

An Introduction to Theoretical Seismology¹

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Magurele-Bucharest, Romania, email: apostol@infp.infp.ro**Abstract**

A few basic facts in the science of the earthquakes are briefly reviewed. An accumulation, or growth, model is put forward for the focal mechanisms and the critical focal zone of the earthquakes, which relates the earthquake average recurrence time to the released seismic energy. The temporal statistical distribution for average recurrence times is introduced for earthquakes, and, on this basis, the Omori distribution in energies is derived, as well as the distribution in magnitudes, by making use of the semi-empirical Gutenberg-Richter law relating seismic energy to earthquake magnitude. On geometrical grounds, the accumulation model suggests the value $r = 1/3$ for the Omori parameter, which leads to $\beta = 1.17$ for the coefficient in the Gutenberg-Richter recurrence law, in agreement with the statistical analysis of the experimental data. Making use of this value, the empirical Bath's law is derived for the most likely highest magnitude of the aftershocks (which is 1.2 less than the magnitude of the main seismic shock), by assuming that the aftershocks are relaxation events of the seismic zone. The time distribution of the earthquakes with a fixed average recurrence time is also derived, the earthquake occurrence prediction is discussed by means of the average recurrence time and the seismicity rate, and application of this discussion to the seismic region Vrancea, Romania, is outlined. Finally, an intriguing effect of non-linear behaviour of the seismic waves is discussed, by describing an exact solution derived recently for the elastic waves equation with cubic anharmonicities, its relevance, and its connection to the approximate quasi-plane waves picture.

Introduction. Empirical observations made along the time on the earthquakes revealed a few salient features of these intriguing natural phenomena which deserve attention and a close scrutiny.

First, the energy released by the earthquakes extends over a huge energy scale, from 10^4J to 10^{17}J . The origin of this energy and the mechanisms of releasing it are still a matter of debate. However, it is widely agreed today that such seismic energies are built up in the earthquake focal zone, centered around a seismic focus, by the movement of the tectonic material (organized probably in plates). The concentration of the seismic energy in the focal zone implies huge values of mechanical strain and stress, which are released suddenly, now and then, here and there, by abrupt displacement, fractures, ruptures, violent movements, along seismic faults of the critical focal zone.

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Second, the critical focal zone seems to be very localized as compared to the spatial range the earthquake effects are felt over. Indeed, if one assumes that the seismic energy E is concentrated in a volume of a characteristic radius R , then we may represent this energy as $E = \mu(\mu/\mu_e)R^3$, where μ stands for a generic rigidity and μ_e represents a generic elasticity modulus. Assuming then representative values $\mu = 10^9 \text{dyn/cm}^2$ and $\mu_e = 10^{11} \text{dyn/cm}^2$, as suggested by laboratory tests on typical geological material, one obtains the linear size R of the critical focal zone ranging from ten meters to a few kilometers. Such an illustrative estimation serves only to help conceiving the existence of a localized critical focal zone, where the seismic energy originates. The depth of the earthquakes foci ranges from a few tens kilometers to several hundreds kilometers in the Earth's crust. The seismic energy is released as elastic waves, which propagate through an inhomogeneous geologic material to the Earth's surface, giving rise to surface waves, with multiple reflections, refractions, diffractions (as for geometrical rays of waves), and a range of various local effects. The seismic waves have a typical velocity of a few kilometers per second, corresponding to an average elastic modulus μ_e of the order of 10^{11}dyn/cm^2 , for an average density 5g/cm^3 of the geological material, and typical wavelengths ranging from meters to kilometers.

The third feature of the earthquakes' studies is the statistical approach. Being given the large extent and variety of the earthquake occurrence, both in space, in time, in energy and in number, the earthquakes may exhibit some regular patterns, or some regularities in their recurrence, as provided by statistical distributions in energy, time, etc. It is estimated that the total amount of seismic energy released annually on the Earth is 10^{18}J , by an average number of 10^6 earthquakes, 80% of this amount coming from shocks whose energy is higher than 10^{16}J . It is also estimated that the total amount of released seismic energy is 0.1% of the total amount of heat produced by the Earth' interior annually.[1] All these great figures may render likely a statistical approach. A word of caution is, however, conceivable in this respect, because many of the small earthquakes may have other causes than being statistically produced, while the extremely great earthquakes might hardly be viewed as statistical events, due to their singular character.

A brief historical excursion. Perhaps the first quantitative knowledge on earthquakes comes from Omori in 1895,[2] who, reportedly, noticed that aftershocks following a major earthquake are distributed in time according to a power-law of the type $\sim t^{-\gamma}$, where the exponent γ is slightly greater than unity. Such power-laws are of current interest today for statistical distributions of various quantities. Seismology made a decisive step forward in the first half of the 20th century, with the discovery of the seismograph and its various improvements.[3] In 1935 Richter defined the magnitude M of an earthquake,[4] as the decimal logarithm of the seismograph's displacement in standard conditions. The procedures for estimating the earthquakes magnitude, as well as other characteristics, have been greatly improved up to 1956, by means of the travel-time and conversion tables, culminating in the Gutenberg-Richter law.[5] Meantime, mechanical models for the focal zone have been put forward,[6] and statistical approaches have been devised, which are being pursued today.[7] An interesting empirical law has been noticed in 1965, known as Bath's law,[8] according to which the highest, most likely magnitude of the aftershocks is 1.2 less than the magnitude of the main seismic shock. Apart from the great interest in causes, mechanisms, patterns and prediction, the modern seismological research deals largely today with territorial zonation, where local seismic effects are analyzed, in order to establish recommendations for construction and to mitigate the seismic risk and hazard.

