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Power-law distributions and avalanche phenomena

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

Events distributed over a large range are avalanche-like governed by power laws providing there exists an isolated primordial event.

Let x run over a huge range of numerical values, from x_0 to D, where $D \gg x_0$. Is may then be natural, and more convenient, to introduce the new variable

$$y = \log_{(D/x_0)}(x/x_0) \ , \tag{1}$$

which lies in the range 0 to 1. At the same time, in order to preserve the original range, it is also natural to consider functions of the type $(D/x_0)^{f(y)}$, where f is an undetermined function of y.

On the other hand, let N be an ensemble of events distributed over x, and write down their probability as given by frequency

$$\frac{dN}{N_0 dx} = A + (D/x_0)^{f(y)} \ , \tag{2}$$

where N_0 is their total number and A is defined by their probability $A + (D/x_0)^{f_0}$ for $x = x_0$, $f_0 = f(0)$. We proceed now to recasting (2) as

$$\frac{1}{A}[1 - \frac{1}{A}(D/x_0)^{f(y)}]\frac{dN}{N_0 dx} = 1 \ . \tag{3}$$

Function $f(y)$ in (3) can be written as $f(y) = f_0 + f_1 y = f_0 + f_1 \log_{(D/x_0)}(x/x_0)$, where $f_1$is the derivative of function f for $y = 0$. We assume $f_1 = f'(0) > 0$. By (3), we get straightforwardly

$$\frac{1}{A}[1 - \frac{1}{A}(D/x_0)^{f_0}(x/x_0)^{f_1}]\frac{dN}{N_0 dx} = 1 \ , \tag{4}$$

or

$$\frac{(D/x_0)^{f_0}}{A^2}(x/x_0)^{f_1}[\frac{A}{(D/x_0)^{f_0}(x/x_0)^{f_1}} - 1]\frac{dN}{N_0 dx} = 1 \ , \tag{5}$$

and define

$$\frac{d\widetilde{N}}{N_0 dx} = [\frac{A}{(D/x_0)^{f_0}(x/x_0)^{f_1}} - 1]\frac{dN}{N_0 dx} \ , \tag{6}$$

such that

$$\frac{d\widetilde{N}}{N_0 dx} = \frac{A^2}{(D/x_0)^{f_0}} \cdot \frac{1}{(x/x_0)^{f_1}} \ . \tag{7}$$

Indeed, making use of (2), equation (6) is identical with equation (7), to the same order of approximation, providing

$$A \gg (D/x_0)^{f_0}(x/x_0)^{f_1} \ . \tag{8}$$

Distribution (7) defines a power-law over the entire range $x_0 < x < D$. Properly normalized over this range it may be associated to the same set of N_0 events, and may be brought to have the same value $A + (D/x_0)^{f_0}$ as distribution (2) for $x = x_0$. The singular behaviour of (7) for $x < x_0$ may be viewed as a condensation-like phenomenon.

We give further on another derivation of the power-law distributions and associated avalanche phenomena. Suppose that probability of the "primordial" event is given by

$$\frac{dN}{N_0 dx} = a \tag{9}$$

at $x = 0$ (or $x = x_0$), and assume that it changes slightly for $x > 0$, by functiion $f(x)$. We write the number of events now as $\widetilde{N} - N$ in order to disentangle the main one from the associated ones, *i.e.*

$$\frac{d(\widetilde{N} - N)}{N_0 dx} = a - f(x) \ . \tag{10}$$

We assume function f be a positive-valued increasing function, such that $A - f(x) > 0$. Now we may write function $f(x)$ as

$$\begin{aligned} f(x) &= g(2x/D) = g[1 + \tfrac{2}{D}(x - D/2)] \simeq g(1) + g'(1)[\tfrac{2}{D}(x - D/2)] = \\ &= g(1)\{1 + \tfrac{g'(1)}{g(1)}[\tfrac{2}{D}(x - D/2)]\} \simeq g(1)[1 + \tfrac{2}{D}(x - D/2)]^{g'(1)/g(1)} = \\ &= g(1)(2x/D)^{g'(1)/g(1)} \ . \end{aligned} \tag{11}$$

This approximation is valid everywhere except for the boundaries of the range D, but it is satisfactory for large values of D. Coefficients $g(1)$ and $g'(1)$ are positive, so that we may re-write (10) as

$$\frac{d(\widetilde{N} - N)}{N_0 dx} = a - x^{\beta} \ , \tag{12}$$

where $\beta = g'(1)/g(1) > 0$ and x is correspondingly re-scaled in the *rhs* of eq. (12). Function $a - x^{\beta}$ can be approximated by $a^2/x^{\beta} - a$, such that the two functions vanish at the same point $x_c = a^{1/\beta}$(we assume $a < [2g(1)]^{\beta}$) and have the same derivative at that point. The singular behaviour for $x \to 0$ of the function $a^2/x^{\beta} - a$ corressponds to disentangling the main event from the rest of the avalanche events. By (9), we can further write down

$$\frac{d(\widetilde{N} - N)}{N_0 dx} = a - x^{\beta} = a^2/x^{\beta} - a = a^2/x^{\beta} - \frac{dN}{N_0 dx} \ , \tag{13}$$

or the power-law

$$\frac{d\widetilde{N}}{N_0 dx} = a^2/x^{\beta} \ . \tag{14}$$

Its singular behaviour for $x = 0$ indicates again the condensation phenomenon of the main event.