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Cut-off parameters of the bosonisation technique in one dimension

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Abstract. The Jordan boson representation and point-splitting regularisation are applied to the one-dimensional two-fermion model (TFM) with momentum-transfer cut-off and zero-mode terms included. It is shown that the correlation functions of the Tomonaga–Luttinger model obtained in this way coincide with those given by diagram summation. The energy gaps entering the well known Luther–Emery–Peschel back-scattering and Umklapp scattering solutions acquire a physical dependence on the momentum-transfer cut-off. First-order perturbation calculations of the charge-density function give different results for the spin-parallel 'back-scattering' contributions to the bosonised TFM and corresponding Fermi gas model, thus indicating that these models are not equivalent.

1. Introduction

The cut-off procedure currently employed for regularising the ultraviolet divergences that occur with boson representation of fermion fields in one dimension (Luther and Peschel 1974, Mattis 1974) poses certain difficult problems (see, for example, Sólyom 1979). The correlation functions of the Tomonaga–Luttinger model obtained by using this cut-off procedure differ from their expressions given by diagram summation. The difference is that the ultraviolet cut-off α^{-1} appears in place of both the band-width cut-off and the momentum-transfer cut-off. It is as if this α^{-1} cut-off is so strong that there is no dissociation between the band-width cut-off and momentum-transfer cut-off. The same cut-off procedure leads to an α^{-1} dependence of the energy gaps appearing in the back-scattering (Luther and Emery 1974) and Umklapp scattering (Emery *et al* 1976) problems. However, the boson representation holds only in the limit $\alpha \to 0$ (which would render these gaps infinite) and the commutation relations of the bosonised fermion fields are preserved under a canonical transformation of the boson operators that diagonalises the Tomonaga–Luttinger model only if a momentum-transfer cut-off r^{-1} is introduced which should be kept finite as α goes to zero (Theumann 1977). In fact, the need to introduce a momentum-transfer cut-off in calculating correlation functions with the boson representation has already been emphasised ever since the discovery of boson representation (Luther and Peschel 1974). Nevertheless, in spite of this, it is preferable in current practice to disregard the momentum-transfer cut-off and to work solely with the ultraviolet cut-off α^{-1} appearing in the boson representation.

The above-mentioned difficulties which are related to the ultraviolet cut-off α^{-1} originate in the infinite particle density of the non-interacting one-dimensional two-fermion model (TFM) (Apostol 1982, 1983) whose regularisation requires the Jordan point-splitting procedure (Jordan 1935, 1936a, b, 1937). The point-splitting regularisation prescribes a pure band-width cut-off α^{-1}, thus allowing the momentum-transfer cut-off r^{-1} required by the interaction Hamiltonian. The Jordan boson representation is exact in the limit $\alpha \to 0$ which should be taken whenever possible while r is kept finite (Apostol 1983). The consistent use of the point-splitting regularisation removes the aforementioned difficulties of the bosonised TFM and, in addition, enables a direct comparison to be made between the results of this model and those of the Fermi gas model (FGM) (Grest 1976, Haldane 1979, Rezayi *et al* 1979, Grinstein *et al* 1979). The previous boson representations (Luther and Peschel 1974, Mattis 1974, Heidenreich *et al* 1975, Haldane 1979, 1981) lead to the same results when proper allowance is made for the band-width cut-off α^{-1} and momentum-transfer cut-off r^{-1}.