The Gutenberg-Richter law. According to the definition given by Richter, the magnitude M of an earthquake is proportional to the decimal logarithm of the seismic energy flux in standard conditions, *i.e.* it is proportional to the decimal logarithm of a surface energy $E_s \sim R^2$, where R is a characteristic radius. On the other hand, the seismic energy goes with the 3rd power of the radius R , so that we may write down $M \sim \lg E_s \sim 2 \lg R \sim (2/3) \lg E$. Hence, $\lg E \sim (3/2)M$, or

the Gutenberg-Richter law[5, 9, 10]

$$\lg E = a + bM \quad , \quad (1)$$

where $b = 1.5$, relating the seismic energy E released in an earthquake to its magnitude M . It is easy to see that this is a semi-empirical law, the coefficient b being derived on theoretical grounds, while the coefficient $a \simeq 4.4$ (for energy measured in J) is obtained by fitting experimental data. The fitting has been performed on large statistical ensembles, with magnitudes in the range from $M \simeq 5$ to $M \simeq 7$. It is worth noting that this range implies a huge energy scale, one unit in magnitudes standing for a factor $10^{1.5} \simeq 32$ in energy. For extreme magnitudes, *i.e.* for magnitudes lower than 5 or higher than 7, there are deviations from this linear semi-logarithmic law, as expected. On the other hand, the experimental data in (1) imply an error of a factor 10 in energy, which may be equally shared both by the coefficient a and by the bM -term. Therefore, one may say that the inaccuracy in magnitudes is about $\delta M \simeq 0.5/b = 0.33$.

The Gutenberg-Richter law can also be written with natural logarithms as

$$\ln E = a + bM \quad , \quad (2)$$

where $a \simeq 10$ and $b = 3.5$. It can also be represented as $E = E_0 e^{bM}$, where E_0 is a threshold energy. This threshold energy may be viewed as the minimal energy needed for an earthquake to occur, or to be produced, or even to be felt or recorded.

Since $E \sim R^3$ one may get a similar law

$$\ln(R/R_0) = (b/3)M = 1.17M \quad (3)$$

for the characteristic radius R , where R_0 is a threshold length scale associated with the threshold energy E_0 . The lengths R and R_0 have a double meaning here. On one hand, they refer to the critical focal zone, which may have a linear extension R for the magnitude M (and the corresponding energy E accumulated in this zone), R_0 being a minimal length scale associated with the threshold seismic energy. On the other hand, R and R_0 may be viewed as characteristic lengths of the geographical zone disrupted by the earthquake, as measured by epicentral distances. With such an interpretation the law given by (3) seems to be well documented by experimental data.[11]

A model of critical focal zone. The rate of the energy accumulation in the seismic focus can be written as

$$\frac{\partial E}{\partial t} = -\mathbf{v} \text{grad} E \quad , \quad (4)$$

where \mathbf{v} is the accumulation velocity. For a point-like focus, this velocity may be taken as being uniform along all three spatial directions, *i.e.* $v_x = v_y = v_z = v$, and the spatial derivatives of the energy E can also be taken as having the same value, $\partial E/\partial x = \partial E/\partial y = \partial E/\partial z = -(E + E_0)/(R + R_0)$, where E_0 is the seismic threshold energy, R is the radius of the focal zone and R_0 is the length scale of the critical focal zone, as discussed above. Equation (4) becomes then

$$\frac{\partial E}{\partial t} = 3v \frac{E + E_0}{R + R_0} \quad . \quad (5)$$

Such an equation is typical for a growth model, with a point-like focus, a localized growth region, and a high growing rate. For deviations from a point-like focus, or for a non-uniform growth in all spatial directions, the coefficient 3 in front of the *rhs* of equation (5) may have a different value (for instance, for a uniform growth model along two spatial directions only the coefficient is 2). In order to preserve such a generality we denote this coefficient by $1/r$ ($= 3$). Introducing the

accumulation times $t = R/v$ and $t_0 = R_0/v$, where t_0 is a corresponding threshold time, equation (5) becomes

$$\frac{\partial E}{\partial t} = \frac{1}{r} \frac{E + E_0}{t + t_0}, \quad (6)$$

whose solution is

$$1 + t/t_0 = (1 + E/E_0)^r. \quad (7)$$

This power-law is typical for an accumulation, or growth, model. Since $E/E_0 = (R/R_0)^3 = (t/t_0)^3$ for a uniform focal zone, it follows $3r = 1$ indeed from (7), in the limit of large energy, large radius and for long times, which checks out the consistency of the model. We assume therefore $r = 1/3$. Equation (7) is the basic equation of the seismic focus model put forward herein.

