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On the molecular forces acting between macroscopic bodies

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ABSTRACT

The eigenfrequencies are identified for two electromagnetically coupled semi-infinite solids with plane-parallel surfaces (two half-spaces) separated by a third, polarizable body. The corresponding van der Waals–London and Casimir forces are calculated from the zero-point energy (vacuum fluctuations) of the normal modes. It is shown how the results can be extended to bodies of any shape; in particular, the force is given for a sphere interacting with a half-space. The calculations are performed using the well-known Drude–Lorentz (plasma) model of (non-magnetic) polarizable matter. The polarization degrees of freedom are explicitly introduced. It is shown that the polarization dynamical variables for the two bodies are coupled through the electromagnetic field, very similar with two infinite sets of coupled harmonic oscillators.

1. Introduction

The molecular forces acting between atoms (molecules), known as van der Waals–London and Casimir forces, have been derived originally by quantum-mechanical calculations in the non-retarded (small distance) [1–3] and, respectively, retarded (large distance) regime [4] (see also Refs. [5,6]). The force acting in the retarded regime between an atom and a semi-infinite conductor (half-space) has also been derived by quantum-mechanical calculations [4], while the retarded force acting between two conducting half-spaces (Casimir force) has been originally derived by advancing arguments related to the zero-point energy (vacuum fluctuations) of the electromagnetic field with suitable boundary conditions at the surfaces of the two half-spaces [7]. It was realized that these molecular forces are related to the internal electrical polarization of matter, and the macroscopic bodies bring their own characteristics with respect to the electrical polarization (like plasmons, polaritons, surface effects, etc), in comparison with individual quantum particles [8–14].

Molecular forces acting between macroscopic bodies, either conductors or dielectrics, have been derived by the theory of the quantum-statistical electromagnetic fluctuations [15–18] as well as within the framework of the field source theory [19,20]. Both theories consider, on one hand, the polarization as an external source, and estimate the response of the electromagnetic field to this source, and, on the other hand, include polarization (via the dielectric function) in the electromagnetic field, viewing the latter as a dynamical variable (coordinate). There was never clearly grasped which are the normal modes which fluctuate and bring the zero-point energy in the molecular forces. On the other hand, a remarkable progress is being recorded recently in a series of publications regarding the computation of molecular forces for various geometries, especially nanomechanical structures, using elegant scattering-matrix formalism, or path integral methods, or boundary-element methods [21–31]. In particular, repulsive Casimir forces have been identified for a third body acting as a medium (instead of vacuum).

We describe here the polarization by a displacement field of the mobile charges in polarizable matter and solve the coupled equations of motion of this field, interacting via the electromagnetic field, for two semi-infinite solids with plane-parallel surfaces (two half-spaces) separated by a third, polarizable body. The calculations are done using the well-known Drude–Lorentz (plasma) model of (non-magnetic) polarizable matter. We show that the polarizations of the two bodies interact with each other via their electromagnetic field, very much alike two infinite sets of coupled harmonic oscillators. The normal modes of the ensemble of the two bodies are identified and the eigenfrequencies are computed. The force is derived from the zero-point energy (vacuum fluctuations) of these normal modes. We compute the van der Waals–London and Casimir forces for two half-spaces, either conductors or dielectrics, separated, in general, by a third polarizable body. In view of the great deal of interest developed recently for the subject [32–55] we show here how to compute such forces between bodies of any shape, and give the result for the force acting between a sphere and a half-space.

Some particular results concerning the derivation of the molecular forces along the lines described above have been previously

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published [56,57]. The method used here has also been previously illustrated in Refs. [58,59].

2. Matter polarization

We adopt a generic model of matter polarization consisting of identical mobile charges q, with mass m and density n, moving in a rigid, neutralizing background of volume V. A small displacement field $\mathbf{u}(\mathbf{R},t)$ in the position $\mathbf{R}$ of these charges gives, at the time t, a local density imbalance $\delta n = -n\, div\, \mathbf{u}$ and a polarization charge density $\rho = -nq\, div\, \mathbf{u}$. We can see that $\mathbf{P} = nq\mathbf{u}$ is the polarization. Therefore, the displacement field $\mathbf{u}(\mathbf{R},t)$ is a representation for the polarization field $\mathbf{P}(\mathbf{R},t)$. The displacement field obeys the Newton law of motion

$$m\ddot{\mathbf{u}} = q(\mathbf{E}+\mathbf{E}_0) - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}}, \tag{1}$$

where $\mathbf{E}$ is the polarization electric field generated by the polarization charges (and currents), ω_c is a characteristic frequency, γ is a (small) damping factor and $\mathbf{E}_0$ is an external electric field. This is the well-known Drude–Lorentz (plasma) model of polarizable matter [60–62], which assumes a homogeneous, isotropic matter, without spatial dispersion, represented by a field of harmonic oscillators of frequency ω_c. Taking the temporal Fourier transform of Eq. (1), with $\mathbf{E}_t = \mathbf{E}+\mathbf{E}_0$ the total electric field, we get the electric susceptibility $\chi(\omega) = P/E_t$ and the dielectric function

