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Bosonisation of the one-dimensional two-fermion model: boson representation

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Abstract. Jordan's theory of the boson representation of the fermion fields in one dimension is reviewed and generalised to the one-dimensional two-fermion model (TFM). It is shown that an additional constraint (Jordan's commutator) on the fermion fields has been overlooked so far by the theory of the TFM. This supplementary condition requires a particular cut-off procedure which has been considered a little in recent times.

1. Introduction

Although the investigation of the one-dimensional problem of interacting fermions started a long time ago it was only recently that contact was made between theory and experiment with the attempts for understanding the unusual properties of quasi-one-dimensional materials. This aroused a great deal of interest in the many-fermion system in one dimension. This paper will deal with the one-dimensional two-fermion model (TFM) proposed many years ago by Luttinger (1963) and generalised by Luther and Emery (1974) to include the backscattering interaction and by Emery *et al* (1976) to include the Umklapp scattering. There is a close analogy between this model and the one-dimensional Fermi gas model (FGM) whose characteristic features are briefly recalled further below.

The one-dimensional FGM consists of weakly interacting spin-half fermions with wavevector p ranging (in the ground state) from $-k_F$ to $+k_F$, k_F being the Fermi momentum. As the low excited states can be built up by superposing particle–hole pairs in the neighbourhood of the $\pm k_F$ points a bandwidth cut-off k_0 is introduced, much smaller than k_F, which restricts the single-particle states participating in the dynamics of the system within the range $2k_0$ around $\pm k_F$, $\pm k_F - k_0 < p < \pm k_F + k_0$. A linear expression is used for the energy of these states, $\varepsilon_p = \mu + v_F(|p| - k_F)$, where μ is the Fermi level and v_F is the Fermi velocity, thus obtaining two linear branches of the fermion spectrum as p lies near $+k_F$ or $-k_F$. The dynamics of the low excited states is governed by two interaction processes. The first one is the forward scattering process that involves a small momentum transfer. This process exites a particle–hole pair in the neighbourhood of $\pm k_F$. The second one is the backward scattering process, with momentum transfer near $\pm 2k_F$, that excites a particle–hole pair across the Fermi sea. The excitation energies associated with these processes are very small and, consequently, both processes play an essential role in the physics of the system. If there is an underlying lattice periodicity and the band is half filled there is one more process whose importance can not be neglected. This is the Umklapp scattering that excites two particle–hole pairs across the Fermi sea. The momentum transfer in this process is near $\pm 2k_F$ and the momentum conservation is ensured by the reciprocal lattice vector $G = 4k_F$. The FGM is further specified by allowing for a momentum transfer cut-off k_D which differs from k_0. This cut-off is imposed on the processes with momentum transfer near $\pm 2k_F$ which may be interpreted as coming from phonon-mediated effective interaction. Thus the momentum transfer cut-off is reminiscent of the Debye cut-off.

The FGM as formulated before is not a model which is exactly soluble. Various attempts have been made to obtain approximate solutions. The model with backscattering and bandwidth cut-off has firstly been treated (Bychkov *et al* 1966, Dzyaloshinsky and Larkin 1971) by summing the most divergent diagrams (parquet approximation), thus leading to a typical problem with logarithmic singularities. This approach predicts a phase transition which can not be accepted in one dimension. The lowest-order logarithmic corrections have been taken into account by using the skeleton graph technique (Ohmi *et al* 1976) and the renormalisation group approach (Menyhárd and Sólyom 1973, 1975, Sólyom 1973, Fukuyama *et al* 1974, Kimura 1973). Beyond the parquet approximation it was found that all the singularities of the vertex and response functions are shifted to zero frequency and temperature. The momentum transfer cut-off was introduced by Chui *et al* (1974) and the renormalisation group technique was applied to this model (Grest *et al* 1976, Sólyom and Szabó 1977) as well as to the model with Umklapp scattering (Sólyom 1975, Kimura 1975). All this work was recently reviewed by Sólyom (1979). The spectrum of the particle-density excitations was also investigated (Apostol 1981, Apostol *et al* 1981) in the model with backscattering in the limit of weak coupling strengths, when the Fermi sea is not too strongly distorted by interaction.

Unlike the FGM with backscattering and Umklapp scattering the model with forward scattering only is an exactly soluble model. Many years ago Tomonaga (1950) showed that those parts of the Fourier components of the particle-density operator which correspond to each of the two branches of the fermion spectrum satisfy boson-like commutation relations in the weak coupling limit. A model Hamiltonian can be derived to describe the collective excitations of the particle density. This Hamiltonian is expressed as a bilinear form of the two types of boson operators and can be straightforwardly diagonalised (Tomonaga model). The FGM with forward scattering was further developed by Dzyaloshinsky and Larkin (1973) in a very interesting way. They assumed that the two linear branches of the fermion spectrum may be interpreted as being approximately described by two independent fields of fermions with a linear spectrum of the form $\mu \pm v_F(p \mp k_F)$. Here p is confined to the whole energy band which is of the order of k_F. In order to obtain physical results for the correlation functions and momentum distribution of the fermions near $\pm k_F$ a momentum transfer cut-off is required. Both these quantities and the structure of the excitation spectrum were derived by means of the Ward identity (Dzyaloshinsky and Larkin 1973, Everts and Schulz 1974, see also Sólyom 1979, Apostol and Barsan 1981) and a version of the functional integral method (Fogedby 1976, Klemm and Larkin 1979). It is known that these methods are equivalent to a direct diagram summation.

The first precise statement of the one-dimensional TFM was made by Luttinger (1963). The Luttinger model consists of two types of fermions whose energy levels are $\pm v_F p$. The non-interacting ground state is filled from $-\infty$ to $+k_F$ with fermions of the first type and from $-k_F$ to $+\infty$ with fermions of the second type. It is argued that this extension of the allowable fermion states does not modify the physical results—at least in the weak coupling case—as the newly introduced states are far away from the Fermi points. Mattis and Lieb (1965) showed that this infinite filling of the Fermi sea causes the Fourier components of the particle-density operator to satisfy rigorously boson-like commutation relations. The kinetic part of the Hamiltonian was shown to be equivalent to a model Hamiltonian which contains only boson operators. The model with forward scattering interaction (expressed as a bilinear form in boson operators) can be easily treated by means of the canonical transformation method and the results turn out to be those of the Tomonaga model. This is why both these models will be hereafter referred to as the Tomonaga–Luttinger model (TLM). However it is worth remarking that there is a difference between these models: whereas in the Tomonaga model the forward scattering process excites a particle–hole pair near $\pm k_F$ in the Luttinger model this excited pair may be placed everywhere. By using the boson algebra the momentum distribution of fermions (Mattis and Lieb 1965, Gutfreund and Schick 1968) and the one-particle Green function (Theumann 1967, 1976) were calculated in the TLM. A momentum transfer cut-off was required in such calculations to obtain finite results. The TLM was recently reviewed by Bohr (1981). An interesting development of this model was attempted by Haldane (1980, 1981a, b) who added non-linear terms to the fermion dispersion relation. The concept of the 'Luttinger liquid' was introduced by this author and argued to apply to a wide class of one-dimensional systems.

The boson algebra of the Fourier components of the particle-density operator was fully exploited when Luther and Peschel (1974) and Mattis (1974) introduced a boson representation for the fermion field operators. This representation was used to treat the model with backscattering (Luther and Emery 1974, Lee 1975, Gutfreund and Klemm 1976) and Umklapp scattering (Emery *et al* 1976). It was shown that for particular values of the coupling constants both these models are exactly soluble. A gap is opened in the spin- and charge-density wave spectrum, respectively, which has an important effect on the infrared behaviour of the correlation functions. It is worth mentioning here that, despite the formal resemblance of the backscattering and Umklapp scattering terms in the Hamiltonian of the TFM to the corresponding terms in the FGM, there are some important differences between these models (Grest 1976, Haldane 1979, Grinstein *et al* 1979). First, an ambiguity reveals itself when one attempts to assign a momentum transfer to these processes in the TFM. Secondly, whereas the momentum transfer involved by these processes in the FGM is near $\pm 2k_F$ there is no such restriction for the momentum transfer, whatever it is, in the TFM.