The Omori distribution. Suppose that a large number N_0 of earthquakes are produced in a long time T . For a statistical ensemble, the average time of occurring one earthquake is $t_0 = T/N_0$, while $t + t_0$ is then the average time of occurring one earthquake in time t , with energy E , as given by (7). Suppose that N such earthquakes occur in time T , such that their frequency is $N/N_0 = T/(t + t_0)N_0 = t_0/(t + t_0)$. Hence, the temporal probability distribution

$$P(t)dt = -d\left[\frac{t_0}{t + t_0}\right] = \frac{dt/t_0}{(1 + t/t_0)^2}. \quad (8)$$

The threshold time t_0 acquires this way the meaning of the inverse of the seismicity rate, *i.e.* the seismicity rate is $1/t_0 = N_0/T$. Making use of (7) and (8) one can get straightforwardly the distribution in energy

$$P(E)dE = \frac{r dE/E_0}{(1 + E/E_0)^{1+r}}. \quad (9)$$

By analogy with Omori's observation these power-laws are called Omori distributions. In particular, the distribution given by (9) is the Omori distribution in energy, and the parameter r ($= 1/3$ in this case) is called Omori's parameter.

The aftershocks following a major earthquake can be viewed as a relaxation of the seismic zone, which allows the linearization $\delta t/t_0 \simeq r(E/E_0)^{r-1} \delta E/E_0$ of the time-energy equation (7), where $\delta t = \tau$ is the time measured from the major earthquake, $\delta E = \varepsilon$ is the corresponding variation in the seismic energy, and E represents the energy of the major earthquake. One can see from this linear relationship that the energy of the aftershocks can be high, even for short times after the major earthquake, due to the small pre-factor $r(E/E_0)^{r-1}$. This linear relationship can be employed in the energy distribution given by (9), leading to a time distribution $r^{1+r}(E/E_0)^{r^2-r}(d\tau/t_0)/(\tau/t_0)^{1+r}$, *i.e.* the Omori distribution for aftershocks, with $\gamma = 1 + r$. Actually, in the limit of vanishing time τ this distribution has a cutoff $(E/E_0)^{1-r} d\tau/t_0$. The formal expansion of the exact expression of this distribution in powers of τ/t_0 leads to the same result as the one given by the temporal distribution (8).

Distribution in magnitudes. Bath's law. Recurrence law. Making use of the Gutenberg-Richter law $E/E_0 = e^{bM}$ the distribution in magnitudes

$$P(M)dM = \frac{rb}{(1 + e^{bM})^{1+r}} e^{bM} dM \quad (10)$$

is obtained from Omori's distribution given by (9), which acquires the well-known exponential form

$$P(M)dM = \beta e^{-\beta M} dM, \quad (11)$$

for $bM \gg 1$, where $\beta = rb = 1.17$. Assuming, as before, that the aftershocks are governed by the same exponential distribution for their difference $m = \delta M$ in magnitudes,[12] but are associated with the relaxation of the seismic zone after a major earthquake, then $\bar{m} = 0$ and the highest, most likely magnitude of the aftershock will be lower than the magnitude of the main shock by the mean square deviation of this distribution $\delta M = (\bar{m}^2)^{1/2}$, *i.e.* $\sqrt{2}/\beta = \sqrt{2}/1.17 = 1.2$, in agreement with Bath's empirical law.[8]

The mean value of the magnitudes, according to the exponential distribution (11), is $\bar{M} = 1/\beta$, so that the deviation of the exponential distribution from this mean magnitude is given by $\delta M = (\bar{M}^2)^{1/2} - \bar{M} = (\sqrt{2} - 1)/\beta = 0.35$, which is in agreement with the experimental error in magnitudes, as discussed in connection with the Gutenberg-Richter law above. This is consistent with the temporal distribution of the average recurrence times introduced in (8), which leads to the exponential distribution in magnitudes.

The differential distribution of the number ΔN of earthquakes with magnitude between M and $M + \Delta M$ produced in a very large time T leads to $P(M) = (\Delta N/T\Delta M)/(N_0/T)$, where N_0 is the total number of earthquakes produced in time T . Hence the semi-logarithmic law

$$\ln(\Delta N/T\Delta M) = \ln(\beta/t_0) - \beta M \quad , \quad (12)$$

for $bM \gg 1$, where $1/t_0$ is the seismicity rate. This relationship is well documented by statistical studies for earthquakes of magnitudes in the range $5 \leq M \leq 7$. [1, 5, 9] For lower magnitudes the exact equation (10) indicates less earthquakes than (12) does, in agreement with empirical data, (though there are considerable uncertainties in estimating the number of small earthquakes), while for very strong earthquakes the empirical data exhibit deviations from equation (12), as an apparent increase in the coefficient β . This might be another indication that strong seismic movements are not part of a statistical ensemble, though a very difficult task in analyzing data is a reliable estimation of the seismicity rate t_0^{-1} , a critical parameter for such fits.