$$\varepsilon(\omega) = 1+4\pi\chi(\omega) = \frac{\omega^2-\omega_c^2-\omega_p^2}{\omega^2-\omega_c^2+i\omega\gamma} = \frac{\omega^2-\omega_L^2}{\omega^2-\omega_T^2+i\omega\gamma}, \tag{2}$$

where $\omega_p = \sqrt{4\pi nq^2/m}$ is the plasma frequency. This is also well known as the Lydanne–Sachs–Teller dielectric function [63], with the longitudinal frequency $\omega_L = \sqrt{\omega_c^2+\omega_p^2}$ and the transverse frequency $\omega_T = \omega_c$. The latter can be taken as the main absorption frequency of the substance. The model can be generalized in multiple ways [56–59], but for our present purpose the simplified form given above is sufficient.

The displacement field $\mathbf{u}$ produces polarization charge and current densities given by

$$\rho = -div\, \mathbf{P} = -nq\, div\, \mathbf{u}, \quad \mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = nq\dot{\mathbf{u}}, \tag{3}$$

which can be used to compute the electromagnetic potentials

$$\Phi(\mathbf{R},t) = \int d\mathbf{R}' \frac{\rho(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|},$$
$$\mathbf{A}(\mathbf{R},t) = \frac{1}{c}\int d\mathbf{R}' \frac{\mathbf{j}(\mathbf{R}', t-|\mathbf{R}-\mathbf{R}'|/c)}{|\mathbf{R}-\mathbf{R}'|} \tag{4}$$

(subjected to the Lorenz gauge $div\mathbf{A}+(1/c)\partial\Phi/\partial t = 0$). These potentials give rise to the electric field $\mathbf{E}$ in Eq. (1), whence we can get the displacement $\mathbf{u}$. This way, we can compute the electromagnetic fields of a polarizable body, subjected to the action of an external electromagnetic field.

3. Half-spaces

For a half-space extending over the region $z > d$ we take the polarization as

$$\mathbf{P} = nq(\mathbf{u}, u_z)\theta(z-d), \tag{5}$$

where $\theta(z) = 0$ for $z < 0$ and $\theta(z) = 1$ for $z > 0$ is the step function. The polarization charge and current densities are given by

$$\rho = -nq\left(div\mathbf{u}+\frac{\partial u_z}{\partial z}\right)\theta(z-d) - nqu_z(d)\delta(z-d),$$
$$\mathbf{j} = nq(\dot{\mathbf{u}}, \dot{u}_z)\theta(z-d). \tag{6}$$

We use Fourier decompositions of the type

$$\mathbf{u}(\mathbf{r},z;t) = \frac{1}{2\pi}\sum_{\mathbf{k}}\int d\omega\mathbf{u}(\mathbf{k},z;\omega)\, e^{-i\omega t+i\mathbf{kr}}, \tag{7}$$

where $\mathbf{R} = (\mathbf{r},z)$, and may omit ocassionally the arguments $\mathbf{k},\omega$, writing simply $\mathbf{u}(z)$, or $\mathbf{u}$. The electromagnetic potentials given by Eqs. (4)) includes the "retarded" Coulomb potential $e^{i(\omega/c)|\mathbf{R}-\mathbf{R}'|}/|\mathbf{R}-\mathbf{R}'|$, for which we use the well-known decomposition [64]

$$\frac{e^{i\lambda|\mathbf{R}-\mathbf{R}'|}}{|\mathbf{R}-\mathbf{R}'|} = \frac{i}{2\pi}\int d\mathbf{k}\frac{1}{\kappa}e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}\, e^{i\kappa|z-z'|}, \tag{8}$$

where $\lambda = \omega/c$ and $\kappa = \sqrt{\lambda^2-k^2}$. It is more convenient to compute first the vector potential $\mathbf{A}$ and then derive the scalar potential Φ from the gauge equation $div\,\mathbf{A} - i\lambda\Phi = 0$. The calculations are straightforward and we get the Fourier transforms of the potentials

$$\Phi(\mathbf{k},z;\omega) = = \frac{2\pi}{\kappa}\int_d^\infty dz'\mathbf{ku}\, e^{i\kappa|z-z'|} - \frac{2\pi i}{\kappa}\frac{\partial}{\partial z}\int_d^\infty dz' u_z\, e^{i\kappa|z-z'|},$$
$$\mathbf{A}(\mathbf{k},z;\omega) = \frac{2\pi\lambda}{\kappa}\int_d^\infty dz'(\mathbf{u},u_z)\, e^{i\kappa|z-z'|} \tag{9}$$