In the present paper the Jordan point-splitting regularisation as applied to the bosonised TFM is dealt with. The aim is to incorporate consistently both the band-width cut-off α^{-1} and the momentum-transfer cut-off r^{-1} into the bosonised TFM. In § 2 the Jordan boson representation and point-splitting regularisation are briefly reviewed and the bosonised TFM is discussed in relation to the FGM. The zero-mode contributions are included not only in the kinetic Hamiltonian (Haldane 1981, Apostol 1982, 1983) of the TFM but also in its interaction part. Consequently, the canonical transformation that disentangles the charge-density degrees of freedom from the spin-density degrees of freedom has to be extended so as to take care of these zero-mode contributions also. This is done in § 3 where the correlation functions of the Tomonaga–Luttinger model are calculated by means of the Jordan bosonisation technique and found to be identical with those given by diagram summation (Dzyaloshinsky and Larkin 1973). The well known solutions of the back-scattering (Luther and Emery 1974) and Umklapp scattering (Emery *et al* 1976) problems are obtained again in § 4 with the only difference that the energy gaps in the fermion spectrum depend on the momentum-transfer cut-off r^{-1} and not on α^{-1}. This is an inevitable consequence of the consistent use of a bosonisation technique which incorporates both the band-width cut-off α^{-1} and the momentum-transfer cut-off r^{-1}. First-order perturbation calculations of the charge-density correlation functions of the bosonised TFM are dealt with in § 5. It is found that the spin-parallel 'back-scattering' $g_{1\parallel}$ contribution differs from both the spin-antiparallel 'back-scattering' $g_{1\perp}$ term in the bosonised TFM and the $g_{1\parallel}$ term in the corresponding FGM. The former discrepancy has already been noticed (Grest 1976) with the use of the ultraviolet cut-off α^{-1} in the boson representation given by Luther and Peschel (1974) and, therefore, it is not related to the boson representation. It arises from the difference between the $g_{1\parallel}$ and $g_{1\perp}$ terms in the bosonised TFM (amongst other features, for example, these terms are not spin rotationally invariant when $g_{1\parallel} = g_{1\perp}$); although the $g_{1\perp}$ term preserves its form from the FGM, the $g_{1\parallel}$ term acquires a form that is peculiar to the bosonised TFM. This discrepancy indicates the difference between the bosonised TFM and the corresponding FGM, a difference that is even more evident in the latter discrepancy ($g_{1\parallel}$ contributions) mentioned above. This difference shows that the $g_{1\parallel}$ terms in the bosonised TFM and the corresponding FGM are not equivalent. Conclusions are given in § 6. The paper ends with a mathematical Appendix on the Fourier transform of the charge-density correlation function.

2. The Jordan boson representation and point-splitting regularisation

The non-interacting one-dimensional TFM is described by the kinetic Hamiltonian (Luttinger 1963)

$$H_0 = -\mathrm{i}v_\mathrm{F} \sum_s \int \mathrm{d}x : \left(\psi_{1s}^+(x) \frac{\partial}{\partial x} \psi_{1s}(x) - \psi_{2s}^+(x) \frac{\partial}{\partial x} \psi_{2s}(x) \right): \tag{2.1}$$

where v_F is the Fermi velocity,

$$\psi_{js}(x) = L^{-1/2} \sum_p a_{jps} \exp(\mathrm{i}\,px)$$

are the fermion fields for each type of fermion $j = 1, 2$ (embedded in a box of length L) and each spin orientation $s = \pm 1$ and $: \cdots :$ means normal ordering relative to the non-interacting ground state $|0\rangle$ which is filled with particles whose wave-vector $p = 2\pi L^{-1} \times$ integer runs from $-\infty$ to 0 for $j = 1$ and from 0 to $+\infty$ for $j = 2$. Owing to the infinite particle density of the model a band-width cut-off α^{-1} should be introduced, which yields the following regularised expressions for $\psi_{js}^+(x)\psi_{js}(x)$ and $\psi_{js}(x)\psi_{js}^+(x)$ (Apostol 1982, 1983):

$$\begin{aligned}
\psi_{js}^+(x \pm \mathrm{i}\alpha/2)\psi_{js}(x \mp \mathrm{i}\,\alpha/2) &= 1/2\pi\alpha + L^{-1}B_{js} + (2\pi)^{-1}(F_{js}(x) + F_{js}^+(x)) \\
&\quad + \pi\alpha : [L^{-1}B_{js} + (2\pi)^{-1}(F_{js}(x) + F_{js}^+(x))]^2 : + \mathrm{O}(\alpha^2) \\
\psi_{js}(x \pm \mathrm{i}\,\alpha/2)\psi_{js}^+(x \mp \mathrm{i}\,\alpha/2) &= 1/2\pi\alpha - L^{-1}B_{js} - (2\pi)^{-1}(F_{js}(x) + F_{js}^+(x)) \\
&\quad + \pi\alpha : [L^{-1}B_{js} + (2\pi)^{-1}(F_{js}(x) + F_{js}^+(x))]^2 : + \mathrm{O}(\alpha^2)
\end{aligned} \tag{2.2}$$

where the upper (lower) sign corresponds to $j = 1$ (2). In (2.2) the 'charge operators' B_{js} (Krönig 1935) are given by