It is convenient to introduce the exceedence rate of earthquakes, starting with the probability $P_{>} = e^{-\beta M}$ for magnitudes higher than M . This leads to the so-called recurrence law

$$\ln(N_{>}/T) = \ln(1/t_0) - \beta M \quad , \quad (13)$$

where $N_{>}$ is the number of earthquakes of magnitude greater than M and $\ln(1/t_0) = \alpha$ is sometimes called the parameter of the seismicity rate. The exact expression for this recurrence law as given by (10) is $\ln(N_{>}/T) = \alpha - r \ln(1 + e^{bM})$. These recurrence laws are well obeyed by the empirical data for magnitudes up to $M \simeq 7$, where considerable discrepancies are recorded.[1, 5, 9] For practical purposes, in seismic risk and hazard assessment analysis, it is customary to cut the magnitude distribution (11) off at an upper bound, for getting more satisfying data fits.

Average recurrence time. Time distribution. According to the basic equation (7), the average recurrence time for an earthquake of energy $E = E_0 e^{bM}$ and magnitude M , is given by

$$t = t_0[(1 + e^{bM})^r - 1] \quad . \quad (14)$$

This time can be viewed as the average succession time for a series of earthquakes of magnitude M . For convenience, it is also denoted by t_r , in order to distinguish it from the current time t . For $bM \gg 1$ the exponential form

$$t_r = t_0 e^{\beta M} \quad (15)$$

can be used, where $t_0 = T/N_0$ is the inverse of the seismicity rate. This seismicity rate can be related to a reference magnitude M_0 , by writing $t_0 = e^{-\beta M_0}$, *i.e.* the parameter of the seismicity rate $\alpha = \ln(1/t_0)$ is written as $\alpha = \beta M_0$, so that the average recurrence time becomes

$$t_r = e^{\beta(M-M_0)} \quad , \quad (16)$$

and also $t_r = T/N_>$, according to the recurrence law (13). Actually, the average recurrence time is defined for the differential number ΔN of earthquakes with magnitude between M and $M + \Delta M$, as given by (12). Making use of equation (12) one gets the average recurrence time

$$t_r = (t_0/\beta\Delta M)e^{\beta M} = \frac{1}{\beta\Delta M}e^{\beta(M-M_0)} \quad (17)$$

for earthquakes with magnitudes in the range M to $M + \Delta M$. The recurrence time given by (16) corresponds formally to $\beta\Delta M = 1$ (and to the exceedence number $N_>$ of earthquakes). The main source of inaccuracy in estimating such average recurrence times originates in the seismicity rate $1/t_0$, and, correspondingly, the reference magnitude M_0 . In addition, the large earthquakes, whose prediction is of greatest interest, can be viewed as statistical events only with a limited confidence, due to the singular nature of their occurrence. The average recurrence time for such rare seismic events is long, and, consequently, the errors made in their estimation are large in absolute value. This indicates that statistical estimation of the average recurrence times is affected by great uncertainties, and it should be viewed with much caution.

Indeed, if one assume that the entropy of the seismic ensemble of earthquakes is given by $S \sim p \ln p$, where p is the probability, then the maximum value of this entropy under the constraint $\bar{t} = t_r$, *i.e.* under the constraint of fixed average recurrence time, leads to the time distribution

$$p(t) = (1/t_r)e^{-t/t_r} \quad , \quad (18)$$

which is also called sometimes the Poisson-like time distribution for earthquake recurrence. An estimation of the errors implied by this distribution leads to $\delta t = (\bar{t}^2)^{1/2} - t_r = (\sqrt{2} - 1)t_r$, *i.e.* $\simeq 41\%$ accuracy in estimating the average recurrence time. An analysis of the associated seismic events, *i.e.* of the foreshock- and aftershock-type seisms which accompany a major earthquake may reduce such a large uncertainty in the prediction of the average recurrence time of the great seismic events. Unfortunately, such an analysis is not available at this time.

Seismic region Vrancea. Vrancea is the main seismic region of Romania, located just on the apex of the Carpathian mountains arch, approximately at 45.7° N latitude and 26.6° E longitude. The chain of Carpathian mountains makes here an abrupt turn from the NS direction to the EW direction. The seismic focus in Vrancea is located between ~ 80 km and ~ 150 km depth, and produces strong earthquakes with typical magnitude $\sim 7 - 7.5$, whose effects propagate, approximately, along the NE-SW and NW-SE fault directions. The seismicity rate, as estimated from the data recorded in the last 25 years, [13] seems to be $1/t_0 \simeq 100/\text{year}$ ($M > 3$). Modern instrumental recording of the earthquakes in Vrancea is performed since 1980, but there seems to be historical recordings from as early as 1000. The Vrancea earthquakes with magnitude higher than $M = 6$ are shown in Fig. 1 from 1800 to the present.