(where we have left aside the factor nq; it is restored in the final formulae). In order to compute the electric field ($\mathbf{E} = i\lambda\mathbf{A} - grad\Phi$) it is convenient to refer the in-plane vectors (*i.e.*, vectors parallel with the surface of the half-space) to the vectors $\mathbf{k}$ and $\mathbf{k}_\perp = e_z\times\mathbf{k}$, where e_z is the unit vector along the z-direction; for instance, we write

$$\mathbf{u} = u_1\frac{\mathbf{k}}{k}+u_2\frac{\mathbf{k}_\perp}{k}, \tag{10}$$

and a similar representation for the electric field parallel with the surface of the half-space. We get the electric field

$$E_1 = 2\pi i\kappa\int_d^\infty dz' u_1\, e^{i\kappa|z-z'|} - \frac{2\pi k}{\kappa}\frac{\partial}{\partial z}\int_d^\infty dz' u_z\, e^{i\kappa|z-z'|},$$
$$E_2 = \frac{2\pi i\lambda^2}{\kappa}\int_d^\infty dz' u_2\, e^{i\kappa|z-z'|},$$
$$E_z = -\frac{2\pi k}{\kappa}\frac{\partial}{\partial z}\int_d^\infty dz' u_1\, e^{i\kappa|z-z'|} + \frac{2\pi i k^2}{\kappa}\int_d^\infty dz' u_z\, e^{i\kappa|z-z'|} - 4\pi u_z\theta(z-d). \tag{11}$$

We use now the equation of motion (1) (with $\gamma = 0$) for E_2 given by Eq. (11) and for the combinations $iku_1+\partial u_z/\partial z$ and $k\partial u_1/\partial z+i\kappa^2 u_z$ in the region $z > d$. We get wave equations with solutions of the form $u_{1,2} = A_{1,2}\, e^{i\kappa' z}$, where $A_{1,2}$ are constants, and $u_z = -(k/\kappa')A_1\, e^{i\kappa' z}$ (we restrict ourselves to outgoing waves, $\kappa' > 0$), where

$$\kappa'^2 = \kappa^2 - \frac{\lambda^2\omega_p^2}{\omega^2-\omega_c^2}. \tag{12}$$

The total electric field inside the half-space is given by the equation of motion (1)

$$\mathbf{E}_t = -\frac{m}{q}(\omega^2-\omega_c^2)\mathbf{u} \tag{13}$$

for $z > d$. We can see that the field propagates in the half-space with a modified wavevector κ', according to the Ewald–Oseen extinction theorem [65]. The modified wavevector κ' given by Eq. (12) can also be written as

$$\kappa'^2 = \varepsilon\frac{\omega^2}{c^2} - k^2, \tag{14}$$

where ε is the dielectric function (as given by Eq. (2)). We can check the well-known polaritonic dispersion relation $\varepsilon\omega^2 = c^2K'^2$, where $\mathbf{K}' = (\mathbf{k},\kappa')$ is the wavevector.

The amplitudes $A_{1,2}$ can be derived from the original Eq. (1) and the field Eqs. (11) (for $z>d$). We get

$$\frac{1}{2}A_1\omega_p^2\frac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}\,e^{i\kappa z}=\frac{q}{m}E_{01},$$
$$\frac{1}{2}A_2\omega_p^2\frac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}\,e^{i\kappa z}=\frac{q}{m}E_{02}. \quad (15)$$

The (polarization) electric field, both inside and outside the half-space, can be computed from Eqs. (11). We get

$$E_1=-4\pi nqA_1\frac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z}-2\pi nqA_1\frac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}\,e^{i\kappa z},\quad z>d,$$
$$E_2=-4\pi nqA_2\frac{\omega^2-\omega_c^2}{\omega_p^2}e^{i\kappa' z}-2\pi nqA_2\frac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}\,e^{i\kappa z},\quad z>d,$$
$$E_z=4\pi nqA_1\frac{k(\omega^2-\omega_c^2)}{\kappa'\omega_p^2}e^{i\kappa' z}+2\pi nqA_1\frac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}\,e^{i\kappa z},\quad z>d, \quad (16)$$

for $z>d$. The (polarization) electric field outside the half-space (in the region $z<d$) is given by

$$E_1=-2\pi nqA_1\frac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}\,e^{-i\kappa z},\quad z<d,$$
$$E_2=-2\pi nqA_2\frac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}\,e^{-i\kappa z},\quad z<d \quad (17)$$

and $E_z=(k/\kappa)E_1$ for $z<d$. We can see that it is the field reflected by the half-space ($\kappa\to-\kappa$).

The amplitudes $A_{1,2}$ can be viewed either as being determined by the external field $\mathbf{E}_0$ (and $\mathbf{H}_0$) through Eqs. (15), or as free parameters. In the latter case Eqs. (15) are not valid anymore, but the (polarization) electric fields (Eqs. (16) and (17)), as well as the associated magnetic fields hold. We can check also that all the fields are continuous at the surface $z=d$, except for E_z and E_{tz}, which exhibit a discontinuity ($E_{tz}(z=d^-)=\varepsilon E_{tz}(z=d^+)$), as expected.