Although the boson representation of the fermion field operators proved to be of great use in treating the one-dimensional TFM there are nevertheless some difficulties in dealing with it. All these difficulties are related to the cut-off parameter α introduced by Luther and Peschel (1974). The boson representation given by Luther and Peschel (1974) is not normal-ordered in boson operators. When normal ordering is attempted factors appear which contain divergent summations over an infinite range of wavevectors. Luther and Peschel (1974) introduced a cut-off parameter α in their boson representation in such a way as to ensure the convergence of these summations in a simple way. The boson representation is shown to be exact only in the limit $\alpha \to 0$. However this cut-off procedure leads to some inconsistencies which will be successively sketched here (Sólyom 1979). The one-particle Green's function and response functions of the TLM can be calculated by using the boson representation of the fermion field operators and the bosonised Hamiltonian. When compared with the same quantities calculated by the usual direct diagram summation one can see that the two cut-offs (bandwidth and momentum transfer) appearing in these latter expressions are both replaced by the cut-off α^{-1}. Thus α^{-1} can be interpreted neither as a bandwidth cut-off nor as a momentum transfer cut-off, but appears in place of both of them. This suggests that the cut-off parameter α is too strong, as it leaves no room for the dissociation of the bandwidth cut-off from the momentum transfer cut-off. Another type of difficulty arises when the backscattering and Umklapp scattering are introduced. As is well known these models are exactly soluble and have a gap in the excitation spectrum of the spin- and charge-density degrees of freedom for particular values of the coupling constants. This gap is proportional to α^{-1} and when α is allowed to go to zero the gap becomes infinite, a physically meaningless result. Instead of making α equal to zero Luther and Emery (1974) kept it finite and interpreted α^{-1} as a bandwidth cut-off. However, Theumann (1977) still showed that, in order to preserve the anticommutation relations of the fermion fields under the canonical transformation on the boson operators that diagonalises the Hamiltonian of the TLM a momentum transfer cut-off r^{-1} is needed which must be kept finite while α goes to zero. The momentum transfer cut-off r^{-1} proves to be essential to the preservation of sum rules for the spectral density (Theumann 1976) and, in fact, the cut-off parameter r was used earlier by Luther and Peschel (1974) for deriving the correlation functions of the TLM by means of the bosonisation technique. However, it was pointed out by Theumann (1977) that the backscattering Hamiltonian (as well as the Umklapp scattering one) can be diagonalised by using Luther and Peschel's boson representation only if the limiting process is inverted, that is by letting $r \to 0$ while keeping α finite. Grest (1976) calculated perturbationally the first-order contributions to the charge-density response function of the TFM with backscattering by using Luther and Peschel's boson representation. He found that the expression of this function does not coincide with that corresponding to the FGM (calculated both with bandwidth cut-off and with bandwidth and momentum transfer cut-offs). The discrepancy relates to the cut-off parameter α which does not apply in the same way to the contributions that differ only by their spin indices ($g_{1\parallel}$ and $g_{1\perp}$). As Grest (1976) correctly pointed out this discrepancy arises from the nature of the parameter α, as it is used by Luther and Peschel (1974), which is not a true bandwidth cut-off parameter but merely a parameter introduced *ad hoc* in order to remove divergences.

Recently Haldane (1979, 1981a) showed that a major failing of the previous boson representations (Luther and Peschel 1974, Mattis 1974) is the zero-mode terms associated with the particle-number operators. He consistently took into account these terms and obtained the complete form of the boson representation. This boson representation looks very much the same as that encountered in the field-theoretical literature (see, for example, Heidenreich *et al* 1975) and, in fact, it was derived a long time ago by Jordan (1935, 1936a, b, 1937) for a single field of fermions with energy levels $+p$ in his attempt to construct a neutrinic theory of light. The boson representation given by Haldane (1979, 1981a) is normal ordered in boson operators so that there is no need for the cut-off parameter α in this expression. However, products of two or more field operators are to be calculated and the normal-ordering problem arises again. In order to make the summations over wavevectors appearing in problems of this type finite Haldane (1979, 1981a) pointed out a cut-off procedure which is essentially the same as that given by Luther and Peschel (1974) although the parameter α has a different interpretation. The boson representation and the cut-off procedure given by Haldane (1979, 1981a) remove all the aforementioned inconsistencies of the TFM (although an explicit proof of this fact has not been produced until now).

However there is a quantity pointed out by Jordan (1935, 1936a, b, 1937) (and hereafter referred to as Jordan's commutator) which has been overlooked so far by all these boson representations (Haldane's included). Owing to the fact that the Fermi sea of the TFM has an infinite number of particles some operators may have infinite values when acting upon the states of the system. Jordan redefined these operators in such a way as they be finite and the infinite c numbers which result were controlled by the cut-off parameter α. As a result the commutator of the Hermitian conjugate fields at the same space point must satisfy a certain relationship. This Jordan's commutator plays the part of an additional condition which must be satisfied by the boson representation. Its importance is directly connected to the renormalisation of the infinitely large density of particles. The fulfilment of Jordan's commutator ensures the complete equivalence of the boson representation to the original formulation of the TFM in terms of the fermion field operators. The cut-off procedure given by Luther and Peschel (1974) does not make the bosonised fermion fields satisfy Jordan's commutator chiefly because this boson representation omits the zero-mode contribution. A slight modification† of Haldane's cut-off prescriptions (Haldane 1981a, 1982) ensures the correct reproduction of Jordan's commutator. This modification is implicitly contained in Haldane (1981a) and, in fact, it had been suggested by Jordan (1936a, b, 1937) (also see Apostol 1982). Moreover, Jordan's theory provides the precise relationship between the kinetic Hamiltonian of the TFM expressed with the fermion operators and its bosonised form. The bosonised form of this Hamiltonian contains zero-mode terms associated with the particle-number operators. These zero-mode contributions are ineffective in the TLM (although they play an important role in the extension of the TLM to include particle-number and current excitations, as Haldane (1981a) recently pointed out) but would presumably lead to non-trivial results in the TFM with backscattering and Umklapp scattering.

It is worth emphasising here the nature of the cut-off parameter α introduced by Jordan and its connection with the cut-off parameter r of the momentum transfer and the cut-off parameters proposed by Luther and Peschel (1974) and Haldane (1981a). In order to obtain a consistent boson representation of the fermion fields in one dimension in a continuum model approach it is necessary not only to regularise the divergences which occur but also to reproduce all the properties of the original fermion fields correctly. Among these properties there is Jordan's commutator of the Hermitean conjugate fermion fields at the same point which is directly related to the products $\psi^+(x)\,\psi(x)$ and $\psi(x)\,\psi^+(x)$, $\psi(x)$ being the fermion field and $\psi^+(x)$ its Hermitian conjugate. Jordan (1936a, b, 1937) pointed out that these products are infinitely large and regularised them by using a particular cut-off procedure of introducing the cut-off parameter α. Regularised in this way these expressions can be expanded in powers of α, the leading contribution being of order α^{-1}. It follows that α^{-1} is a bandwidth cut-off and that the renormalised expressions of these quantities are given by the next-to-leading contributions in α, which do not depend on α. These 'physical' contributions reveal themselves in the one-particle Green function and their correct evaluation is essential for deriving the bosonised form of the kinetic Hamiltonian. When Jordan's commutator is computed the leading contributions to $\psi^+(x)\,\psi(x)$ and $\psi(x)\,\psi^+(x)$ cancel one another and only the relevant contributions appear in this commutator. This is why this commutator plays a particular role in the regularisation of the boson representation. It is noteworthy that the cut-off parameter α is altogether distinct from the cut-off parameter

† The author is deeply indebted to one of the referees for useful comments on this point.

r appearing in the coupling constants of the TLM: this latter cut-off parameter belongs entirely to the interaction and corresponds to a momentum transfer cut-off. While the cut-off parameter r is an ingredient of the interaction and has to be kept finite in calculations the parameter α is essential to the regularisation scheme of the non-interacting model and the physical results should not depend on it in the limit $\alpha \rightarrow 0$. All these features of the regularisation scheme are to be correctly reproduced by the boson representation. Luther and Peschel (1974) introduced a cut-off parameter in their non-normal-ordered boson representation with the aim of ensuring the convergence of infinitely large factors. This cut-off procedure correctly reproduces the leading contributions to $\psi^+(x)\,\psi(x)$ and $\psi(x)\,\psi^+(x)$ but it fails in reproducing Jordan's commutator. The presence of the cut-off parameter in this boson representation causes all the aforementioned inconsistencies of the TFM. Great progress was achieved by Haldane (1979, 1981a) who took into account the zero-mode terms and constructed a normal-ordered boson representation. Due to the normal ordering there is no need of the cut-off parameter in the boson representation. This boson representation can be used together with Haldane's cut-off prescriptions (1981a) to remove all the inconsistencies reported for the TFM (although the explicit calculations of this kind have not been performed until now). Moreover, a slight modification of Haldane's cut-off procedure (Haldane 1982) correctly reproduces Jordan's commutator as well. Actually, this modified cut-off procedure has implicitly been used by Haldane (1981a) in deriving the bosonised form of the kinetic Hamiltonian and, in fact, it is but another version of Jordan's regularisation scheme (Apostol 1982).

The aim of this paper is to generalise Jordan's theory to the TFM (which is described by four fermion operators, spin included) and to introduce Jordan's regularisation scheme. The paper is organised as follows. In § 2 Jordan's theory is reviewed for a single fermion field in one dimension with linear energy levels $+p$. Here the complete equivalence between the boson representation and the original formulation of the problem in terms of the fermion operators is emphasised. Jordan's theory is generalised to the TFM in § 3 and conclusions are given in § 4. Four objects are introduced in Appendix 1 in such a way as to ensure the anticommutation relations of the four different field operators. Appendix 2 is devoted to the interplay between the cut-off α^{-1} and the momentum transfer cut-off r^{-1} (Theumann 1977). It is shown that the anticommutation relations of the fermion operators and Jordan's commutator are invariant under the canonical transformation on the boson operators that diagonalises the Hamiltonian of the TLM. The effect of Jordan's cut-off procedure on the TFM as well as the effect of the zero-mode contributions on the backscattering and Umklapp scattering problems are discussed in Apostol (1983). The relationship between the TFM and the FGM are discussed there.