The seismicity rate indicated above corresponds to a reference magnitude $M_0 = 3.9$, so that, making use of equation (17) one gets an average recurrence time $t_r \sim 40$ years for earthquakes with magnitudes in the range $M \sim 7$ to $M \sim 7.9$. It corresponds formally to equation (16) ($\beta\Delta M \simeq 1$). Such a succession time seems to agree qualitatively with data shown in Fig. 1 for the greatest earthquakes in Vrancea. However, as said before, such an estimation must be viewed with great caution. Indeed, first we consider earthquakes with high magnitude because, for lower magnitude, it is difficult to distinguish the regular seisms from those associated with major earthquakes, as foreshock- and aftershock-type seisms (if such a distinction exists, and to what extent and relevance). Second, it is difficult to fully view the earthquakes with high magnitude as statistical events. A great source of errors is brought by the seismicity rate, a parameter which is very difficult to estimate. For instance, if one includes the recorded seisms with magnitude lesser

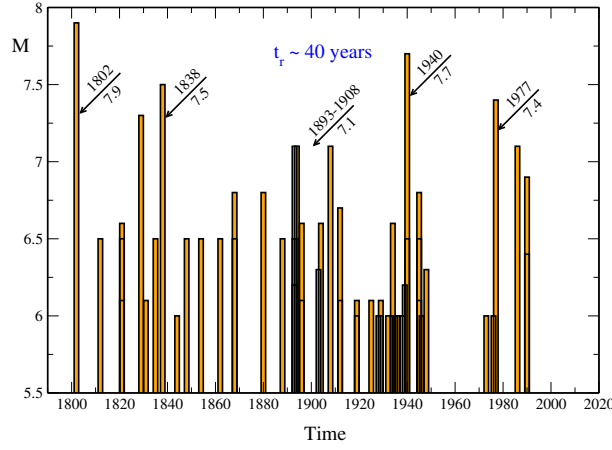


Figure 1: Vrancea earthquakes with magnitude $M > 6$ from 1800 to the present. The greatest seismic events are shown by arrows, and the average recurrence time $t_r = 40$ years corresponds approximately to $M \sim 7.2 - 7.9$

than $M = 3$ one obtains a seismicity rate $1/t_0 \simeq 130/\text{year}$ for Vrancea region, which corresponds to a reference magnitude $M_0 = 4.15$ and leads to the same average recurrence time $t_r \sim 40\text{years}$ for earthquakes with magnitudes in the range $M = 7.2$ to $M = 7.9$, which seems to be in closer agreement with the data shown in Fig. 1.

The succession of the great seismic events in Vrancea, according to Fig. 1, seems to obey qualitatively the average recurrence time $t_r \sim 40\text{years}$, with one exception between 1838 and 1893-1894-1908, where the time elapsed between the two major successive earthquakes is about 60years. This looks like an irregular situation, which might be explained by the great seismic activity during the whole this period. One may also see in Fig. 1 certain seismic activities which might be associated with weaker seisms accompanying major earthquakes, of foreshock- and aftershock-type. If the 41% maximal error in estimating the average recurrence time indicated before is assigned to this accompanying seismic activity in the anomalous period 1838-1908 in Vrancea, then one should allow for at least an equal time in order to have another major earthquake. This makes $\sim 16\text{years}$, which, indeed, corresponds roughly to the approximate time by which the earthquakes in 1893-1894-1908 have been delayed with respect to the one in 1838 (one must also notice in Fig. 1 the great earthquakes in 1893 and 1894, each of magnitude $M = 7.1$; such doublets are intriguingly exhibited from time to time in Vrancea, as shown in Fig. 1). Such an anomalous situation could be identified by constantly monitoring the seismic activity, in order to detect intense accompanying seismic activities, of foreshock- and aftershock-type. Assuming that the next seismic period in Vrancea is a regular one, then the next major earthquake after the one in 1977 ($M = 7.4$) would be expected there around 2017. Indeed, as shown in Fig. 1, one can say that the aftershock-type activity in Vrancea ended around 1990 at the latest, after the great doublets in 1977 ($M = 7.4$) and 1986 ($M = 7.1$). If, starting around 2010, an intense seismic activity will be recorded in Vrancea, of the foreshock-type, then the situation might be an anomalous one, similar to the period 1838-1908, and the next major earthquake would then be postponed until around 2030; if, on the contrary, there will be a rather moderate seismic activity in this period, then the situation is likely to be a regular one, and the next major earthquake would be expected around 2017. All these, however, are only very qualitative estimations, made here for the purpose of temptingly testing the theoretical results presented herein, and for illustrating the kind of analytical discussion they may allow for.

Non-linear effects. Apart from statistical theories, another topic much debated in seismology is

the understanding of the non-linear effects associated with the propagation of the seismic energy, and with the seismic waves in general. This is an issue in non-linear elasticity, and an instance of an exact solution to a non-linear wave equation is briefly presented here, as well as its relation to the quasi-plane waves.[14] The intriguing issue in this connection is that the non-linearities are present, exact solutions, as the one presented below are unphysical, and still the empirical observations are compatible with a limited type of quasi-linear behaviour in the propagation of the seismic energy and the associated effects. It is shown below that indeed, there are local amplification factors in the non-linear effects of the propagation of the seismic energy, which, however, still allow for a quasi-linear regime.

The first non-linear correction to the wave equation comes from the cubic anharmonicities, which lead to an elastic energy

$$E = \int d\mathbf{r} \left(\frac{\lambda}{2} u_{ii}^2 + \mu u_{ij}^2 + \frac{1}{3} A u_{ij} u_{jk} u_{ki} + B u_{ij}^2 u_{kk} + \frac{1}{3} C u_{ii}^3 \right) , \quad (19)$$

for an isotropic elastic body, where λ and μ are the usual Lamé coefficients, A , B , C are constant coefficients, and $u_{ij} = (1/2)(\partial u_i / \partial x_j + \partial u_j / \partial x_i + \partial u_k / \partial x_i \cdot \partial u_k / \partial x_j)$ is the cartesian (finite-) strain tensor. It is assumed that the coefficients in (19) are such as the stability conditions are satisfied. First, a transverse displacement, say, $u_2(x_1)$ is not affected by the cubic anharmonicities above, so that the corresponding linear wave equation is left unchanged (the transverse waves propagate with velocity $v_t = \sqrt{\mu/\rho}$, where ρ is the density of the body).