For a half-space extending in the region $z<-d$ we can repeat the calculations described above. The displacement field in this case is written as $(\mathbf{v},v_z)\theta(-z-d)$. It is easy to see that we can get the results for the half space extending in the region $z<-d$ from those pertaining to the half-space extending in the region $z>d$ by changing z into $-z$. For instance, the displacement field is given by $v_{1,2}=B_{1,2}e^{-i\kappa' z}$ and $v_z=(k/\kappa')B_1e^{-i\kappa' z}$, where $B_{1,2}$ are constant amplitudes; the electric field is given by

$$E_1=-4\pi nqB_1\frac{\omega^2-\omega_c^2}{\omega_p^2}e^{-i\kappa' z}-2\pi nqB_1\frac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z},\quad z<-d,$$
$$E_2=-4\pi nqB_2\frac{\omega^2-\omega_c^2}{\omega_p^2}e^{-i\kappa' z}-2\pi nqB_2\frac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z},\quad z<-d,$$
$$E_z=-4\pi nqB_1\frac{k(\omega^2-\omega_c^2)}{\kappa'\omega_p^2}e^{-i\kappa' z}-2\pi nqB_1\frac{k(\kappa\kappa'+k^2)}{\kappa\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z},\quad z<-d \quad (18)$$

for $z<-d$ and

$$E_1=-2\pi nqB_1\frac{\kappa\kappa'-k^2}{\kappa'(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{i\kappa z},\quad z>-d,$$
$$E_2=-2\pi nqB_2\frac{\lambda^2}{\kappa(\kappa'+\kappa)}e^{i(\kappa'+\kappa)d}e^{i\kappa z},\quad z>-d \quad (19)$$

and $E_z=-(k/\kappa)E_1$ for $z>-d$; and the amplitudes $B_{1,2}$ are given by

$$\frac{1}{2}B_1\omega_p^2\frac{\kappa\kappa'+k^2}{\kappa'(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z}=\frac{q}{m}E_{01},$$
$$\frac{1}{2}B_2\omega_p^2\frac{\lambda^2}{\kappa(\kappa'-\kappa)}e^{i(\kappa'-\kappa)d}e^{-i\kappa z}=\frac{q}{m}E_{02}. \quad (20)$$

We consider now two half-spaces, one, denoted by 1, extending in the region $z>d/2$, another, denoted by 2, occupying the region $z<-d/2$. The field pertaining to these half-spaces is given here, with d replaced by $d/2$. We focus on the amplitudes Eqs. (15) and (20). The external field for the half-space 2 (Eqs. (20)) is the field given by Eq. (17) produced by half-space 1 in the region $z<d/2$; similarly, the external field for the half-space 1 (Eqs. (15)) is the field given by Eq. (19), produced by half-space 2 in the region $z>-d/2$. All the quantities pertaining to half-spaces 1,2 will get a suffix 1 or, respectively, 2. Introducing these fields in Eqs. (15) and (20) we get the dispersion equations

$$\frac{\kappa_1'-\kappa}{\kappa_1'+\kappa}\cdot\frac{\kappa_2'-\kappa}{\kappa_2'+\kappa}e^{2i\kappa d}=1,$$
$$\frac{\kappa_1'-\kappa}{\kappa_1'+\kappa}\cdot\frac{\kappa_2'-\kappa}{\kappa_2'+\kappa}\cdot\frac{\kappa\kappa_1'-k^2}{\kappa\kappa_1'+k^2}\cdot\frac{\kappa\kappa_2'-k^2}{\kappa\kappa_2'+k^2}e^{2i\kappa d}=1. \quad (21)$$

The solutions of these equations give the eigenfrequencies of the two electromagnetically coupled half-spaces. Since

$$(\kappa'\pm\kappa)(\kappa\kappa'\pm k^2)=\lambda^2(\varepsilon\kappa\pm\kappa'), \quad (22)$$

according to Eq. (14), the second dispersion Eq. (21) can also be written as

$$\frac{\kappa_1'-\varepsilon_1\kappa}{\kappa_1'+\varepsilon_1\kappa}\cdot\frac{\kappa_2'-\varepsilon_2\kappa}{\kappa_2'+\varepsilon_2\kappa}e^{2i\kappa d}=1, \quad (23)$$

where $\varepsilon_{1,2}(\omega)$ are the dielectric functions of the two half-spaces. These dispersion equations have been established in Refs. [8,10,11], using continuity conditions for the electromagnetic field at the surfaces of the two half-spaces.