2. Jordan's theory

Let α_{jq}, $j = 1, 2$, $q = 2\pi L^{-1}(n + \frac{1}{2})$, n integer, be the destruction operators of two types of fermions with the properties

$$\alpha_{jq} = \alpha^{+}_{j-q} \qquad \{\alpha_{jq}, \alpha_{j'q'}\} = \delta_{jj'}\,\delta_{q-q'} \tag{2.1}$$

L being the length of the box the system is confined to. Under such circumstances Jordan (1935) proved that the operator

$$b_k = \mathrm{i}\sum_q \alpha_{1q}\,\alpha_{2k-q} \qquad b_k = b^{+}_{-k} \tag{2.2}$$

where $k = 2\pi L^{-1}n$, n integer, satisfies boson-like commutation relations:

$$[b_k, b^{+}_{k'}] = (2\pi)^{-1}Lk\,\delta_{kk'}. \tag{2.3}$$

The proof is as follows. Let us firstly suppose that $k, k' \geqslant 0$. The operators b_k and $b^{+}_{k'}$ may be written as

$$b_k = \mathrm{i}\sum_{q>0} \alpha^{+}_{1q}\,\alpha_{2k+q} + \mathrm{i}\sum_{0<q<k} \alpha_{1q}\,\alpha_{2k-q} + \mathrm{i}\sum_{q>k} \alpha_{1q}\,\alpha^{+}_{2q-k}$$

$$b^{+}_{k'} = -\mathrm{i}\sum_{q>0} \alpha^{+}_{2k'+q}\,\alpha_{1q} - \mathrm{i}\sum_{0<q<k'} \alpha^{+}_{2k'-q}\,\alpha^{+}_{1q} - \mathrm{i}\sum_{q>k'} \alpha_{2q-k'}\,\alpha^{+}_{1q}.$$

For $k \geqslant k' \geqslant 0$ we have

$$[b_k, b^{+}_{k'}] = \sum_{j,q>0} \alpha^{+}_{jq}\,\alpha_{jq+k-k'} - \sum_{j,0<q<k'} \alpha^{+}_{jq}\,\alpha_{jq+k-k'} - \sum_{j,q>k'} \alpha^{+}_{jq}\,\alpha_{jq+k-k'}$$
$$+ \sum_{j,0<q<k-k'} \alpha_{jq}\,\alpha_{jk-k'-q} + \sum_{0<q<k} \delta_{kk'} = (2\pi)^{-1}Lk\,\delta_{kk'}$$

since we noticed that

$$\sum_{0<q<k-k'} \alpha_{jq}\,\alpha_{jk-k'-q} = \sum_{0<q<k-k'} \alpha_{jk-k'-q}\,\alpha_{jq} = -\sum_{0<q<k-k'} \alpha_{jq}\,\alpha_{jk-k'-q} = 0.$$

Similarly we have for $k' \geqslant k \geqslant 0$

$$[b_k, b^{+}_{k'}] = \sum_{j,q>k'-k} \alpha^{+}_{jq}\,\alpha_{jq+k-k'} - \sum_{j,k'-k<q<k'} \alpha^{+}_{jq}\,\alpha_{jq+k-k'} - \sum_{j,q>k'} \alpha^{+}_{jq}\,\alpha_{jq-k-k'}$$
$$+ \sum_{j,0<q<k'-k} \alpha^{+}_{jq}\,\alpha^{+}_{jk'-k-q} + \sum_{0<q<k} \delta_{kk'} = (2\pi)^{-1}Lk\,\delta_{kk'}.$$

For $k, k' \leqslant 0$ it follows immediately

$$[b_k, b^{+}_{k'}] = [b^{+}_{-k}, b_{-k'}] = (2\pi)^{-1}Lk\,\delta_{kk'}.$$

For completing the proof we have still to consider $k \geqslant 0$, $k' \leqslant 0$. In this case we have $[b_k, b^{+}_{k'}] = [b_k, b_{-k'}]$ and for $k, k' \geqslant 0$ we obtain

$$[b_k, b_{k'}] = \sum_q \alpha_{1q}\,\alpha_{1k+k'-q} - \sum_q \alpha_{2q}\,\alpha_{2k+k'-q} = \sum_q \alpha_{1k+k'-q}\,\alpha_{1q}$$
$$- \sum_q \alpha_{2k+k'-q}\,\alpha_{2q} = -\sum_q \alpha_{1q}\,\alpha_{1k+k'-q} + \sum_q \alpha_{2q}\,\alpha_{2k-k'-q} = 0.$$

Let

$$\psi(x) = L^{-1/2}\sum_p a_p \exp(\mathrm{i}px)$$

be the fermion field operator whose Fourier components a_p obey the anticommutation relations

$$\{a_p, a_{p'}\} = 0 \qquad \{a^{+}_p, a_{p'}\} = \delta_{pp'} \tag{2.4}$$

the wavevector p being given by $p = 2\pi L^{-1}n$, n integer. We define the operators α_{jq} by the following relations:

$$\begin{aligned}
\alpha_{1q} &= 2^{-1/2}(a_{q-\pi L^{-1}} + a^{+}_{-q-\pi L^{-1}}) \qquad & a_p &= 2^{-1/2}(\alpha_{1p+\pi L^{-1}} + \mathrm{i}\alpha_{2p+\pi L^{-1}}) \\
\alpha_{2q} &= \mathrm{i}2^{-1/2}(a^{+}_{-q-\pi L^{-1}} - a_{q-\pi L^{-1}}) \qquad & a^{+}_p &= 2^{-1/2}(\alpha_{1-p-\pi L^{-1}} - i\alpha_{2-p-\pi L^{-1}})
\end{aligned} \tag{2.5}$$

where $q = \pm(p + \pi L^{-1}) = 2\pi L^{-1}(n + \frac{1}{2})$, n integer. One can easily see that the operators α_{jq} fulfil the conditions (2.1). Let us introduce the Fourier components $\rho(-k)$ of the particle-density operator

$$\rho(-k) = \sum_p a_p^+ a_{p+k} \qquad \rho^+(-k) = \sum_p a_p^+ a_{p-k} = \rho(k) \qquad k > 0. \tag{2.6}$$

With the aid of (2.5) we get

$$\rho(-k) = \sum_p a_p^+ a_{p+k} = \mathrm{i} \sum_q \alpha_{1q} \alpha_{2k-q} = b_k \tag{2.7}$$

where we again used the property

$$\sum_q \alpha_{jq} \alpha_{jk-q} = -\sum_q \alpha_{jq} \alpha_{jk-q} = 0$$

for $k > 0$. It follows from (2.3) and (2.7) that

$$[\rho(-k), \rho^+(-k')] = (2\pi)^{-1} L k \delta_{kk'} \qquad [\rho(-k), \rho(-k')] = 0 \qquad k, k' > 0. \tag{2.8}$$

These are the well-known boson-like commutation relations of the Fourier components of the fermion-density operator in one dimension. Tomonaga (1950) derived these relations within the approximation of weak coupling strengths (when the Fermi sea is not too strongly distorted by interaction) and Mattis and Lieb (1965) used a 'unitarily inequivalent' particle–hole representation to obtain them.

We pass now to Jordan's boson representation. Let us assume that the field operator $\psi(x)$ corresponds to a one-dimensional many-fermion system with cylic boundary conditions on the box of length L, $-L/2 < x \leqslant L/2$. Throughout this paper the calculations are performed under the assumption $L \to \infty$ so that the sum Σ_p may be replaced by $(2\pi)^{-1} L \int \mathrm{d}p$. The single-particle energy levels are $v_F p$, v_F being the Fermi velocity and $p = 2\pi L^{-1} n$, n integer, the wavevector. This system is governed by the kinetic Hamiltonian

$$H_0 = v_F \sum_{p>0} p a_p^+ a_p - v_F \sum_{p \leqslant 0} p a_p a_p^+ = v_F \sum_{p>0} p a_p^+ a_p + v_F \sum_{p \leqslant 0} p(a_p^+ a_p - 1) \tag{2.9}$$

where $a_p\,(a_p^+)$ is the destruction (creation) operator of the single-particle state labelled by the wavevector p. These operators obey the anticommutation relations given by (2.4). The ground state $|0\rangle$ is filled with particles from $-\infty$ to k_F, k_F being the Fermi momentum, so that the ground-state energy is $E_0 = (4\pi)^{-1} L v_F k_F^2$. Instead of working with the particle-number operator $\Sigma_p a_p^+ a_p$ which has an infinite value when acting upon the states of the system the 'charge' operator is used

$$B = \sum_{p>0} a_p^+ a_p - \sum_{p \leqslant 0} a_p a_p^+ = \sum_{p>0} a_p^+ a_p + \sum_{p \leqslant 0} (a_p^+ a_p - 1) \tag{2.10}$$

which counts the particles with $p > 0$ minus the holes with $p \leqslant 0$. When applied to the ground state this operator yields $B|0\rangle = (Lk_F/2\pi)|0\rangle$. Let us also introduce the quantities

$$V(x) = -\mathrm{i}2\pi L^{-1} \sum_{k>0} k^{-1} \exp(\mathrm{i}kx)\,\rho(-k)$$

$$F(x) = \frac{\partial V(x)}{\partial x} = 2\pi L^{-1} \sum_{k>0} \exp(\mathrm{i}kx)\,\rho(-k) \tag{2.11}$$

where $\rho(-k)$ is defined by (2.6). With these definitions the particle-density operator can easily be expressed as

$$\psi^+(x)\,\psi(x) = L^{-1}\sum_{p,k}\exp(\mathrm{i}kx)\,a_p^+ a_{p+k}$$
$$= L^{-1}\sum_{p\leqslant 0} 1 + L^{-1}B + (2\pi)^{-1}[F(x) + F^+(x)]. \tag{2.12}$$