A longitudinal displacement $u_1(x_1) = u(x)$ is, however, subjected to the non-linear equation $\partial^2 u / \partial t^2 = (\partial^2 u / \partial x^2)(v_l^2 + v^2 \partial u / \partial x)$, where $v_l = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the longitudinal waves, and $v^2 = [3(\lambda + 2\mu) + 2(A + 3B + c)]/\rho$ is a characteristic square velocity. Leaving aside again the stability conditions,[14] and denoting $U = \partial u / \partial x + v_l^2 / v^2$, this non-linear equation becomes

$$\partial^2 U / \partial t^2 = (v^2 / 2) \partial^2 U^2 / \partial x^2 . \quad (20)$$

This equation is the continuum limit of the Fermi-Pasta-Ulam equation.[14] Its solution, and solutions of other, similar, equations have been analyzed recently by making use of the Lie algebra of the equation symmetry group and the prolongation technique.[15] The solution $U(t, x) = g(t)f(x)$ of equation (20) can be obtained by elementary quadratures. The time dependence is given by[14]

$$g(t) = |s| \left[\sqrt{3} \frac{1 - cn(\sqrt{\sqrt{3}}|s||\omega t|)}{1 + cn(\sqrt{\sqrt{3}}|s||\omega t|)} - 1 \right] sgn(\omega^2) , \quad (21)$$

where $s = -g(0)$ ($\dot{g}(0) = 0$), ω is a constant of integration and cn is the Jacobi elliptic cosine-amplitude. Function $g(t)$ given by (21) is a periodic function with period $\sqrt{\sqrt{3}}|s||\omega t| = 4K$, where K is the complete elliptic integral $F(\pi/2, k)$ (~ 4) for $k^2 = (2 + \sqrt{3})/4$. It has singularities at $\sqrt{\sqrt{3}}|s||\omega t| = 4K(n + 1/2)$, where n is an integer. These singularities make the solution of eq. (20) unphysical. The spatial dependence $f(x)$ is given by the implicit equation $\sqrt{(|f/h|)^3 - 1} F(1/2, 1/3, 3/2; 1 - (|f/h|)^3) = 3|\omega x/v|/2\sqrt{|h|}$, where F is the Gauss hypergeometric function and $h = f(0)$ is another constant of integration ($f'(0) = 0$).[14] Function f goes like $f \sim |h| sgn(\omega/v)^2 + (\omega/2v)^2 x^2$ for $x \sim 0$, and $f \sim (\omega/2v)^2 x^2$ at infinite ($x \rightarrow \pm\infty$). It is worth noting that $f(x)$ is boundless for spatial boundaries placed at infinite, which adds to the unphysical character of the solution. The general solution of the non-linear equation for the longitudinal strain $u(t, x)$ reads then

$$u(t, x) = g(t - t_0) \int_0^x dx f(x - x_0) - (v_l/v)^2 x + c , \quad (22)$$

where the origin of time t_0 and the origin of space x_0 are introduced, and c is another constant of integration. The nature of this solution is worth discussing. First, it is worth noting that the displacement given by (22) implies large strain (and stress) values at the boundary of the spatial region, which is consistent with the accumulation model of the critical focal zone employed herein. Second, these large strain and stress values may lead in time to ruptures at the boundaries of the focal zone (or at the boundaries of the critical seismic region), as a consequence of the boundless increase of the time dependence (which is singular at certain times, as noted above). These ruptures may propagate, with a non-uniform velocity, which represents a distinct mechanism of dissipation of the seismic energy in the critical zone affected by non-linearities. It is not restricted to cubic anharmonicities, higher-order non-linear contributions to the wave equation leading to a similar behaviour.[14] Third, it is worth noting that the total energy conserves, but it is non-uniformly distributed, such that ruptures may appear in time at the boundaries of the spatial region. The energy flow at the boundaries increases also boundlessly in time. The process looks rather like a vibration than a wave propagation. All these features make the solution unphysical. Exact, unphysical solutions of non-linear type described above are therefore more appropriate for the critical focal zone and for the seismic region disrupted by the earthquakes.