$$B_{js} = \int \mathrm{d}x : \psi_{js}^+(x)\psi_{js}(x) :$$

and

$$F_{js}(x) = 2\pi L^{-1} \sum_{k>0} \rho_{js}(\mp k) \exp(\pm \mathrm{i}\,kx)$$

where $\rho_{js}(\mp k)$ are the Fourier components of the particle density. The normal ordering in (2.2) refers to $\rho_{js}(\mp k)$ which satisfy boson-like commutation relations (Tomonaga 1950, Mattis and Lieb 1965) and $\rho_{js}(\mp k)|0\rangle = 0$. These boson-like commutation relations make possible the boson representation (Jordan 1935, 1936a, b, 1937, Luther and Peschel 1974, Mattis 1974, Heidenreich *et al* 1975, Haldane 1979, 1981) of the fermion field operators

$$\begin{aligned}
\psi_{js}(x) &= c_{js} L^{-1/2} S_{js}^{\mp 1} \exp[\pm \mathrm{i}\, 2\pi L^{-1}(B_{js} - \tfrac{1}{2})x] \\
&\quad \times \exp(-\Omega_{js}^+(x)) \exp(\Omega_{js}(x))
\end{aligned} \tag{2.3}$$

where the coefficients c_{js} have been given by Apostol (1983), $S_{js} = \exp(\mathrm{i}\,N_{js})$, $N_{js}^+ = N_{js}$, $[N_{js}, B_{j's'}] = \pm \mathrm{i}\,\delta_{jj'}\delta_{ss'}$ and

$$\Omega_{js}(x) = 2\pi L^{-1} \sum_{k>0} k^{-1}\rho_{js}(\mp k) \exp(\pm \mathrm{i}kx).$$

All the commutation relations among the operators $\psi_{js}(x)$, $\rho_{js}(\mp k)$, B_{js} and H_0, including the Jordan commutator obtained from (2.2) and given by

$$[\psi^+_{js}(x), \psi_{js}(x)] = \lim_{\alpha\to 0} [\psi^+_{js}(x \pm \mathrm{i}\alpha/2)\psi_{js}(x \mp \mathrm{i}\,\alpha/2) - \psi_{js}(x \pm \mathrm{i}\,\alpha/2)\psi^+_{js}(x \mp \mathrm{i}\,\alpha/2)]$$
$$= 2L^{-1}B_{js} + \pi^{-1}(F_{js}(x) + F^+_{js}(x)) \quad (2.4)$$

are satisfied by the boson representation (2.3) provided that the Jordan point-splitting regularisation

$$\begin{aligned}\psi^+_{js}(x)\psi_{js}(y) &\to \psi^+_{js}(x \pm \mathrm{i}\,\alpha/2)\psi_{js}(y \mp \mathrm{i}\,\alpha/2)\\ \psi_{js}(x)\psi^+_{js}(y) &\to \psi_{js}(x \pm \mathrm{i}\,\alpha/2)\psi^+_{js}(y \mp \mathrm{i}\,\alpha/2)\end{aligned} \quad (2.5)$$

is used, the limit $\alpha \to 0$ being taken whenever possible.

It is worthwhile mentioning here that the Jordan boson representation (2.3) differs from the boson representation given by Luther and Peschel (1974) not only in the normal ordering of the boson operators but also in the fact that the cut-off α^{-1} is absent in (2.3) unlike the ultraviolet cut-off which explicitly appears in the boson representation given by Luther and Peschel (1974). The specific use of the band-width cut-off α^{-1} is required by the original fermion problem (in particular the fulfilment of (2.2)) and is prescribed by the point-splitting regulation (2.5) (Apostol 1983).

If the Jordan boson representation (2.3) and the point-splitting regulations (2.5) are made use of, the bosonised form of the kinetic Hamiltonian (2.1) is obtained:

$$H_0 = \pi L^{-1} v_{\mathrm{F}} \sum_{js} B^2_{js} + 2\pi L^{-1} v_{\mathrm{F}} \sum_{js,k>0} \rho^+_{js}(\mp k)\rho_{js}(\mp k) \quad (2.6)$$

a relationship that is often called the 'Krönig identity' (Uhlenbrok 1967, Dover 1968, Heidenreich *et al* 1980). The standard form of the bosonised interaction Hamiltonian of the TFM can be obtained by starting with the following interaction Hamiltonian of the FGM:

$$\begin{aligned}H_{\mathrm{i}} = (g_{2\|} - g_{1\|}) \sum_s \int & \mathrm{d}x\,\mathrm{d}y\, \psi^+_{1s}(x)\psi_{1s}(x)v(x-y)\psi^+_{2s}(y)\psi_{2s}(y)\\ - g_{1\perp} \int & \mathrm{d}x\, (\psi^+_{11}(x)\psi_{1-1}(x)\psi^+_{2-1}(x)\psi_{21}(x) + \mathrm{HC})\\ + g_{2\perp} \sum_s \int & \mathrm{d}x\,\mathrm{d}y\, \psi^+_{1s}(x)\psi_{1s}(x)v(x-y)\psi^+_{2-s}(y)\psi_{2-s}(y)\\ + g_3 \int & \mathrm{d}x\, (\psi^+_{11}(x)\psi^+_{1-1}(x)\psi_{2-1}(x)\psi_{21}(x) + \mathrm{HC})\\ + \tfrac{1}{2}g_{4\|} \sum_{js} \int & \mathrm{d}x\,\mathrm{d}y\, \psi^+_{js}(x)\psi_{js}(x)v(x-y)\psi^+_{js}(y)\psi_{js}(y)\\ + \tfrac{1}{2}g_{4\perp} \sum_{js} \int & \mathrm{d}x\,\mathrm{d}y\, \psi^+_{js}(x)\psi_{js}(x)v(x-y)\psi^+_{j-s}(y)\psi_{j-s}(y)\end{aligned} \quad (2.7)$$

where the reciprocal-lattice vector has been omitted in the Umklapp scattering term g_3 and the potential function is

$$v(x) = \frac{1}{2\pi}\frac{r}{x^2 + (r/2)^2}$$

r^{-1} being therefore the momentum-transfer cut-off. Indeed, replacing the quantities $\psi_{js}^{+}(x)\psi_{js}(x)$ in (2.7) by

$$\lim_{\alpha\to 0}[\psi_{js}^{+}(x\pm \mathrm{i}\alpha/2)\psi_{js}(x\mp \mathrm{i}\alpha/2)-\langle 0|\psi_{js}^{+}(x\pm \mathrm{i}\alpha/2)\psi_{js}(x\mp \mathrm{i}\alpha/2)|0\rangle]$$
$$= L^{-1}B_{js} + (2\pi)^{-1}(F_{js}(x) + F_{js}^{+}(x)) \tag{2.8}$$

one obtains

$$\begin{aligned} H_{\mathrm{i}}^{\mathrm{b}} = {} & (g_{2\|} - g_{1\|})L^{-1}\sum_{s} B_{1s}B_{2s} + g_{2\perp}L^{-1}\sum_{s} B_{1s}B_{2-s} \\ & + \tfrac{1}{2}g_{4\|}L^{-1}\sum_{js} B_{js}^{2} + \tfrac{1}{2}g_{4\perp}L^{-1}\sum_{js} B_{js}B_{j-s} \\ & + (g_{2\|} - g_{1\|})L^{-1}\sum_{s,k>0} v(k)(\rho_{1s}(-k)\rho_{2s}(k) + \mathrm{HC}) \\ & + g_{2\perp}L^{-1}\sum_{s,k>0} v(k)(\rho_{1s}(-k)\rho_{2-s}(k) + \mathrm{HC}) \\ & + g_{4\|}L^{-1}\sum_{js,k>0} v(k)\rho_{js}^{+}(\mp k)\rho_{js}(\mp k) \\ & + g_{4\perp}L^{-1}\sum_{js,k>0} v(k)\rho_{js}^{+}(\mp k)\rho_{j-s}(\mp k) \\ & - g_{1\perp}\int \mathrm{d}x\,(\psi_{11}^{+}(x)\psi_{1-1}(x)\psi_{2-1}^{+}(x)\psi_{21}(x) + \mathrm{HC}) \\ & + g_{3}\int \mathrm{d}x\,(\psi_{11}^{+}(x)\psi_{1-1}^{+}(x)\psi_{2-1}(x)\psi_{21}(x) + \mathrm{HC}) \end{aligned} \tag{2.9}$$

where $v(k) = \exp(-r|k|/2)$ is the Fourier transform of $v(x)$.

As the bosonised TFM given by (2.6) and (2.9) will be discussed in relation to the corresponding FGM given by (2.1) and (2.7), a few comments on the latter are in order here. Significant differences occur (Haldane 1979) between the results from this FGM and those corresponding to the original FGM (Dzyaloshinsky and Larkin 1971, Grest *et al* 1976, Sólyom 1979), the most important arising from the $g_{1\|}$ and $g_{1\perp}$ terms which do not represent a true back-scattering contribution. A momentum transfer r^{-1} has been introduced in the $g_{1\|}$ term in (2.7) via the potential $v(x)$ and in this respect the present FGM resembles the model proposed by Rezayi *et al* (1979, 1981) and Grinstein *et al* (1979). However, a contact (zero-range) interaction has been allowed for in the $g_{1\perp}$ term in (2.7) in contrast with the model used by the above-mentioned researchers.