In order to control the divergent sum in (2.12) Jordan introduced the cut-off parameter $\alpha > 0$ by

$$\psi^+(x)\,\psi(y) = \lim_{\alpha\to 0}\,[\psi(x - \mathrm{i}\alpha/2)]^+\,\psi(y - \mathrm{i}\alpha/2) \tag{2.13}$$

and found

$$[\psi(x-\mathrm{i}\alpha/2)]^+\,\psi(x-\mathrm{i}\alpha/2) = L^{-1}\sum_{p>0}\exp(p\alpha)\,a_p^+ a_p - L^{-1}\sum_{p\leqslant 0}\exp(p\alpha)\,(a_p a_p^+ - 1)$$
$$+ L^{-1}\sum_{p,k>0}\exp[(p + k/2)\,\alpha]\,\exp(\mathrm{i}kx)\,a_p^+ a_{p+k}$$
$$+ L^{-1}\sum_{p,k>0}\exp[(p + k/2)\,\alpha]\,\exp(-\mathrm{i}kx)\,a_{p+k}^+ a_p \tag{2.14}$$

which for small α can be written as

$$[\psi(x-\mathrm{i}\alpha/2)]^+\,\psi(x-\mathrm{i}\alpha/2)$$
$$= (1/2\pi\alpha) + L^{-1}B + (2\pi)^{-1}[F(x) + F^+(x)] + \mathrm{O}(\alpha). \tag{2.15}$$

Similarly we define

$$\psi(x)\,\psi^+(y) = \lim_{\alpha\to 0}\,\psi(x + \mathrm{i}\alpha/2)[\psi(y + \mathrm{i}\alpha/2)]^+ \tag{2.16}$$

and have

$$\psi(x + \mathrm{i}\alpha/2)[\psi(x + \mathrm{i}\alpha/2)]^+$$
$$= (1/2\pi\alpha) - L^{-1}B - (2\pi)^{-1}[F(x) + F^+(x)] + \mathrm{O}(\alpha) \tag{2.17}$$

so that

$$[\psi^+(x), \psi(x)] = \lim_{\alpha\to 0}\{[\psi(x-\mathrm{i}\alpha/2)]^+\,\psi(x-\mathrm{i}\alpha/2) - \psi(x+\mathrm{i}\alpha/2)[\psi(x+\mathrm{i}\alpha/2)]^+\}$$
$$= 2L^{-1}B + \pi^{-1}[F(x) + F^+(x)]. \tag{2.18}$$

One can see that the regularised expressions (2.15) and (2.17) of the infinitely large products $\psi^+(x)\,\psi(x)$ and $\psi(x)\,\psi^+(x)$ may be expanded in powers of the cut-off parameter α, the leading contribution being $(2\pi\alpha)^{-1}$. In the limit $\alpha\to 0$ the relevant contributions are the next-to-leading ones which are independent of α. These renormalised quantities are the only ones which contribute to Jordan's commutator (2.18), whose expression is thus independent of α. This commutator was pointed out by Jordan (1936a, b, 1937) and it has been overlooked so far by the theory of the TFM. Jordan's commutator (2.18) represents an additional condition which must be satisfied by the boson representation of the fermion field. Let us note a useful relation which can be derived from (2.14) and (2.15):

$$L^{-1}\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^+\,\psi(x-\mathrm{i}\alpha/2)$$
$$= L^{-1}\sum_{p>0}\exp(p\alpha)\,a_p^+ a_p - L^{-1}\sum_{p\leqslant 0}\exp(p\alpha)\,(a_p a_p^+ - 1)$$
$$= \frac{1}{2\pi\alpha} + L^{-1}B + \mathrm{O}(\alpha). \tag{2.19}$$

Using the anticommutator $\{\psi^+(x), \psi(y)\} = \delta(x-y)$ and (2.15), (2.18) we remark that $(\pi\alpha)^{-1}$ stands for $\delta(0)$.

One can easily verify that the conditions

$$[\psi(x), \rho(-k)] = \exp(-\mathrm{i}kx)\,\psi(x)$$
$$[\psi(x), \rho^+(-k)] = \exp(\mathrm{i}kx)\,\psi(x) \qquad [\psi(x), B] = \psi(x) \tag{2.20}$$

are satisfied if $\psi(x)$ is of the form

$$\psi(x) = \chi(x)\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x)) \tag{2.21}$$

where $\chi(x)$ should be chosen in such a way as

$$[\chi(x), \rho(-k)] = [\chi(x), \rho^+(-k)] = 0 \qquad [\chi(x), B] = \chi(x). \tag{2.22}$$

We have used here the fact that B commutes with $\rho(-k)$ and $\rho^+(-k)$. Let us introduce the unitary operator S which is defined by

$$Sa_pS^{-1} = a_{p+2\pi L^{-1}} \qquad S\psi(x)S^{-1} = \exp(-\mathrm{i}2\pi L^{-1}x)\,\psi(x)$$
$$Sa_p^+S^{-1} = a^+_{p-2\pi L^{-1}} \qquad S\psi^+(x)S^{-1} = \exp(\mathrm{i}2\pi L^{-1}x)\,\psi^+(x). \tag{2.23}$$

One can easily see that

$$[S, \rho(-k)] = [S, \rho^+(-k)] = 0,$$
$$SBS^{-1} = \sum_{p>2\pi L^{-1}} a_p^+ a_p - \sum_{p\leqslant 2\pi L^{-1}} a_p a_p^+ = B - 1 \tag{2.24}$$

that is

$$[S, B] = -S \qquad [S^{-1}, B] = S^{-1}. \tag{2.25}$$

We have similarly†

$$SH_0S^{-1} = v_{\mathrm{F}} \sum_{p>2\pi L^{-1}} p a_p^+ a_p + v_{\mathrm{F}} \sum_{p\leqslant 2\pi L^{-1}} p(a_p^+ a_p - 1)$$
$$- 2\pi L^{-1} v_{\mathrm{F}}\Big(\sum_{p>2\pi L^{-1}} a_p^+ a_p + \sum_{p\leqslant 2\pi L^{-1}} (a_p^+ a_p - 1)\Big)$$
$$= H_0 - v_{\mathrm{F}} \sum_{0<p\leqslant 2\pi L^{-1}} p - 2\pi L^{-1} v_{\mathrm{F}}(B-1) = H_0 - 2\pi L^{-1} v_{\mathrm{F}}(B - \tfrac{1}{2})$$

or

$$[S, H_0] = -2\pi L^{-1} v_{\mathrm{F}}(B - \tfrac{1}{2})S = -2\pi L^{-1} v_{\mathrm{F}} S(B + \tfrac{1}{2}). \tag{2.26}$$

Looking at (2.22) and (2.25) we find that $\chi(x)$ must be of the form

$$\chi(x) = S^{-1}\chi_0(B, x) \tag{2.27}$$

† Strictly speaking we may not replace the sum $\Sigma^p_{0<p\leqslant 2\pi L^{-1}}$ by the integral $L(2\pi)^{-1}\int_0^{2\pi L^{-1}} p\,\mathrm{d}p = \pi L^{-1}$ as we have done in deriving (2.26). However this apparent inaccuracy leads to the correct result which can be obtained rigorously as follows. Let us introduce the set S_n of unitary operators defined by $S_n a_p S_n^{-1} = a_{p+2\pi L^{-1}\alpha_n}$ etc, $\alpha_n = n$, n integer. We have $S_n\psi(x)S_n^{-1} = \exp(-\mathrm{i}2\pi L^{-1}\alpha_n x)\psi(x)$, $S_nBS_n^{-1} = B - \alpha_n$, $S_nH_0S_n^{-1} = H_0 - 2\pi L^{-1}v_{\mathrm{F}}\alpha_n B + \pi L^{-1}v_{\mathrm{F}}\beta_n$, where $\beta_n = n(n-1)$. The operator S will be defined by $Sa_pS^{-1} = a_{p-2\pi L^{-1}\alpha}$, $S\psi(x)S^{-1} = \exp(-\mathrm{i}2\pi L^{-1}\alpha x)\psi(x)$, $SBS^{-1} = B - \alpha$, $SH_0S^{-1} = H_0 - 2\pi L^{-1}v_{\mathrm{F}}\alpha B + \pi L^{-1}v_{\mathrm{F}}\beta$ where $\alpha = \lim_{n\to\infty}(\alpha_n)^{1/n} = 1$ and $\beta = \lim_{n\to\infty}(\beta_n)^{1/n} = 1$. It follows that S defined in this way has the same effect as that of S given by (2.23) provided that the sum $\Sigma^p_{0<p\leqslant 2\pi L^{-1}}$ is replaced by the integral $L(2\pi)^{-1}\int_0^{2\pi L^{-1}} p\,\mathrm{d}p = \pi L^{-1}$. It is noteworthy that this definition of S may be extended to the real powers of this operator by allowing for a continuous range of the wavevector p.

where $\chi_0(B, x)$ has to be further specified. Moreover

$$S\chi(x)S^{-1} = S^{-1}\chi_0(B-1, x) = \exp(-\mathrm{i}2\pi L^{-1}x)S^{-1}\chi_0(B, x)$$

whence

$$\chi_0(B, x) = \exp(\mathrm{i}2\pi L^{-1}x)\chi_0(B-1, x)$$

that is

$$\chi_0(B, x) = K(x)\exp(\mathrm{i}2\pi L^{-1}Bx) \tag{2.28}$$

$K(x)$ being an undetermined function of x. In order to find $K(x)$ we investigate the equation of motion for the fermion field

$$\begin{aligned}
[\psi(x), H_0] = -\mathrm{i}v_\mathrm{F}\frac{\partial}{\partial x}\psi(x) &= -\mathrm{i}v_\mathrm{F}\frac{\partial\chi(x)}{\partial x}\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x)) \\
&\quad - \mathrm{i}v_\mathrm{F}\chi(x)\frac{\partial}{\partial x}[\exp(\mathrm{i}V^-(x))\exp(\mathrm{i}V(x))] \\
&= [\chi(x), H_0]\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x)) \\
&\quad + \chi(x)[\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x)), H_0].
\end{aligned} \tag{2.29}$$