After all this seismic energy is dissipated in ruptures and damage of the elastic body, the non-linear contributions to the wave equation may be viewed as perturbation to the plane wave solutions of the linear equation. Indeed, introducing the perturbation parameter $\varepsilon = (v/v_l)^2$ the equation for the longitudinal displacement may be written as $\ddot{u} - v_l^2 u'' = \varepsilon v_l^2 u' u''$, whose solution reads[14]

$$u = a \cos(\omega t - kx) + \frac{1}{16} \varepsilon a^2 k^2 (x + v_l t) \cos[2(\omega t - kx)] + \dots$$

$$+ \frac{1}{128} \varepsilon^2 a^3 k^4 (x + v_l t)^2 [\cos[3(\omega t - kx)] - \cos(\omega t - kx)] + \dots \quad (23)$$

where a is the amplitude, $\omega = v_l k$ is the frequency and k is the wavevector of the elementary plane wave. The solution given by (23) is, actually, a triple expansion in powers of the perturbation parameter ε , the ratio ak of the amplitude to the wavelength, and the ratio lk of a characteristic length $l = x + v_l t$ to the wavelength. The solution (23) is actually an asymptotic series, and it has a limited validity over finite distances and times, providing that the amplitude is much smaller than the wavelength. Such a wave may be viewed as a quasi-plane wave, *i.e.* a plane wave distorted by higher-order harmonics of limited validity in space and time. It is worth noting the amplification factor F of the order of $F \simeq 1 + \varepsilon alk^2/16$ (in displacement) brought by the non-linear effects to such quasi-plane waves, amplification which is well-documented in the analysis of the local seismic effects due to non-linearities. An estimation of the distribution of the seismic energy originating in a localized focal zone shows that the long wavelengths and small amplitudes are favoured, the ratio ak being of the order of $10^{-2} - 10^{-4}$. Therefore, one may use the quasi-plane waves pictures up to distances l very large in comparison with the wavelengths.[14]

Another worth noting non-linear phenomenon appears in the non-linear coupling between a longitudinal displacement and a transverse one, propagating in the same direction. Beside higher-order harmonics and amplification factors, there may appear resonances at certain frequencies, due to the combined-frequency phenomenon, as, for instance, at the transverse wave frequency $\omega_2 = (\omega_1/2)(1 + v_t/v_l)$, where ω_1 is the frequency of the longitudinal wave.[14] Such resonances depend on the ratio v_t/v_l of the waves velocities. Another non-linear coupling arises, for instance, from longitudinal displacements of the type $u_1(x_1)$, $u_2(x_2)$, $u_3(x_3)$, which might be relevant for the dynamics of the accumulation model of the critical focal zone. There seems not to be a simple treatment of such coupled non-linear equations.

Conclusions. There seems to be at least three basic features pertaining to the science of the earthquakes, according to the present image of this science. First, the energy of the earthquakes is

distributed over a huge scale, according to the semi-empirical Gutenberg-Richter law, as given by (1) or (2), relating the seismic energy to magnitude M . The Gutenberg-Richter coefficient $b = 3.5$ is worth noting here (equation (2)). Second, the seismic energy originates in a rather restricted critical focal zone, of a characteristic linear size given by (3); at the same time, equation (3) refers also to an epicentral length scale characteristic of the seismic region disrupted by the earthquake. Third, the large variety of the earthquakes in energy, magnitude, number, space and time suggests a statistical approach, as based on their various distributions. Such a statistical approach is also suggested by the distribution in magnitudes of the differential number of earthquakes (equation (12)), by a similar distribution of the earthquakes with magnitudes exceeding a given value (excedence, or recurrence law given by equation (13)), by the Omori temporal distribution of the aftershocks (which goes like $t^{-\gamma}$, where $\gamma = 1+r$, the positive Omori parameter r acquiring a small value), by the highest, most likely aftershock magnitude (Bath's law, this aftershock magnitude being 1.2 less than the magnitude of the main shock), and by the time Poisson-like distribution of the recurrence times (equation (18)). All these laws are semi-empirical, having a limited validity. Such a limitation comes mainly from the fact that very small seisms, or very great earthquakes, by their own nature, do not reliably belong to a statistical ensemble. It is also worth noting an intriguing issue much debated today in seismology, regarding the effects of the non-linearities on the propagation of the seismic energy, and the corresponding estimation of such local effects, especially in studies of seismic risk and hazard.

An attempt of a systematic understanding of such basic features in seismology is made here, by introducing an accumulation, or growth, model for the concentration of the seismic energy in the critical focal zone. This model relates the accumulation time to the seismic energy (equation (7)), and introduces a characteristic parameter r , whose value $r = 1/3$ is derived on geometrical grounds. It turns out that this parameter r is the Omori parameter. Indeed, the second main theoretical point made here is the interpretation of the accumulation time as the average recurrence time of the earthquakes with corresponding energy (and magnitude), as given by the accumulation model. On this basis, the temporal distribution (8) of the earthquake average recurrence times is derived, the Omori distribution in energy (equation (9)) and the exponential distribution in magnitudes with the exponent $\beta = br = 3.5/3 = 1.17$ (equations (10) and (11)). By linearizing the time-energy dependence for the seismic activity following a major earthquake one gets the temporal Omori distribution of the aftershocks with $\gamma = 1 + r = 1.33$, consistent with experimental observations. By assuming the aftershocks as reflecting the relaxation of the seismic zone one gets Bath's law $\delta M = \sqrt{2}/\beta = 1.2$ for the amount by which the most likely, highest magnitude of the aftershocks is lesser than the magnitude of the main earthquake. The differential distribution of the earthquakes in magnitudes (12), as well as the excedence rate (recurrence law) (13) are derived from the exponential distribution in magnitudes with $\beta = 1.17$, in agreement with empirical observations. The time Poisson-like distribution of earthquakes is also derived for a fixed average recurrence time (equation (18)), and the differential average recurrence time is given for earthquakes with magnitude in the range M to $M + \Delta M$ (equation (17)). The errors in estimating both the magnitude and time distributions are discussed, and the errors associated with the seismicity rate $1/t_0$ are shown to be critical for the statistical prediction of long succession times of the great earthquakes (t_0 being also a threshold time introduced by the accumulation model). An application of these results is made to the great earthquakes in the seismic region Vrancea, Romania, in the past 200years.