Perturbation calculations can be performed with the interaction Hamiltonian (2.7), the divergent quantities being regularised by means of both the momentum-transfer cut-off r^{-1} and the band-width cut-off α^{-1}. The latter enters the calculations quite simply via the free Green functions

$$\begin{aligned} G_{js}^{0}(x,t) &= -\mathrm{i}\,\langle 0|T\psi_{js}[x \pm \mathrm{i}\,\alpha(t)/2, t]\psi_{js}^{+}[\mp \mathrm{i}\,\alpha(t)/2, 0]|0\rangle \\ &= (1/2\pi)[\pm x - v_{\mathrm{F}}t + \mathrm{i}\,\alpha(t)]^{-1} \end{aligned} \tag{2.10}$$

where $\alpha(t) = \alpha \operatorname{sgn} t$ and the point-splitting regularisation (2.5) has been used. Indeed, let us calculate the first-order contributions of the first two terms in (2.7) to the charge-density correlation function:

$$N(x, t) = N_1(x, t) + N_2(x, t)$$
$$N_1(x, t) = -2\,\mathrm{i}\langle\tilde{0}|T\psi^+_{21}(x, t)\psi_{11}(x, t)\psi^+_{11}(0, 0)\psi_{21}(0, 0)|\tilde{0}\rangle \tag{2.11}$$
$$N_2(x, t) = -2\,\mathrm{i}\,\langle\tilde{0}|T\psi^+_{21}(x, t)\psi_{11}(x, t)\psi^+_{1-1}(0, 0)\psi_{2-1}(0, 0)|\tilde{0}\rangle$$

$|\tilde{0}\rangle$ being the exact ground state. The zeroth-order expression of $N(x, t)$, i.e.

$$N_0(x, t) = N^0_1(x, t) = -2\,\mathrm{i}\,G^0_{11}(x, t)G^0_{21}(-x, -t) \tag{2.12}$$

has the Fourier transform (see Appendix)

$$N_0(\omega) = -2\,\mathrm{i}\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\,G^0_{11}(x, t) \times G^0_{21}(-x, -t) \underset{\alpha\omega/v_\mathrm{F}\to 0}{\propto} (1/\pi v_\mathrm{F})\ln(\alpha\omega/v_\mathrm{F}). \tag{2.13}$$

The $g_{1\perp}$ contribution to $N_2(x, t)$ can be obtained straightforwardly by means of the standard perturbation calculations:

$$N^1_2(x, t) = -2g_{1\perp}\int \mathrm{d}x'\,\mathrm{d}t\,G^0_{1-1}(x', t')G^0_{2-1}(-x', -t') \times G^0_{11}(x - x', t - t')G^0_{21}(x' - x, t' - t) \tag{2.14}$$

where the point-splitting regularisation is used in the free Green functions by means of the band-width cut-off α^{-1}. The Fourier transform of (2.14) is

$$N^1_2(\omega) = \tfrac{1}{2}g_{1\perp}N^2_0(\omega) \underset{\alpha\omega/v_\mathrm{F}\to 0}{\propto} (g_{1\perp}/2\pi^2 v^2_\mathrm{F})\ln^2(\alpha\omega/v_\mathrm{F}). \tag{2.15}$$

In contrast with (2.15), which is controlled only by the band-width cut-off α^{-1}, the $g_{1\parallel}$ contribution depends on both the α^{-1} and the momentum-transfer cut-off r^{-1}. In the limit $\alpha \to 0$, one obtains

$$N^1_1(\omega) \underset{r\omega/4v_\mathrm{F}\to 0}{\propto} [(g_{1\parallel} - g_{2\parallel})/2(2\pi v_\mathrm{F})^2]\ln^2(r\omega/4v_\mathrm{F}) \tag{2.16}$$

a result that agrees with those obtained by Rezayi *et al* (1979). In the opposite limit $r \to 0$, the perturbation calculations yield

$$N^1_1(\omega) \underset{\alpha\omega/v_\mathrm{F}\to 0}{\propto} [(g_{1\parallel} - g_{2\parallel})/2\pi^2 v^2_\mathrm{F}]\ln^2(\alpha\omega/v_\mathrm{F}). \tag{2.17}$$

One can see from (2.15) and (2.17) that the band-width cut-off α^{-1} applies in the same way to both $g_{1\perp}$ and $g_{1\parallel} - g_{2\parallel}$ contributions, as might be expected from the form of the first two terms in (2.7) which differ only in their spin orientations when $r \to 0$ (Grest 1976).