Using (2.26) we obtain straightforwardly

$$[\chi(x), H_0] = 2\pi L^{-1}v_\mathrm{F}S^{-1}(B-\tfrac{1}{2})\chi_0(B, x)$$

where we have used the commutator $[B, H_0] = 0$. Taking into account the relation

$$[\rho(-k), H_0] = v_\mathrm{F}k\rho(-k) \tag{2.30}$$

we obtain similarly

$$[\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x)), H_0] = -\mathrm{i}v_\mathrm{F}\frac{\partial}{\partial x}[\exp(\mathrm{i}V^+(x))\exp(\mathrm{i}V(x))].$$

Introducing these results into (2.29) we obtain the equation

$$-\mathrm{i}\partial\chi_0(B, x)/\partial x = 2\pi L^{-1}(B-\tfrac{1}{2})\chi_0(B, x)$$

whose solution is

$$\chi_0(B, x) = c\exp[\mathrm{i}2\pi L^{-1}(B-\tfrac{1}{2})x] \tag{2.31}$$

c being a constant. Therefore $K(x) = c\exp(-\mathrm{i}\pi L^{-1}x)$ as one can see by comparing (2.28) and (2.31). Bringing together the results given by (2.11), (2.21), (2.27) and (2.31) we obtain Jordan's boson representation

$$\begin{aligned}
\psi(x) = cS^{-1}\exp[\mathrm{i}2\pi L^{-1}(B-\tfrac{1}{2})x]\exp\Big(-2\pi L^{-1}\sum_{k>0}k^{-1}\exp(-\mathrm{i}kx)\rho^+(-k)\Big) \\
\times\exp\Big(2\pi L^{-1}\sum_{k>0}k^{-1}\exp(\mathrm{i}kx)\rho(-k)\Big).
\end{aligned} \tag{2.32}$$

It still remains to check whether the anticommutation relations

$$\{\psi^+(x), \psi(y)\} = \delta(x-y) \qquad \{\psi(x), \psi(y)\} = 0 \tag{2.33}$$

and Jordan's commutator (2.18) are satisfied by this boson representation. In order to do this we follow Jordan's prescriptions (2.13) and (2.16) of introducing the cut-off parameter α. When using this cut-off procedure and the boson representation (2.32) for calculating products of two fermion fields we encounter sums of the type

$$f(z) = 2\pi L^{-1} \sum_{k>0} k^{-1} \exp(-kz) \qquad \mathrm{Re}\, z \geqslant 0, \qquad z \neq 0. \tag{2.34}$$

For $L^{-1}|z| \ll 1$ (condition fulfilled for any fixed z and $L \to \infty$) this sum may be approximated by

$$f(z) \simeq -\ln(2\pi L^{-1} z) + \pi L^{-1} z \tag{2.35}$$

and this approximation will be used throughout this paper. By straightforward calculation we obtain for $x \neq y$

$$\begin{aligned}\{\psi(x), \psi(y)\} = {}& c^2 S^{-2} \exp[\mathrm{i}2\pi L^{-1}(B - \tfrac{1}{2})(x + y)] \\ & \times \exp\left(-2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(-\mathrm{i}kx) + \exp(-\mathrm{i}ky)]\rho^{+}(-k)\right) \\ & \times \exp\left(2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(\mathrm{i}kx) + \exp(\mathrm{i}ky)]\rho(-k)\right) \\ & \times [\![\exp\{-\mathrm{i}2\pi L^{-1}x - f[-\mathrm{i}(x-y)]\} \\ & + \exp\{-\mathrm{i}2\pi L^{-1}y - f[\mathrm{i}(x-y)]\}]\!] = 0\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}\psi^2(x) = {}& c^2 S^{-2} \exp[\mathrm{i}4\pi L^{-1}(B - \tfrac{1}{2})x] \exp\left(-4\pi L^{-1} \sum_{k>0} k^{-1} \exp(-\mathrm{i}kx)\,\rho^{+}(-k)\right) \\ & \times \exp\left(4\pi L^{-1} \sum_{k>0} k^{-1} \exp(\mathrm{i}kx)\,\rho(-k)\right) \exp(-f(0)) = 0\end{aligned} \tag{2.37}$$

due to the last exponential factor which is equal to zero. Using the cut-off procedure given by (2.13) and (2.16) we obtain

$$\begin{aligned}[\psi(x - \mathrm{i}\alpha/2)]^{+} \psi(y - \mathrm{i}\alpha/2) = {}& |c|^2 \exp[-\mathrm{i}2\pi L^{-1}(B - \tfrac{1}{2})(x - y)] \exp[2\pi L^{-1}(B - \tfrac{1}{2})\alpha] \\ & \times \exp\left(2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(-\mathrm{i}kx + \alpha k/2) - \exp(-\mathrm{i}ky - \alpha k/2)]\rho^{+}(-k)\right) \\ & \times \exp\left(-2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(\mathrm{i}kx - \alpha k/2) - \exp(\mathrm{i}ky + \alpha k/2)]\rho(-k)\right) \\ & \times \exp\{f[\alpha - \mathrm{i}(x-y)]\}\end{aligned} \tag{2.38}$$

and

$$\begin{aligned}\psi(y + \mathrm{i}\alpha/2)[\psi(x + \mathrm{i}\alpha/2)]^{+} = {}& |c|^2 \exp[-\mathrm{i}2\pi L^{-1}(B + \tfrac{1}{2})(x - y)] \exp[-2\pi L^{-1}(B + \tfrac{1}{2})\alpha] \\ & \times \exp\left(2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(-\mathrm{i}kx - \alpha k/2) - \exp(-\mathrm{i}ky + \alpha k/2)]\rho^{+}(-k)\right) \\ & \times \exp\left(-2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(\mathrm{i}kx + \alpha k/2) - \exp(\mathrm{i}ky - \alpha k/2)]\rho(-k)\right) \\ & \times \exp\{f[\alpha + \mathrm{i}(x-y)]\}\end{aligned} \tag{2.39}$$

so that

$$\begin{aligned}\{\psi^{+}(x), \psi(y)\} = {}& |c|^2 \exp[-\mathrm{i}2\pi L^{-1} B(x - y)] \\ & \times \exp\left(2\pi L^{-1} \sum_{k>0} k^{-1}[\exp(-\mathrm{i}kx) - \exp(-\mathrm{i}ky)]\rho^{+}(-k)\right)\end{aligned}$$

$$\times \exp\left(-2\pi L^{-1}\sum_{k>0} k^{-1}[\exp(\mathrm{i}kx)-\exp(\mathrm{i}ky)]\rho(-k)\right)$$
$$\times \lim_{\alpha\to 0}\{\alpha\pi^{-1}[\alpha^2+(x-y)^2]^{-1}\} = |c|^2 L\delta(x-y). \tag{2.40}$$

If follows that $c = c_0 L^{-1/2}$, c_0 being a constant with $|c_0| = 1$. We have similarly from (2.38)

$$[\psi(x-\mathrm{i}\alpha/2)]^+\psi(x-\mathrm{i}\alpha/2) = L^{-1}\exp[2\pi L^{-1}(B-\tfrac{1}{2})\alpha + f(\alpha)]$$
$$\times\exp\left(4\pi L^{-1}\sum_{k>0}k^{-1}\exp(-\mathrm{i}kx)\sinh\frac{\alpha k}{2}\rho^+(-k)\right)$$
$$\times\exp\left(4\pi L^{-1}\sum_{k>0}k^{-1}\exp(\mathrm{i}kx)\sinh\frac{\alpha k}{2}\rho(-k)\right)$$
$$=\frac{1}{2\pi\alpha}+L^{-1}B+(2\pi)^{-1}[F(x)+F^+(x)]$$
$$+\pi\alpha:\{L^{-1}B+(2\pi)^{-1}[F(x)+F^+(x)]\}^2: + \mathrm{O}(\alpha^2) \tag{2.41}$$

where : . . . : means the normal ordering of the boson operators; from (2.39) we obtain

$$\psi(x+\mathrm{i}\alpha/2)[\psi(x+\mathrm{i}\alpha/2)]^+ = (1/2\pi\alpha) - L^{-1}B - (2\pi)^{-1}[F(x)+F^+(x)]$$
$$+\pi\alpha:\{L^{-1}B+(2\pi)^{-1}[F(x)+F^+(x)]\}^2: \mathrm{O}(\alpha^2). \tag{2.42}$$