It is shown, by analyzing the cubic anharmonic corrections to the elastic waves equation corresponding to longitudinal displacements, that the non-linearities have a disruptive effect on the critical focal zone, or the epicentral region greatly affected by the earthquake. The exact solution of this equation (equations (21) and (22)) has an unphysical character, exhibiting time singularities

and a boundless increase at the boundaries of the spatial region. Such an unphysical behaviour is also specific to higher-order non-linearities. Consequently, ruptures may appear at the boundaries of the critical zone, which may propagate in the whole body of the region. However, the propagating seismic energy is distributed mainly on long wavelengths and small amplitudes, such that for small values of the ratio of the amplitude to the wavelength the linear picture of quasi-plane waves is still valid, in a perturbational picture, for limited distances and times (as controlled by the ratio of a characteristic length l to the wavelength, according to equation (23)). As a consequence of the non-linearities the quasi-plane waves are distorted by higher-order harmonics, and exhibit local amplification factors in displacement, velocity and acceleration, as documented by empirical evidence. The non-linearities may lead, in this perturbational approach, to other effects, as resonances, combined-frequency phenomenon, or non-linear coupling between various kinds of elastic waves, which enriches considerably the linear phenomenology of waves propagation.

A technical Appendix regarding statistical distributions. Suppose that we have a quantity x , which we believe is close in value to some value x_0 . We further assume that x is distributed around this x_0 value according to some probability density $p(x)$. It is then natural to view the mean value $\bar{x} = \int xp(x)dx$ as a measure for x_0 . However, we have immediately the problem that, obviously, $x_0 = (x_0^2)^{1/2}$, which would mean that $(x^2)^{1/2}$ is also a measure for x_0 . Therefore, we have at least two measures for x_0 , which differ in general, and the only meaning to this situation is to admit that $(x^2)^{1/2} - \bar{x}$ is a measure for the errors we made in estimating with a statistical distribution the value x_0 . The situation is amplified by the other, higher-order momenta of the statistical distribution, of the form $(x^n)^{1/n}$, where n is any integer, which may even diverge with increasing n . A satisfying situation would be a distribution $p(x)$ very peaked on the value x_0 , as, for instance, in the limiting case $p(x) = \delta(x - x_0)$, where all these momenta have the same values. It is such an approach we employed herein in estimating the errors in magnitude, or in the average recurrence times of the earthquakes. The large errors we found in these estimations indicate that such statistical approaches have only to be used with great caution for the analysis of the earthquakes. The situation is similar (but not identical) with the distribution of the fluctuations, which measure the deviation of a function $f(x)$ with respect to its mean value. Indeed, we may expand such a function around \bar{x} as $f(x) = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + (x - \bar{x})^2 f''(\bar{x})/2 + \dots$, and get $\bar{f} = f(\bar{x}) + (x - \bar{x})^2 f''(\bar{x})/2 + \dots$, hence $(x - \bar{x})^2 = \bar{x}^2 - \bar{f}^2$ as a measure of the fluctuations of x . Though the convergence problem remains, in principle, the series for \bar{f} may be rendered convergent by a suitable decreasing of its derivatives.

The analysis of the foreshock- and aftershock-type seismic activity deserves further attention. According to the accumulation model, the aftershock-type seismic energy ε obeys a linear relationship with the increasing time τ measured from the main seismic shock (linearization of equation (7)), over a limited range of variation of these two quantities. These quantities are variations of the main seismic energy and of the main accumulation time, *i.e.* $\delta E = \varepsilon$ and $\delta t = \tau$. Consequently, according to the Omori distribution (9), it is natural to view the distribution of this seismic energy as being described by the exponential law $\beta e^{-\beta m}$, where $m = \delta M$ is the variation in magnitude.[12] A similar, symmetric distribution holds however for the negative values of the variation m in magnitudes, according to the relaxation model of the seismic zone, so that $\bar{m} = \delta \bar{M} = 0$. It remains that the most likely, highest magnitude of such a seismic relaxation process is measured by $\delta M = [(\delta \bar{M})^2]^{1/2} = (m^2)^{1/2} = \sqrt{2}/\beta = 1.2$, *i.e.* Bath's law. If the linearization procedure is employed here, *i.e.* if we assume $\tau/t_0 = r(E/E_0)^r(\varepsilon/E)$ as described after equation (9), then we get the time $\tau = rt_r(\varepsilon/E)$ as corresponding to this "main" aftershock, or, for $\varepsilon = E_0 e^{bM} - E_0 e^{b(M-\delta M)}$ and $E = E_0 e^{bM}$, we get $\tau \simeq rt_r = t_r/3$ for the expectation time of the "main" aftershock. A similar analysis holds also for the foreshock-type seismic activity.

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