4. Molecular forces

In general, the dispersion Eqs. (21) have not solutions. However, there exist particular conditions, corresponding precisely to physically interesting cases, which ensure solutions for the dispersion Eqs. (21). For instance, conductors are characterized by $\omega_c=0$ and large values of ω_p. In this case, the z-component κ' of the wavevector is purely imaginary and its magnitude acquires large values in comparison with κ (*i.e.*, λ). Purely imaginary wavevectors κ' correspond to damped surface plasmon-polariton modes in conductors (see, for instance, Refs. [56,58]), in agreement with the original Casimir's assumption concerning the boundary conditions at the surfaces of two semi-infinite metals. In this retarded regime of interaction the electromagnetic field is propagating between the two half-spaces (κ real), but it is damped along the z-direction inside the conducting half-spaces. Good dielectrics are characterized by $\omega\ll\omega_c\ll\omega_p$, so that κ' (which is real) acquires again large values. This condition is usually referred to as the condition of long wavelengths in comparison with the main (characteristic) absorption wavelength of the substance (see, for instance, Ref. [18]). It is easy to see that Eqs. (21) have solutions $\kappa d=\pi n$, n any integer, for $|\kappa_{1,2}'|\gg\kappa_{1,2},|\varepsilon_{1,2}|\kappa_{1,2}$. Solutions $\kappa d=\pi n$ can be easily understood. In the in-between region there is a field produced by the half-space 1, which goes like $\mathbf{E}^{(1)},\mathbf{H}^{(1)}\sim e^{-i\kappa z}$ and a field produced by the half-space 2, which goes like $\mathbf{E}^{(2)},\mathbf{H}^{(2)}\sim e^{i\kappa z}$. Cross-terms of the form $\mathbf{E}^{(1)*}\mathbf{E}^{(2)}$, integrated over z from $-d/2$ to $d/2$, in the energy of the electromagnetic field in this region give rise to the factor $\sin\kappa d$. The condition $\kappa d=\pi n$ ensures the vanishing of this interaction energy. There is also an interaction electromagnetic energy inside the two half-spaces (involving cross-terms), which cannot, in general, be removed, except in those cases where it is practically negligible. This condition correspond to $|\kappa_{1,2}'|\gg\kappa_{1,2},|\varepsilon_{1,2}|\kappa_{1,2}$.

The solutions $\kappa d=\pi n$ ($\kappa=\sqrt{\lambda^2-k^2}$) imply the eigenfrequencies

$$\Omega_n(k)=c\sqrt{k^2+\frac{\pi^2n^2}{d^2}}, \quad (24)$$

according to Eqs. (15) and (20); the corresponding amplitudes can be written as

$$A_{1,2,n}=2\pi a_{1,2,n}\delta(\omega-\Omega_n(k)), \tag{25}$$

where $u_{1,2,n}(\mathbf{k},z;t)=a_{1,2,n}e^{i\Omega_n(k)t}e^{i\kappa'_{1,2}z}$. We can see that $a_{1,2,n}$ are displacements, according to Eq. (7), and they correspond to the coordinates of harmonic-oscillators with frequencies $\Omega_n(k)$. According to Eq. (20) a similar representation holds for the amplitudes $B_{1,2}$ of the displacement field in the half-space 2, as well as for the associated electromagnetic fields. In effect, the coordinates of the $a_{1,2,n}$-type are the coordinates of the normal modes (labeled by $\mathbf{k}$ and n) of the two electromagnetically coupled half-spaces. The motion of the normal modes can be quantized, according to standard rules, so that the ground-state energy is given by

$$E=\sum_{n=0}^{\infty}\sum_{\mathbf{k}}\hbar\Omega_n(k)=\frac{S\hbar c}{2\pi}\sum_{n=0}\int_0 dk\cdot k\sqrt{k^2+\frac{\pi^2n^2}{d^2}}, \tag{26}$$

where S denotes the area of the surface and a factor 2 has been introduced in order to account for the two labels 1 and 2.

We estimate the change brought about by the finite distance d in the energy E using the Euler–Maclaurin formula [66]:

$$\Delta E=\sum_{m=1}\frac{(-1)^m B_m(\pi/d)^{2m-1}}{(2m)!}f^{(2m-1)}(0), \tag{27}$$

where B_m are the Bernoulli's numbers and

$$f(\kappa)=\frac{S\hbar c}{2\pi}\int dkk\sqrt{k^2+\kappa^2}, \tag{28}$$

introducing $u=k^2+\kappa^2$, Eq. (27) becomes

$$\Delta E=\frac{\hbar cS}{4\pi}\sum_{m=1}\frac{(-1)^m B_m(\pi/d)^{2m-1}}{(2m)!}\left(\int_{\kappa^2}du\sqrt{u}\right)_0^{(2m-1)}, \tag{29}$$

The only contribution to Eq. (29) comes from the third-order derivative. We get ($B_2=1/30$)

$$\Delta E=-\frac{\pi^2\hbar cS}{720}\cdot\frac{1}{d^3}, \tag{30}$$

and an attractive force

$$F=-\frac{\pi^2\hbar cS}{240}\cdot\frac{1}{d^4}, \tag{31}$$

which is the well-known Casimir force, acting between two half-spaces with parallel surfaces separated by distance d. We can see that it is the same for dielectrics and conductors (under the conditions given before), including the pair conductor-dielectric, does not depend on the nature of the two semi-infinite bodies and arises from the zero-point (vacuum) fluctuations of the motion of the charge carriers in the two polarizable bodies. We may say that it has a universal character.