These expressions agree with those given by (2.15) and (2.17) and one can easily see that Jordan's commutator (2.18) is obtained by this bosonisation technique. We notice that the factor $\exp(\alpha k/2)$ appearing in these calculations may be considered as a short-hand notation for its first-order power expansion $1+\alpha k/2$. In this way the limit $\alpha\to 0$ may be safely transposed with the summation over k. This done, the validity of Jordan's boson representation (2.32) and the cut-off prescriptions (2.13) and (2.16) are completely established. We should now obtain the bosonised form of the kinetic Hamiltonian H_0 given by (2.5). By straightforward calculation we have

$$-\mathrm{i}\int \mathrm{d}x\,[\psi(x-\mathrm{i}\alpha/2)]^+\frac{\partial}{\partial x}\psi(x-\mathrm{i}\alpha/2)$$
$$=\sum_{p>0}\exp(p\alpha)pa_p^+a_p - \sum_{p\leq 0}\exp(p\alpha)p(a_pa_p^+-1)$$
$$=\frac{\partial}{\partial\alpha}\left(\sum_{p>0}\exp(p\alpha)a_p^+a_p - \sum_{p\leq 0}\exp(p\alpha)(a_pa_p^+-1)\right) \tag{2.43}$$

and comparing with

$$\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^+\psi(x-\mathrm{i}\alpha/2) = \sum_{p>0}\exp(p\alpha)a_p^+a_p - \sum_{p\leq 0}\exp(p\alpha)(a_pa_p^+-1) \tag{2.44}$$

we obtain

$$\sum_{p>0}\exp(p\alpha)pa_p^+a_p - \sum_{p\leq 0}\exp(p\alpha)pa_pa_p^+$$
$$=\frac{L}{2\pi\alpha^2}+\frac{\partial}{\partial\alpha}\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^+\psi(x-\mathrm{i}\alpha/2). \tag{2.45}$$

From (2.41) we obtain

$$\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^{+}\psi(x-\mathrm{i}\alpha/2)$$
$$=\frac{L}{2\pi\alpha}+B+\pi L^{-1}\alpha\left(B^{2}+2\sum_{k>0}\rho^{+}(-k)\rho(-k)\right)+\mathrm{O}(\alpha^{2}) \tag{2.46}$$

and introducing this into (2.45) we obtain

$$\sum_{p>0}\exp(p\alpha)pa_p^{+}a_p-\sum_{p\leqslant 0}\exp(p\alpha)pa_pa_p^{+}$$
$$=\pi L^{-1}B^{2}+2\pi L^{-1}\sum_{k>0}\rho^{+}(-k)\rho(-k)+\mathrm{O}(\alpha) \tag{2.47}$$

whence

$$H_0=v_{\mathrm{F}}\sum_{p>0}pa_p^{+}a_p-v_{\mathrm{F}}\sum_{p\leqslant 0}pa_pa_p^{+}$$
$$=\pi L^{-1}v_{\mathrm{F}}B^{2}+2\pi L^{-1}v_{\mathrm{F}}\sum_{k>0}\rho^{+}(-k)\rho(-k). \tag{2.48}$$

One can see that (2.26) and (2.29) are satisfied by this bosonised form of H_0. From (2.43), (2.44), (2.46) and (2.48) one obtains also

$$\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^{+}\psi(x-\mathrm{i}\alpha/2)=\frac{L}{2\pi\alpha}+B+\alpha v_{\mathrm{F}}^{-1}H_0+\mathrm{O}(\alpha^{2})$$

which agrees with (2.19) and

$$-\mathrm{i}\int \mathrm{d}x[\psi(x-\mathrm{i}\alpha/2)]^{+}\frac{\partial}{\partial x}\psi(x-\mathrm{i}\alpha/2)=\frac{\partial}{\partial\alpha}\int \mathrm{d}x[x-\mathrm{i}\alpha/2)]^{+}\psi(x-\mathrm{i}\alpha/2)$$
$$=-\frac{L}{2\pi\alpha^{2}}+v_{\mathrm{F}}^{-1}H_0+\mathrm{O}(\alpha).$$

This latter relation can also be obtained by using the boson representation of the fermion field directly. The expectation value of the product $[\psi(x-\mathrm{i}\alpha/2)]^{+}\psi(x-\mathrm{i}\alpha/2)$ given by (2.41) on the ground state is $(2\pi\alpha)^{-1}+(2\pi)^{-1}k_{\mathrm{F}}+\mathrm{O}(\alpha)$, whence one may interpret α^{-1} as a bandwidth cut-off.

Concluding this section we remark that Jordan's theory provides the exact and complete correspondence between the original problem formulated in terms of the fermion fields and its bosonised form.

3. Boson representation for the TFM

We pass now to the generalisation of Jordan's boson representation to the set of four fermion operators appearing in the theory of the TFM

$$\psi_{js}(x)=L^{-1/2}\sum_{p}a_{jps}\exp(\mathrm{i}px)\qquad \{a_{jps}^{+},a_{j'p's'}\}=\delta_{jj'}\delta_{pp'}\delta_{ss'}\qquad \{a_{jps},a_{j'p's'}\}=0 \tag{3.1}$$

where $j=1,2$, $p=2\pi L^{-1}n$, n integer and $s=\pm 1$ is the spin index. The Hamiltonian of the system is given by

$$H_0 = v_F \sum_{s,p>0} p a_{1ps}^+ a_{1ps} - v_F \sum_{s,p\leq 0} p a_{1ps} a_{1ps}^+ - v_F \sum_{s,p<0} p a_{2ps}^+ a_{2ps} + v_F \sum_{s,p\geq 0} p a_{2ps} a_{2ps}^+$$

$$= v_F \sum_{s,p>0} p a_{1ps}^+ a_{1ps} + v_F \sum_{s,p\leq 0} p (a_{1ps}^+ a_{1ps} - 1)$$

$$- v_F \sum_{s,p<0} p a_{2ps}^+ a_{2ps} - v_F \sum_{s,p\geq 0} p (a_{2ps}^+ a_{2ps} - 1) \tag{3.2}$$

and the Fermi sea is filled with particles of the first type ($j = 1$) from $p = -\infty$ to $p = +k_F$ and with particles of the second type ($j = 2$) from $p = -k_F$ to $p = +\infty$. The 'charge' operators are

$$B_{1s} = \sum_{p>0} a_{1ps}^+ a_{1ps} + \sum_{p\leq 0} (a_{1ps}^+ a_{1ps} - 1)$$

$$B_{2s} = \sum_{p<0} a_{2ps}^+ a_{2ps} + \sum_{p\geq 0} (a_{2ps}^+ a_{2ps} - 1) \tag{3.3}$$

which commute with H_0. One can easily see that the operators α_{jqs} and β_{jqs} defined by

$$\begin{aligned}
\alpha_{1qs} &= 2^{-1/2}(a_{1q-\pi L^{-1}s} + a_{1-q-\pi L^{-1}s}^+)\\
\alpha_{2qs} &= \mathrm{i}2^{-1/2}(a_{1-q-\pi L^{-1}s}^+ - a_{1q-\pi L^{-1}s})\\
\beta_{1qs} &= 2^{-1/2}(a_{2-q-\pi L^{-1}s} + a_{2-q-\pi L^{-1}s}^+)\\
\beta_{2qs} &= \mathrm{i}2^{-1/2}(a_{2q-\pi L^{-1}s}^+ - a_{2-q-\pi L^{-1}s})\\
a_{1ps} &= 2^{-1/2}(\alpha_{1p+\pi L^{-1}s} + \mathrm{i}\alpha_{2p+\pi L^{-1}s})\\
a_{1ps}^+ &= 2^{-1/2}(\alpha_{1-p-\pi L^{-1}s} - \mathrm{i}\alpha_{2-p-\pi L^{-1}s})\\
a_{2ps} &= 2^{-1/2}(\beta_{1-p-\pi L^{-1}s} + \mathrm{i}\beta_{2-p-\pi L^{-1}s})\\
a_{2ps}^+ &= 2^{-1/2}(\beta_{1p+\pi L^{-1}s} - \mathrm{i}\beta_{2p+\pi L^{-1}s})
\end{aligned} \tag{3.4}$$

where $q = \pm(p + \pi L^{-1}) = 2\pi L^{-1}(n + \frac{1}{2})$, n integer, satisfy the conditions (2.1) so that the Fourier components of the particle-density operators

$$\begin{aligned}
\rho_{1s}(-k) &= \rho_{1s}^+(k) = \sum_p a_{1ps}^+ a_{1p+ks} = \mathrm{i}\sum_q \alpha_{1qs}\alpha_{2k-qs}\\
\rho_{2s}(k) &= \rho_{2s}^+(-k) = \sum_p a_{2p+ks}^+ a_{2ps} = \mathrm{i}\sum_q \beta_{1qs}\beta_{2k-qs} \qquad k>0
\end{aligned} \tag{3.5}$$

obey the boson-like commutation relations

$$[\rho_{js}(\mp k), \rho_{j's'}^+(\mp k')] = (2\pi)^{-1} L k \delta_{jj'} \delta_{ss'} \delta_{kk'}$$

$$[\rho_{js}(\mp k), \rho_{j's'}(\mp k')] = 0 \qquad k, k' > 0$$

where the upper (lower) sign corresponds to $j, j' = 1(2)$. In addition any $\rho_{js}(\mp k)$ commutes with any B_{js} and

$$[\rho_{js}(\mp k), H_0] = v_F k \rho_{js}(\mp k) \qquad [B_{js}, H_0] = 0. \tag{3.7}$$

