_x}{\partial t} &= \omega_k a_y \\ \frac{\partial a_y}{\partial t} &= -\omega_k a_x \end{aligned} \tag{5}$$

where ω_k is given by (3) and

$$a_{x,y}(k,t) = \int dz m_{x,y}(z,t)e^{-ikz}$$

are the Fourier components of the magnetization. Equations (5) lead to an oscillator type equation of motion by taking the time derivative in one equation and making use of the other,

$$\frac{\partial^2 a_x}{\partial t^2} = \omega_k \frac{\partial a_y}{\partial t} = -\omega_k^2 a_x$$

This would suggest a parametric resonance, providing ω_k has a time dependence in the above equation of a harmonic oscillator.

Suppose that this is true, as, for instance, the applied field H_i has a time dependence:

$$\begin{aligned} \frac{\partial a_x}{\partial t} &= \omega(t) a_y \\ \frac{\partial a_y}{\partial t} &= -\omega(t) a_x \end{aligned} \tag{6}$$

Making the change of variable $d\tau = \omega(t)dt$ the above system of equations become

$$\begin{aligned} \frac{\partial a_x(\tau)}{\partial \tau} &= a_y(\tau) \\ \frac{\partial a_y(\tau)}{\partial \tau} &= -a_x(\tau) \end{aligned}$$

whose solution

$$a_{x,y} = \exp(\pm i\tau) = \exp\left\{\pm i \int \omega(t)dt\right\}$$

has no parametric resonance however.

In order to have a parametric resonance it is necessary to have different ω in the above two equations but this kind of anisotropy is difficult to be obtained. Equations (6) should have the form

$$\begin{aligned} \frac{\partial a_x}{\partial t} &= \omega_1(t)a_y \\ \frac{\partial a_y}{\partial t} &= -\omega_2(t)a_x \end{aligned} \tag{7}$$

with $\omega_1/\omega_2 \neq$const.

In the case of a a classical parametric oscillator described by the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

where m and k have time dependence (they are the parameters), we have the canonic equations

$$\begin{aligned} \dot{x} &= \frac{1}{m(t)}p \\ \dot{p} &= -k(t)x \end{aligned}$$

which have the form given by equations (7). Making the change of variable $d\tau = dt/m(t)$ we obtain

$$\begin{aligned} \frac{dx}{d\tau} &= p(\tau) \\ \frac{dp}{d\tau} &= -m(\tau)k(\tau)x \end{aligned}$$

which lead to the parametric resonance equation

$$\frac{d^2x}{d\tau^2} = -\omega^2(t)x$$

where $\omega^2(\tau) = m(\tau)k(\tau)$.

Therefore we must emphasize that the requirements $\omega_1 \neq \omega_2$ in (7) is essential.

It can be easily verified that the parametric resonance do not even occur if both the current (X) and the external field (or more generally, ω_k) have time dependence.

5 Notes

5.1 Solution of the equation of motion in Ref. 1

We consider the equation of motion

$$\frac{\partial m_\pm}{\partial t} = -i\gamma H_i m_\pm + iDM_0 \frac{\partial^2 m_\pm}{\partial z^2} + X_0 e^{i\omega t} \frac{\partial m_\pm}{\partial z} \tag{8}$$

For $X_0 = 0$ the eigenmodes are plane waves with the dispersion relation

$$\omega_k = \gamma H_i + Dk^2 M_0$$

For $X_0 \neq 0$ the eigenmodes are

$$e^{\pm ikz} e^{-if_k(t)}$$

where f_k has a nonzero imaginary part which give time dependent amplitude.

From (8), $f_k(t)$ satisfy the equation

$$\begin{aligned} \frac{\partial f_k}{\partial t} &= \gamma H_i + DM_0 k^2 \mp kX_0 e^{i\omega t} \\ &= \omega_k \mp kX_0 e^{i\omega t} \end{aligned}$$

Up to a constant, the solution is

$$\begin{aligned} f_k(t) &= \omega_k t \mp kX_0 \frac{e^{i\omega t} - 1}{i\omega} \\ &= \omega_k t \mp \frac{2kX_0}{\omega} e^{i\frac{\omega t}{2}} \sin \frac{\omega t}{2} \end{aligned}$$

Therefore, the eigenmodes are

$$\exp\left\{ \mp \frac{2kX_0}{\omega} \sin^2 \frac{\omega t}{2} \right\} \cdot e^{\pm ik\left(z + \frac{kX_0}{\omega} \sin \omega t\right)} e^{-i\omega_k t}$$

The amplitude is time dependent but remains finite even for $\omega t \gg 1$; therefore the equation has no parametric resonance.

5.2 An equation with parametric resonance

If we assume the equation of motion

$$\frac{\partial m_\pm}{\partial t} = -i\gamma H_i m_\pm + iDM_0 \frac{\partial^2 m_\pm}{\partial z^2} + X_0 e^{\pm i\phi(t)} \frac{\partial m_\pm}{\partial z}$$

with $\phi(-t) = \phi(t)$. The eigenmodes have the form

$$m_\pm = e^{ikz} e^{-if_\pm}$$

where $f_\pm$ staisfy the equation

$$\frac{\partial f_\pm}{\partial t} = \omega_k - kX_o \exp(\pm i\phi(t))$$

where ω_k is given by (3). The above equation has the solution

$$f_\pm = \omega_k t - kX_0 \int_0^t \exp(\pm i\phi(t'))dt'$$

Therefore, the eigenmodes are

$$m_\pm = \exp\left\{\mp kX_0 \int_0^t dt' \sin\phi(t')\right\} e^{ik\left(z+X_0\int_0^t \cos\phi(t')dt'\right)} e^{-i\omega_k t}$$

We can observe that the two amplitude satisfy the relation $A_+A_- = 1$, situation similar to that occurred in the parametric resonance of the classical oscillator.

Because ϕ is a periodic function we have

$$\begin{aligned}\int_0^t dt' \sin\phi(t') &= \int_0^{[\frac{t}{T}]T} dt' \sin\phi(t') + \int_{[\frac{t}{T}]}^t dt' \sin\phi(t') \\ &= \left[\frac{t}{T}\right]\int_0^T dt' \sin\phi(t') + \int_0^{\{\frac{t}{T}\}} dt' \sin\phi(t') \end{aligned} \tag{9}$$

where [...] and {...} are the integer, respectively the fractional part. The second term is always finite.

Neglecting the finite term in (9) we have the dominant contribution

$$A_\pm = e^{\mp\frac{t}{T}kX_0\int_0^T \sin\phi(t')dt'} = \mu_\pm^{\frac{t}{T}}$$

where $\mu_+\mu_- = 1$

$$\mu_\pm = \exp\left\{\mp kX_0 \int_0^T dt \sin\phi(t)\right\}$$

If the above integral is finite, we have $\mu_+ > 1$ or $\mu_- > 1$, which means that one of the above amplitude is divergent. There are many possible choices. All periodic functions which satisfy $0 < \phi < \pi$ are a good choices. For example

$$\phi(t) = \frac{\pi}{2}(1 + \cos\frac{2\pi t}{T})$$

give

$$\int_0^T \sin\phi(t)dt \simeq 3\frac{T}{2\pi}$$

or

$$\phi(t) = \frac{\pi}{2}\sqrt{1 + \cos\frac{2\pi t}{T}}$$

References

[1] J-Ph Ansermet, *Classical description of spin wave excitation by currents in bulk magnets,* preprint, 2003.

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