It should be emphasised at this point that the quantities $\psi^+_{js}(x)\psi_{js}(x)$ are not regularised in the Hamiltonian (2.7) but, instead, the regularisation is achieved via the free Green functions according to (2.10). This is quite different from the bosonised Hamiltonian (2.9) where the $g_{1\parallel} - g_{2\parallel}$ term is no longer of the same type as it is in (2.7); it has been regularised by means of the point-splitting procedure (2.8) while the $g_{1\perp}$ term has been left unchanged. Consequently, it is reasonable to expect differences in their contributions to the perturbation calculations; the contributions arising from the $g_{1\parallel} - g_{2\parallel}$ term in (2.9) will be regularised by means of both the momentum-transfer cut-off r^{-1} and the band-width cut-off α^{-1} while those corresponding to the $g_{1\perp}$ term in (2.9) will be regularised solely by the band-width cut-off α^{-1}, as in the FGM (2.7). Similar differences will occur also between the $g_{1\parallel} - g_{2\parallel}$ contributions of the bosonised TFM (2.9) and those of the corresponding FGM (2.7). In the former the quantities $\psi_{js}^{+}(x)\psi_{js}(x)$ have been replaced by (2.8). Consequently, they are expressed in terms of B_{js} and $\rho_{js}(\mp k)$ operators; the fermion fields entering them can no longer be separated from one another. In contrast, the fermion fields in the FGM (2.7) may participate in any contraction involved in perturbation calculations, the regularisation being achieved via the free Green functions (2.10). These differences are explicitly shown in § 5 for the first-order perturbation calculations of the charge-density correlation function $N(x, t)$ within the TFM. They indicate that the FGM (2.7) and the bosonised TFM (2.9), are not equivalent; in fact, this is already evident in the mathematical expressions (2.7) and (2.9) of the two Hamiltonians.

3. Correlation functions of the Tomonaga–Luttinger model

The Green function

$$G_{1s}(x,t) = -\mathrm{i}\langle\tilde{0}|T\psi_{1s}[x+\mathrm{i}\,\alpha(t)/2,t]\psi_{1s}^{+}[-\mathrm{i}\alpha(t)/2,0]|\tilde{0}\rangle$$

of the Tomonaga–Luttinger model $H_{\mathrm{TL}} = H_0 + H_{\mathrm{i}}^{\mathrm{b}}(g_{1\parallel} = g_{1\perp} = g_3 = 0)$ can be calculated by using the boson representation (2.3) provided that the standard canonical transformations (Mattis and Lieb 1965, Luther and Emery 1974) that diagonalise this model are extended to include the zero-mode contributions also. Straightforward calculations lead to

$$\begin{aligned} G_{1s}(x,t) = (1/2\pi)&[(x-u_{\rho}^{0}t+\mathrm{i}\alpha(t))(x-u_{\sigma}^{0}t+\mathrm{i}\alpha(t))]^{-1/2}\\ &\times[(x-u_{\rho}^{0}t+\mathrm{i}r(t))(x-u_{\sigma}^{0}t+\mathrm{i}\,r(t))]^{1/2}\\ &\times[(x-u_{\rho}t+\mathrm{i}\,r(t))(x-u_{\sigma}t+\mathrm{i}\,r(t))]^{-1/2}\\ &\times[r^{-2}(x-u_{\rho}t+\mathrm{i}\,r(t))(x+u_{\rho}t-\mathrm{i}\,r(t))]^{-\alpha_{\rho}}\\ &\times[r^{-2}(x-u_{\sigma}t+\mathrm{i}\,r(t))(x+u_{\sigma}t-\mathrm{i}\,r(t))]^{-\alpha_{\sigma}} \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &\alpha(t) = \alpha\operatorname{sgn} t \qquad r(t) = r\operatorname{sgn} t \qquad \alpha_{\rho,\sigma} \approx \tfrac{1}{2}(\gamma_{\rho,\sigma}/2v_{\mathrm{F}})^2\\ &u_{\rho,\sigma}^{0} = v_{\mathrm{F}} + (2\pi)^{-1}(g_{4\parallel} \pm g_{4\perp}) \qquad u_{\rho,\sigma}^{2} = u_{\rho,\sigma}^{02} - \gamma_{\rho,\sigma}^{2}\\ &\gamma_{\rho,\sigma} = (2\pi)^{-1}(g_{2\parallel} \pm g_{2\perp}) \end{aligned} \tag{3.2}$$

and the k dependence of $\gamma_{\rho,\sigma} \sim \exp(-rk/2)$ has explicitly been used.