Likewise as before we introduce the unitary operators $S_{js} = (S_{js}^{-1})^+$

$$S_{js} a_{j'ps'} S_{js}^{-1} = \delta_{jj'}\delta_{ss'} a_{jp+2\pi L^{-1}s} + (1 - \delta_{jj'}\delta_{ss'}) a_{j'ps'} \tag{3.8}$$

with the properties

$$S_{js} B_{j's'} S_{js}^{-1} = \delta_{jj'} \delta_{ss'} (B_{js} \mp 1) + (1 - \delta_{jj'} \delta_{ss'}) B_{j's'},$$
$$S_{js} H_0 S_{js}^{-1} = H_0 \mp 2\pi L^{-1} v_F (B_{js} \mp \tfrac{1}{2}) \tag{3.9}$$

and $[S_{js}, \rho_{j's'}(\mp k)] = [S_{js}, \rho^+_{j's'}(\mp k)] = 0$. One can straightforwardly check that all the properties of the field operators listed below

$$[\psi_{js}(x), \rho_{j's'}(\mp k)] = \delta_{jj'} \delta_{ss'} \exp(\mp ikx)\, \psi_{js}(x) \qquad [\psi_{js}(x), B_{j's'}] = \delta_{jj'} \delta_{ss'} \psi_{js}(x)$$
$$S_{js} \psi_{j's'}(x) S_{js}^{-1} = \delta_{jj'} \delta_{ss'} \exp(-i2\pi L^{-1} x)\, \psi_{js}(x) + (1 - \delta_{jj'} \delta_{ss'})\, \psi_{j's'}(x)$$
$$[\psi_{js}(x), H_0] = \mp i v_F \partial \psi_{js}(x)/\partial x \qquad \{\psi^+_{js}(x), \psi_{j's'}(y)\} = \delta_{jj'} \delta_{ss'} \delta(x - y) \tag{3.10}$$
$$\{\psi_{js}(x), \psi_{j's'}(y)\} = 0$$
$$[\psi^+_{js}(x), \psi_{js}(x)] = 2L^{-1} B_{js} + \pi^{-1} [F_{js}(x) + F^+_{js}(x)]$$

where

$$F_{js}(x) = 2\pi L^{-1} \sum_{k>0} \exp(\pm ikx)\, \rho_{js}(\mp k)$$

are satisfied by the boson representation

$$\psi_{js}(x) = c_{js} L^{-1/2} S_{js}^{\mp 1} \exp[\pm i2\pi L^{-1} (B_{js} - \tfrac{1}{2}) x] \exp\left(-2\pi L^{-1} \sum_{k>0} k^{-1} \exp(\mp ikx)\, \rho^+_{js}(\mp k)\right)$$
$$\times \exp\left(2\pi L^{-1} \sum_{k>0} k^{-1} \exp(\pm ikx)\, \rho_{js}(\mp k)\right) \tag{3.11}$$

provided that Jordan's prescriptions are used for introducing the cut-off parameter α:

$$\psi^+_{js}(x)\, \psi_{js}(y) = \lim_{\alpha \to 0} [\psi_{js}(x \mp i\alpha/2)]^- \psi_{js}(y \mp i\alpha/2)$$
$$\psi_{js}(x)\, \psi^-_{js}(y) = \lim_{\alpha \to 0} \psi_{js}(x \pm i\alpha/2) [\psi_{js}(y \pm i\alpha/2)]^+. \tag{3.12}$$

The coefficients c_{js} are chosen in such a way as to satisfy the relations

$$c^+_{js} c_{js} = c_{js} c^+_{js} = 1 \qquad \{c_{js}, c_{j's'}\} = \{c_{js}, c^+_{j's'}\} = 0 \qquad (js) \neq (j's'). \tag{3.13}$$

Their construction is given in Appendix 1.†

Jordan's boson representation (3.11) is normal ordered in boson operators and consistently includes the modes corresponding to $k = 0$ (through the B_{js} operators). This boson representation has also been derived recently by Haldane (1979, 1981a) by means of an entirely different technique. It is noteworthy that the ladder operators constructed by Haldane (1981a) have all the properties of the S_{js} operators introduced here (including their action on the field operators and on the kinetic Hamiltonian). It is easy to verify that the boson representation of Luther and Peschel (1974) does not satisfy Jordan's commutator (last line in equation (3.10)). Indeed, for $j = 1$ and dropping the spin index this boson representation reads (Luther and Peschel 1974)

† The conditions (3.13) are satisfied by the Dirac matrices as well as by the operational representations of the coefficients c_{js} in terms of the 'charge' operators B_{js} (see for example Heidenreich *et al* 1975, Sólyom 1979). However, in order to diagonalise the Luther–Emery Hamiltonian and the umklapp scattering Hamiltonian (as is done in Apostol (1983)) the coefficients c_{js} must be subjected further to additional conditions which are satisfied neither by the Dirac matrices nor by the operatorial representations of c_{js}.

$$\psi(x) = (2\pi\alpha)^{-1/2} \exp(ik_F x + 2\pi \sum_{k>0} k^{-1} \exp(-\alpha k/2)$$
$$\times [\exp(ikx)\rho(-k) - \exp(-ikx)\rho^+(-k)]$$

so that we have

$$\psi^+(x)\psi(x) = \psi(x)\psi^+(x) = [1 - \exp(-2\pi\alpha)]^{-1} \simeq (2\pi\alpha)^{-1}.$$

One can see that this cut-off procedure correctly reproduces only the first term in (2.15) and (2.17) and gives $[\psi^+(x), \psi(x)] = 0$ in contrast to (2.18). Haldane's boson representation (1981a) is normal ordered so that the parameter α may be taken out from its expression. However Haldane regularises the divergent factors appearing in products of field operators by means of a cut-off $\exp(-\varepsilon k)$ $(\varepsilon \to 0^+)$ which is essentially the same so that again only the first term is correctly obtained in $\psi^+(x)\psi(x)$ and $\psi(x)\psi^+(x)$. It is obvious that the cut-off parameter α should be introduced in such a way as not only to simply regularise the divergent factors but also to reproduce the properties of the original fermion fields correctly. The expressions (2.15) and (2.17) are correctly obtained, and hence the commutator (2.18), by a slight modification of Haldane's cut-off prescriptions (Haldane 1982), namely

$$\psi^+(x)\psi(x) = \lim_{a\to 0} \psi^+(x + a/2)\psi(x - a/2)$$

$$\psi(x)\psi^+(x) = \lim_{a\to 0} \psi(x + a/2)\psi^+(x - a/2)$$

(where the limit $\varepsilon \to 0^+$ in Haldane's cut-off prescriptions is taken). Obviously this is nothing but another version of Jordan's cut-off procedure (2.13) and (2.16) and, in fact, it has been used by Haldane (1981a) to derive the bosonised form of the kinetic Hamiltonian. One may see that Jordan's cut-off procedure is more specific than the usual one in which only the factor $\exp(-\alpha k/2)$ appears.

Finally we note that the kinetic Hamiltonian H_0 given by (3.2) becomes in the boson representation

$$H_0 = \pi L^{-1} v_F \sum_{js} B_{js}^2 + 2\pi L^{-1} v_F \sum_{js,k>0} \rho_{js}^+(\mp k)\rho_{js}(\mp k). \tag{3.14}$$

As $[B_{js}, H_0] = 0$ the additional zero-mode contribution appearing in H_0 has no notable effect on the energy spectrum of H_0 which can be described either in terms of one-fermion excitations or in terms of ρ excitations (Mattis and Lieb 1965).

4. Conclusions

It has been shown that Jordan's commutator has been overlooked so far by the theory of the TFM. This commutator plays the part of an additional condition which must be satisfied by the boson representation of the fermion fields in one dimension. Including Jordan's commutator in the theory of the TFM amounts to introducing a well-determined cut-off procedure for calculating products of two fermion fields at the same space point. This cut-off procedure differs from that usually given in the literature (Luther and Peschel 1974) and, in fact, it can be obtained by a slight modification from that proposed by Haldane (1981a). Jordan's theory of the boson representation of fermion fields in one dimension has been reviewed in the present paper and generalised to the TFM. The precise connection between the fermion problem and its bosonised counterpart has been emphasised.

Appendix 1. The coefficients

Let us consider four types of fermions labelled by $(js) \equiv i = 1, 2, 3, 4$ so that $(1, +1) = 1, (1, -1) = 2, (2, +1) = 3$ and $(2, -1) = 4$ each with the energy levels p = integer. The ground state $|\bar{0}\rangle$ of this system is filled with particles from $p = -\infty$ to $p = 0$ (or any other constant, not necessarily the same for all particles; in this case the definition of b_i below should be changed correspondingly). Let us define the 'charge' operators

$$b_i = \sum_{p>0} n_p^i + \sum_{p\leq 0} (n_p^i - 1)$$

where n_p^i is the occupation number of the p level with i-type particles, $n_p^i = 0, 1$. All the b_i yield zero when acting upon the ground state $b_i|\bar{0}\rangle = 0$. We consider the states $|b_1 b_2 b_3 b_4\rangle$ characterised by specified eigenvalues b_i (integers) of the 'charge' operators and define the operators c_i by

$$c_i|b_1 b_2 b_3 b_4\rangle = (-1)^{\sum_{0\leq j<i} b_j} |b_1 \ldots b_i - 1 \ldots b_4\rangle$$

$$c_i^+|b_1 b_2 b_3 b_4\rangle = (-1)^{\sum_{0\leq j<i} b_j} |b_1 \ldots b_i + 1 \ldots b_4\rangle$$

where $i = 1, 2, 3, 4$ and $b_0 \equiv 0$. It is easy to check that the commutation relations (3.13) are satisfied by the operators $c_i = c_{js}$ defined on the space spanned by the states $|b_1 b_2 b_3 b_4\rangle$.