The effect of the temperature $T=1/\beta$ can be incorporated in Eq. (29) by the change

$$\int_{\kappa^2}du\sqrt{u}\rightarrow\int_{\kappa^2}du\sqrt{u}\coth\left[\frac{1}{2}\beta\hbar c\sqrt{u}\right]. \tag{32}$$

For realistic values of the parameters we have $\beta\hbar c/d\gg 1$, so we get a small temperature correction factor $\simeq\coth(\beta\hbar c/d)$ in the expression of the force.

For shorter distances d, the electromagnetic field acquires the non-retarded regime corresponding to $\lambda\rightarrow 0$; it follows that $\kappa\simeq ik$, *i.e.*, the electromagnetic field is damped along the z-direction, both inside and outside the half-spaces. In this limit we have

$$\kappa'\simeq\kappa-\frac{\lambda^2\omega_p^2}{2\kappa(\omega^2-\omega_c^2)},\quad \kappa\kappa'+k^2\simeq\lambda^2\left[1-\frac{\omega_p^2}{2(\omega^2-\omega_c^2)}\right], \tag{33}$$

and $\kappa\kappa'-k^2\simeq -2k^2$. Making use of these approximations, the second Eq. (21) leads to

$$(\omega^2-\omega_{c1}^2-\tfrac{1}{2}\omega_{p1}^2)(\omega^2-\omega_{c2}^2-\tfrac{1}{2}\omega_{p2}^2)=\tfrac{1}{4}\omega_{p1}^2\omega_{p2}^2e^{-2kd}. \tag{34}$$

We solve this equation for large values of the kd, which bring the main contribution to integrals over $\mathbf{k}$. Within this approximation, the *rhs* of Eq. (34) may be treated as a small perturbation. From the zero-point energy, we get the van der Waals–London force (per unit area) for distinct bodies

$$F=-\frac{\hbar\omega_{p1}\omega_{p2}}{16\pi\sqrt{2}C_1C_2(\omega_{p1}C_1+\omega_{p2}C_2)}\cdot\frac{1}{d^3}, \tag{35}$$

where

$$C_{1,2}=\sqrt{\frac{\varepsilon_{01,2}+1}{\varepsilon_{01,2}-1}}, \tag{36}$$

$\varepsilon_{01,2}$ being the static dielectric constants (for conductors, $C_{1,2}\rightarrow 1$). For identical bodies, the force becomes

$$F=-\frac{\hbar\omega_p}{32\pi\sqrt{2}}\left(\frac{\varepsilon_0-1}{\varepsilon_0+1}\right)^{3/2}\cdot\frac{1}{d^3} \tag{37}$$

(for conductors $|\varepsilon_0|\rightarrow\infty$).

5. A third body

We assume now that a slab of thickness d and parameters ω_{p3},ω_{c3} (body 3) is inserted in the gap between the two half-spaces. All the calculations given in the previous sections are repeated for this body, which brings its own component κ'_3 of the wavevevector along the z-axis, given by

$$\kappa_3'^2=\kappa^2-\frac{\lambda^2\omega_{p3}^2}{\omega^2-\omega_{c3}^2}=\varepsilon_3\lambda^2-k^2, \tag{38}$$

ε_3 being the dielectric function of this body. The first dispersion Eq. (21) becomes now

$$\left(\frac{\kappa_1'+\kappa}{\kappa_1'-\kappa}\cdot\frac{1}{\kappa_3'+\kappa}-\frac{1}{\kappa_3'-\kappa}\right)\left(\frac{\kappa_2'+\kappa}{\kappa_2'-\kappa}\cdot\frac{1}{\kappa_3'+\kappa}-\frac{1}{\kappa_3'-\kappa}\right)e^{2i\kappa_3'd}$$
$$=\left(\frac{\kappa_1'+\kappa}{\kappa_1'-\kappa}\cdot\frac{1}{\kappa_3'-\kappa}-\frac{1}{\kappa_3'+\kappa}\right)\left(\frac{\kappa_2'+\kappa}{\kappa_2'-\kappa}\cdot\frac{1}{\kappa_3'-\kappa}-\frac{1}{\kappa_3'+\kappa}\right), \tag{39}$$

while the second dispersion Eq. (21) can be written as

$$(a_1b_--b_+)(a_2b_--b_+)e^{2i\kappa_3'd}=(a_1b_+-b_-)(a_2b_+-b_-), \tag{40}$$

where

$$a_i=\frac{\kappa\kappa_i'+k^2}{\kappa\kappa_i'-k^2}\cdot\frac{\kappa_i'+\kappa}{\kappa_i'-\kappa}=\frac{\varepsilon_i\kappa+\kappa_i'}{\varepsilon_i\kappa-\kappa_i'},\quad i=1,2, \tag{41}$$

and

$$b_\pm=\frac{\kappa\kappa_3'\pm k^2}{\kappa_3'\mp\kappa}. \tag{42}$$

We can see that the dispersion Eqs. (21) can be retrieved from Eqs. (39) and (40) by putting formally $\kappa_3'=\kappa$, as for vacuum.