It can be seen that the Green function (3.1) calculated by means of the boson representation (2.3) and point-splitting regularisation (2.5) contains both the momentum-transfer cut-off r^{-1} and the band-width cut-off α^{-1} and reproduces the result obtained by diagram summation (Dzyaloshinsky and Larkin 1973, Sólyom 1979). The same result can be obtained by using the previous boson representations (Luther and Peschel 1974, Mattis 1974, Haldane 1979, 1981) provided that the two cut-offs α^{-1} and r^{-1} are included and, in fact, a particular form of (3.1) has already been obtained by Luther and Peschel (1974).

The charge- and spin-density correlation functions can be obtained in the same way. We limit ourselves here to giving only the results of these calculations performed with the present bosonisation technique:

$$\begin{aligned} N(x,t) &= -2\,\mathrm{i}\,\langle\tilde{0}|T\psi_{21}^{+}(x,t)\psi_{11}(x,t)\psi_{11}^{+}(0,0)\psi_{21}(0,0)|\tilde{0}\rangle \\ &= -2\mathrm{i}\,G_{11}(x,t)G_{21}(-x,-t)(g_\rho(x,t))^{\beta_\rho}(g_\sigma(x,t))^{\beta_\sigma} \\ \chi(x,t) &= -2\,\mathrm{i}\langle\tilde{0}|T\psi_{21}^{+}(x,t)\psi_{1-1}(x,t)\psi_{1-1}^{+}(0,0)\psi_{21}(0,0)|\tilde{0}\rangle \\ &= -2\,\mathrm{i}\,G_{11}(x,t)G_{21}(-x,-t)(g_\rho(x,t))^{\beta_\rho}(g_\sigma(x,t))^{-\beta_\sigma} \\ \Delta_s(x,t) &= -2\,\mathrm{i}\langle\tilde{0}|T\psi_{21}(x,t)\psi_{1-1}(x,t)\psi_{1-1}^{+}(0,0)\psi_{21}^{+}(0,0)|\tilde{0}\rangle \\ &= 2\,\mathrm{i}\,G_{11}(x,t)G_{21}(x,t)(g_\rho(x,t))^{-\beta_\rho}(g_\sigma(x,t))^{\beta_\sigma} \\ \Delta_t(x,t) &= -2\,\mathrm{i}\langle\tilde{0}|T\psi_{21}(x,t)\psi_{11}(x,t)\psi_{11}^{+}(0,0)\psi_{21}^{+}(0,0)|\tilde{0}\rangle \\ &= 2\,\mathrm{i}\,G_{11}(x,t)G_{21}(x,t)(g_\rho(x,t))^{-\beta_\rho}(g_\sigma(x,t))^{-\beta_\sigma} \end{aligned} \tag{3.3}$$

where

$$g_{\rho,\sigma}(x,t) = r'^{-2}(x - u_{\rho,\sigma}t + \mathrm{i}r'(t))(x + u_{\rho,\sigma}t - \mathrm{i}r'(t)).$$

Also $r'(t) = \frac{1}{2}r\,\mathrm{sgn}\,t$ and $\beta_{\rho,\sigma} = \gamma_{\rho,\sigma}/2u_{\rho,\sigma}$. Similar results are obtained for the $4k_F$ correlation function (Klemm and Larkin 1979).

The common feature of these correlation functions (including the Green functions) is the dissociation between the band-width cut-off α^{-1} and the momentum-transfer cut-off r^{-1}; the former occurs, as expected, only in the non-interaction contributions to the correlation functions while the latter is associated exclusively with their interacting parts.

4. Back-scattering and Umklapp scattering Hamiltonians

When the boson representation (2.3) is made use of, the $g_{1\perp}$ term in (2.9) becomes

$$\begin{aligned} h_\sigma(x) &= \psi_{11}^{+}(x)\psi_{1-1}(x)\psi_{2-1}^{+}(x)\psi_{21}(x) \\ &= c_{1\sigma}^{+}c_{2\sigma}c_{1\rho}c_{2\rho}L^{-2}\exp[2^{1/2}\,\mathrm{i}(N_{1\sigma}+N_{2\sigma})] \\ &\quad\times\exp[-\mathrm{i}\,2^{3/2}\pi L^{-1}(B_{1\sigma}+B_{2\sigma})x]\exp(2^{1/2}\Omega_{1\sigma}^{+}(x)) \\ &\quad\times\exp(-2^{1/2}\Omega_{1\sigma}(x))\exp(-2^{1/2}\Omega_{2\sigma}^{+}(x))\exp(2^{1/2}\Omega_{2\sigma}(x)) \end{aligned} \tag{4.1}$$