Let us define the operators $c_{j\rho}$ and $c_{j\sigma}$ by $c_{1\rho} = c_{2-1}^+$, $c_{2\rho} = c_{21}$, $c_{1\sigma} = c_{11}$ and $c_{2\sigma} = c_{1-1}$. Taking the superposition

$$|\Phi\rangle_{\varphi_1\varphi_2} = \sum_{b_1 b_2 b_3 b_4} \exp[\mathrm{i}(b_1\varphi_1 + b_3\varphi_2)](-1)^{[b_1(b_1-1)+b_3(b_3-1)]/2}|b_1 b_2 b_3 b_4\rangle$$

where $\Phi_{1,2}$ are real parameters, one can easily verify the relations

$$c_{1\rho}c_{2\rho}|\Phi\rangle_{\varphi_1\varphi_2} = c_{1\rho}^+ c_{2\rho}|\Phi\rangle_{\varphi_1\varphi_2} = \exp(\mathrm{i}\varphi_2)|\Phi\rangle_{\varphi_1\varphi_2},$$

$$c_{1\sigma}c_{2\sigma}|\Phi\rangle_{\varphi_1\varphi_2} = c_{1\sigma}c_{2\sigma}^+|\Phi\rangle_{\varphi_1\varphi_2} = -\exp(\mathrm{i}\varphi_1)|\Phi\rangle_{\varphi_1\varphi_2}$$

$$c_{1\rho}c_{2\rho}^+|\Phi\rangle_{\varphi_1\varphi_2} = c_{1\rho}^+ c_{2\rho}^+|\Phi\rangle_{\varphi_1\varphi_2} = -\exp(-\mathrm{i}\varphi_2)|\Phi\rangle_{\varphi_1\varphi_2},$$

$$c_{1\sigma}^+ c_{2\sigma}|\Phi\rangle_{\varphi_1\varphi_2} = c_{1\sigma}^+ c_{2\sigma}^+|\Phi\rangle_{\varphi_1\varphi_2} = \exp(-\mathrm{i}\varphi_1)|\Phi\rangle_{\varphi_1\varphi_2}$$

which are useful in diagonalising the Hamiltonian of the TFM with backscattering and Umklapp scattering.

Appendix 2. The interplay between the bandwidth cut-off α^{-1} and the momentum transfer cut-off r^{-1}

We investigate the effect of the canonical transformation

$$\rho_{js}(\mp k) \rightarrow \tilde{\rho}_{js}(\mp k) = v_s(k)\rho_{js}(\mp k) + w_s(k)\rho_{\bar{j}s}^+(\mp k)$$

where $\bar{j} = 1$ for $j = 2$ and $\bar{j} = 2$ for $j = 1$, $v_s^2(k) - w_s^2(k) = 1$, $w_s(k) = w_s \exp(-rk/2)$, r^{-1} being a momentum transfer cut-off, on the anticommutation relations of the field operators and on Jordan's commutator. We shall prove that these relations are preserved by such a transformation provided that $\alpha \rightarrow 0$ while r is held finite. This invariance was proved by Theumann (1977) for the usual cut-off procedure introduced by Luther and Peschel (1974) and it is shown here that it also holds for Jordan's present prescription of introducing the cut-off parameter α. By a straightforward calculation we obtain

$$
\begin{aligned}
[\tilde{\psi}_{js}(x \mp \mathrm{i}\alpha/2)]^{+}\tilde{\psi}_{js}(y \mp \mathrm{i}\alpha/2) = {}& L^{-1}\exp[\mp \mathrm{i}2\pi L^{-1}B_{js}(x-y)]\exp[2\pi L^{-1}(B_{js}-\tfrac{1}{2})\alpha] \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\mp \mathrm{i}kx+\alpha k/2)-\exp(\mp \mathrm{i}ky-\alpha k/2)]\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\pm \mathrm{i}ky+\alpha k/2)-\exp(\pm \mathrm{i}kx-\alpha k/2)]\rho_{j\bar{s}}^{+}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\pm \mathrm{i}ky+\alpha k/2)-\exp(\pm \mathrm{i}kx-\alpha k/2)]\rho_{js}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\mp \mathrm{i}kx+\alpha k/2)-\exp(\mp \mathrm{i}ky-\alpha k/2)]\rho_{j\bar{s}}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}w_s^2\sum_{k>0} k^{-1}\exp(-rk)\{2-\exp[\mp \mathrm{i}k(x-y)+\alpha k] \\
& - \exp[\pm \mathrm{i}k(x-y)-\alpha k]\}\Big) \\
& \times \exp\{\pm \mathrm{i}\pi L^{-1}(x-y)+f[\mp \mathrm{i}(x-y)+\alpha]\}
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{\psi}_{js}(y \pm \mathrm{i}\alpha/2)[\tilde{\psi}_{js}(x \pm \mathrm{i}\alpha/2)]^{+} = {}& L^{-1}\exp[\mp \mathrm{i}2\pi L^{-1}B_{js}(x-y)]\exp[-2\pi L^{-1}(B_{js}+\tfrac{1}{2})\alpha] \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\mp \mathrm{i}ky+\alpha k/2)-\exp(\mp \mathrm{i}kx-\alpha k/2)]\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\pm \mathrm{i}kx+\alpha k/2)-\exp(\pm \mathrm{i}ky-\alpha k/2)]\rho_{j\bar{s}}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\pm \mathrm{i}kx+\alpha k/2)-\exp(\pm \mathrm{i}ky-\alpha k/2)]\rho_{js}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\mp \mathrm{i}ky+\alpha k/2)-\exp(\mp \mathrm{i}kx-\alpha k/2)]\rho_{j\bar{s}}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}w_s^2\sum_{k>0} k^{-1}\exp(-rk)[2-\exp[\pm \mathrm{i}k(x-y)+\alpha k] \\
& - \exp[\mp \mathrm{i}k(x-y)-\alpha k]\}\Big) \\
& \times \exp\Big(\mp \mathrm{i}\pi L^{-1}(x-y)+f[\pm \mathrm{i}(x-y)+\alpha]\}
\end{aligned}
$$

whence

$$
\begin{aligned}
\{\tilde{\psi}_{js}^{+}(x), \tilde{\psi}_{js}(y) = {}& \exp[\mp \mathrm{i}2\pi L^{-1}B_{js}(x-y)] \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\mp \mathrm{i}kx)-\exp(\mp \mathrm{i}ky)]\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\pm \mathrm{i}kx)-\exp(\pm \mathrm{i}ky)]\rho_{j\bar{s}}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\sum_{k>0} k^{-1}v_s(k)[\exp(\pm \mathrm{i}kx)-\exp(\pm \mathrm{i}ky)]\rho_{js}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\sum_{k>0} k^{-1}w_s(k)[\exp(\mp \mathrm{i}kx)-\exp(\mp \mathrm{i}ky)]\rho_{j\bar{s}}(\mp k)\Big) \\
& \times \left(\frac{r^2}{r^2+(x-y)^2}\right)^{w_s^2}\lim_{\alpha\to 0}\frac{\pi^{-1}\alpha}{\alpha^2+(x-y)^2} = \delta(x-y)
\end{aligned}
$$

and

$$
\begin{aligned}
[\psi_{js}^{+}(x),\ \psi_{js}(x)] = \lim_{\alpha\to 0} L^{-1}\Big[& \exp[2\pi L^{-1}(B_{js}-\tfrac{1}{2})\alpha] \\
& \times \exp\Big(2\pi L^{-1}\alpha \sum_{k>0} v_s(k)\exp(\mp \mathrm{i}kx)\,\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\alpha \sum_{k>0} w_s(k)\exp(\pm \mathrm{i}kx)\,\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\alpha \sum_{k>0} v_s(k)\exp(\pm \mathrm{i}kx)\,\rho_{js}(\mp k)\Big) \\
& \times \exp\Big(2\pi L^{-1}\alpha \sum_{k>0} w_s(k)\exp(\mp \mathrm{i}kx)\,\rho_{js}(\mp k)\Big) \\
& - \exp[-2\pi L^{-1}(B_{js}+\tfrac{1}{2})\alpha] \\
& \times \exp\Big(-2\pi L^{-1}\alpha \sum_{k>0} v_s(k)\exp(\mp \mathrm{i}kx)\,\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\alpha \sum_{k>0} w_s(k)\exp(\pm \mathrm{i}kx)\,\rho_{js}^{+}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\alpha \sum_{k>0} v_s(k)\exp(\pm \mathrm{i}kx)\,\rho_{js}(\mp k)\Big) \\
& \times \exp\Big(-2\pi L^{-1}\alpha \sum_{k>0} w_s(k)\exp(\mp \mathrm{i}kx)\,\rho_{js}(\mp k)\Big)\Big] \left(\frac{r^2}{r^2-\alpha^2}\right)^{w_s^2} \exp[f(\alpha)] \\
= \lim_{\alpha\to 0} L^{-1} 2\alpha\{2\pi L^{-1}B_{js} & + [\tilde{F}_{js}(x)+\tilde{F}_{js}^{+}(x)]\} \left(\frac{r^2}{r^2-\alpha^2}\right)^{w_s^2} \frac{1}{2\pi L^{-1}\alpha} \\
= 2L^{-1}B_{js} + \pi^{-1} & [\tilde{F}_{js}(x)+\tilde{F}_{js}^{+}(x)].
\end{aligned}
$$

Similarly one can see that $\{\tilde{\psi}_{js}(x),\ \tilde{\psi}_{js}(y)\}=0$, so that we may conclude that all the aforementioned anticommutation relations and Jordan's commutator are invariant under the transformation $\rho_{js}(\mp k)\to\tilde{\rho}_{js}(\mp k)$ provided that $\alpha\to 0$ while r is kept finite.

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