For large values of $|\kappa'_{1,2}|$ (either conductors or dielectrics), Eqs. (39) and (40) have solution $\kappa_3'd=\pi n$, n integer, which implies $\varepsilon_3(\omega)\lambda^2=c^2K_3'^2$, where $\mathbf{K}_3'=(\mathbf{k},\pi n/d)$. This equation has two branches of solutions, one starting at $\sqrt{\omega_{p3}^2+\omega_{c3}^2}$ with an asymptote $\simeq cK_3'$, and another starting as vK_3' and asymptote ω_{c3}, where

$$v=c\frac{\omega_{c3}}{\sqrt{\omega_{p3}^2+\omega_{c3}^2}}=\frac{c}{\sqrt{\varepsilon_{30}}}, \tag{43}$$

ε_{30} being the (static) dielectric constant of the body 3. These are the well-known polariton branches in a polarizable body.

It follows that the Casimir force is given by the same Eq. (31) with the renormalized light velocity (polariton velocity) v, as expected. For a conducting body inserted in the gap (κ_3' purely imaginary), the force is vanishing.

In the non-retarded regime $\kappa \simeq ik$ the situation is more complicated. Eq. (40) leads to

$$[4(\omega^2-D_1)(\omega^2-D_3)-\omega_{p1}^2\omega_{p3}^2][4(\omega^2-D_2)(\omega^2-D_3)-\omega_{p2}^2\omega_{p3}^2]$$
$$=4[\omega_{p1}^2(\omega^2-D_3)-\omega_{p3}^2(\omega^2-D_1)][\omega_{p2}^2(\omega^2-D_3)-\omega_{p3}^2(\omega^2-D_2)]e^{-2kd}, \tag{44}$$

where

$$D_i=\frac{1}{2}\omega_{pi}^2\frac{\varepsilon_{0i}+1}{\varepsilon_{0i}-1}, \quad i=1,2,3. \tag{45}$$

The zero-point energy associated with the solutions of this equation leads to the van der Waals–London force. It is easy to see that for large values of D_3 (weak dielectric in-between), Eq. (44) becomes Eq. (34), which means that the effect of a weak dielectric introduced in the gap between the two half-spaces is a second-order correction. For two identical conductors 1 and 2 and a distinct conductor 3 in-between the force is given by

$$F=-\frac{\hbar}{32\pi\sqrt{2}}\frac{\omega_p^2-\omega_{p3}^2}{(\omega_p^2+\omega_{p3}^2)^{3/2}}\cdot\frac{1}{d^3}. \tag{46}$$

More complicated situations can be treated by solving Eq. (44).

6. Formulae of the theory of the electromagnetic fluctuations

We give here a formal deduction of the formulae obtained within the framework of the theory of the electromagnetic fluctuations, following Refs. [8,10,11].

Suppose that the eigenvalues $\Omega_n(\mathbf{k})$ are given by the roots of an equation written as $G(\omega,k)=0$, like one of Eqs. (21). Then, the zero-point energy can be written as

$$E=\frac{1}{2}\hbar\sum_{n\mathbf{k}}\Omega_n(k)=\frac{\hbar}{4\pi i}\sum_{n\mathbf{k}}\int d\omega\frac{\omega}{\omega-\Omega_n(k)}, \tag{47}$$

or

$$E=\frac{\hbar}{2i}\int dkk\int d\omega\omega\frac{\partial}{\partial\omega}\ln G \tag{48}$$

(per unit area), where the integration with respect to ω is performed around the positive ω-axis (we assume that function G has no poles). We pass from the variables (ω,k) to the variables (ξ,p) defined by

$$\omega=i\xi, \quad p=\sqrt{1+c^2k^2/\xi^2}\,sgn(\xi). \tag{49}$$

The jacobian of this transformation is

$$\frac{\partial(\omega,k)}{\partial(\xi,p)}=\frac{i\xi p}{c(p^2-1)^{1/2}}, \tag{50}$$

and the integration is represented as

$$\int_{-\infty}^{-1}dp\int_{-\infty}^{0}d\xi-\int_{1}^{\infty}dp\int_{0}^{\infty}d\xi. \tag{51}$$

We take for $G=0$ Eqs. (21), which, with the new variables, become

$$G_1=\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}e^{2\xi pd/c}-1=0,$$
$$G_2=\frac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)}e^{2\xi pd/c}-1=0, \tag{52}$$

where $s_i=(\varepsilon_i-1+p^2)^{1/2}$, $i=1$, 2 and κ is replaced by $\kappa=-i\xi p/c$. The derivative with respect to ω in Eq. (48) becomes

$$\frac{\partial G}{\partial\omega}=-i\frac{\partial G}{\partial\xi}+i\frac{p^2-1}{p\xi}\frac{\partial G}{\partial p}. \tag{53}$$

In order to get the force, we take the (minus) derivative with respect to d in Eq. (48) and make use of

$$\frac{\partial G}{\partial d}=\frac{2\xi p}{c}(G+1). \tag{54}$$

Combining Eqs. (53) and (54), we get easily

$$\frac{\partial}{\partial d}\left(\frac{1}{G}\frac{\partial G}{\partial\omega}\right)=\frac{2}{ic}\left(\frac{1}{p}+\frac{1}{pG}-\frac{\xi p}{G^2}\frac{\partial G}{\partial\xi}+\frac{p^2-1}{G^2}\frac{\partial G}{\partial p}\right). \tag{55}$$

An integration by parts in $F=\partial E/\partial d$ leads to the force

$$F=-\frac{\hbar}{2\pi^2c^3}\int_1^\infty dpp^2\int_0^\infty d\xi\xi^3\left(\frac{1}{G_1}+\frac{1}{G_2}\right), \tag{56}$$

which is the well-known formula given in Refs. [15–20]. The formal equivalence given here can be found entirely in Ref. [10]. For finite temperatures the integration over ξ is replaced by a summation over the integers n, such as $\beta\hbar\xi_n=2\pi n$, where $\beta=1/T$ is the reciprocal of the temperature T.

For conductors, in the retarded limit, Eq. (56) leads to the Casimir force given by Eq. (31). For poor dielectrics, or combinations of poor dielectrics with conductors, Eq. (56) brings a small correction factor to the Casimir force (see, for instance, Eq. (82.6) in Ref. [18]), which indicates, in fact, that the force is vanishing in this case. In the limit of good dielectrics, Eq. (56) leads to the same universal Casimir force given by Eq. (31).

In the non-retarded limit $\omega\to0$ ($\xi\to0$), the most important contribution to the p-integral in Eq. (56) comes from $p\gg1$, due to the presence of the exponential in the denominator. Consequently, we may take $s_{1,2}\simeq p$, which leads to

$$F\simeq-\frac{\hbar}{16\pi^2d^3}\int_0^\infty dxx^2\int_0^\infty d\xi\left[\frac{(1+\varepsilon_1)(1+\varepsilon_2)}{(1-\varepsilon_1)(1-\varepsilon_2)}e^x-1\right]^{-1}, \tag{57}$$

which is the well-known formula given in Refs. [15–20] for the van der Waals–London force. The evaluation of the ξ-integral is difficult, so we cannot compare the result with Eq. (35).

Both Eqs. (56) and (57) can be extended to very rarefied bodies, leading to well-known forces computed quantum-mechanically for two interacting atoms (molecules) [18]. In general, Eqs. (56) and (57) are valid provided equation $G(\omega,k)=0$ has solutions (*i.e.*, Eqs. (21) have solutions). Unfortunately, Eqs. (56) and (57) may also indicate false solutions (as for poor dielectrics).

7. Concluding remarks: sphere and half-space

Let us denote by $F_{1/2}=CS/d^n$ the van der Waals–London or Casimir force acting between two half-spaces separated by distance d, where C is a constant, S is the transverse area of the two half-spaces, $n=3$ for the van der Waals–London force and $n=4$ for the Casimir force. We look for a force $df=C_1/|z|^{n_1}\,dV$, acting between the half-space and a "macroscopically infinitesimal" element of volume dV placed at distance $|z|$ from the half-space, such as

$$\int df=F_{1/2}, \tag{58}$$

where the integration is performed over the other half-space. We find easily $C_1=Cn$ and $n_1=n+1$. Now we compute the force

$$F_s=\int df=Cn\int dV\frac{1}{(R+d-r\cos\theta)^{n+1}} \tag{59}$$

acting between the half-space and a sphere of radius R placed at distance d from the half-space (the distance between the half-space and the surface of the sphere); the integration in Eq. (59) is performed over the volume of the sphere. The integration in Eq. (59) is elementary, and, for $R \gg d$, we get the force

$$F_s \simeq \frac{2\pi CR}{(n-1)d^{n-1}}. \tag{60}$$

The force acting between a half-space and a spherical shell of radius R is $2\pi CR^2/d^n$. In a similar way we can derive the force acting between two bodies of any shape. The force acting between two macroscopic particles is given by

$$f = \frac{n(n+1)(n+2)C}{2\pi d^{n+4}} v_1 v_2, \tag{61}$$

where $v_{1,2}$ are the volumes of the two particles.

In conclusion we may say that the van der Waals–London and Casimir forces are calculated here explicitly for two semi-infinite solids (half-spaces) separated by a third, polarizable body inserted in the gap between the two half-spaces. In contrast with previous, well-known treatments of the problem, the polarization degrees of freedom are introduced here explicitly, and their dynamics is included, beside the Maxwell equations of the electromagnetic field. The equations off motion are solved (both for polarization and the electromagnetic field) for these electromagnetically coupled bodies, the normal modes are identified as harmonic-oscillators modes, and the corresponding eigenfrequencies are computed. The force is calculated from the zero-point energy of the vacuum fluctuations of the polarization. The extension of the results to bodies of any shape is done, and the force acting between a sphere and a half-space is calculated explicitly.

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