where the coefficients $c_{j\rho,\sigma}$ are given by $c_{1\rho} = c_{2-1}^{+}, c_{2\rho} = c_{21}, c_{1\sigma} = c_{11}, c_{2\sigma} = c_{1-1}$ (Apostol 1983), $B_{j\sigma} = (1/\sqrt{2})(B_{j1} - B_{j-1})$, $N_{j\sigma} = (1/\sqrt{2})(N_{j1} - N_{j-1})$ and $\sigma_j(\mp k) = (1/\sqrt{2})(\rho_{j1}(\mp k) - \rho_{j-1}(\mp k))$. When the projection of $h_\sigma(x)$ on $|\phi\rangle_{\varphi_1=\varphi_2=0}$ is taken (according to Appendix 1 of Apostol 1983) the product $c_{1\rho}c_{2\rho}$ can be replaced by 1, so

$h_\sigma(x)$ depends only on the σ degrees of freedom. Under the canonical transformation $\exp(S^\sigma + S^0)$,

$$S_\sigma = 2\pi L^{-1} \sum_{k>0} k^{-1}\varphi_\sigma(k)(\sigma_1(-k)\sigma_2(k) - \sigma_2^+(k)\sigma_1^+(-k))$$

$$S_\sigma^0 = \mathrm{i}\,\varphi_\sigma(0)(N_{2\sigma}B_{1\sigma} - N_{1\sigma}B_{2\sigma})$$

$h_\sigma(x)$ becomes

$$\begin{aligned}\tilde{h}_\sigma(x) = c_{1\sigma}^+ c_{2\sigma} L^{-2} &\exp[2^{1/2}\,\mathrm{i}\,\mathrm{e}^{\varphi_\sigma(0)}(N_{1\sigma} + N_{2\sigma})]\\ &\times \exp[-\mathrm{i}\,2^{3/2}\pi L^{-1}\,\mathrm{e}^{\varphi_\sigma(0)}(B_{1\sigma} + B_{2\sigma})x]\\ &\times \exp\Big(-8\pi L^{-1}\sum_{k>0} k^{-1} w_\sigma(w_\sigma + v_\sigma)\Big)\exp(\tilde{\Omega}_{1\sigma}^+(x))\\ &\times \exp(-\tilde{\Omega}_{1\sigma}(x))\exp(-\tilde{\Omega}_{2\sigma}^+(x))\exp(\tilde{\Omega}_{2\sigma}(x))\end{aligned} \tag{4.2}$$

where $v_\sigma = \cosh\ \varphi_\sigma(k)$, $w_\sigma = \sinh\ \varphi_\sigma(k) \sim \exp(-rk/2)$, $\tanh 2\varphi_\sigma(k) = -\gamma_k/u_\sigma^0$, $k \geq 0$ and

$$\tilde{\Omega}_{j\sigma}(x) = 2\pi L^{-1}\sum_{k>0} 2^{1/2}\,k^{-1}\exp(\varphi_\sigma(k) \pm \mathrm{i}kx)\,\sigma_j(\mp k).$$

It should be emphasised at this point that $\tilde{h}_\sigma(x)$ does not contain either the ultraviolet cut-off α^{-1} (Luther and Emery 1974) or the band-width cut-off α^{-1} given by the present point-splitting regularisation. The divergences resulting from the non-normal ordering of the boson operators in (4.1) have been regularised in (4.2) solely by the momentum-transfer cut-off r^{-1} contained in

$$\exp\Big(-8\pi L^{-1}\sum_{k>0} k^{-1} w_\sigma(w_\sigma + v_\sigma)\Big).$$

The main contribution of these divergences is obtained by omitting the dependence on r in $\tilde{\Omega}_{j\sigma}(x)$ (but not in the exponential pre-factor) and setting $2^{1/2}\exp(\varphi_\sigma(k)) = 1$ for any $k \geq 0$. We get then

$$\begin{aligned}\exp\Big(-8\pi L^{-1}&\sum_{k>0} k^{-1} w_\sigma(w_\sigma + v_\sigma)\Big)\\ &= \exp\Big(3\pi L^{-1}\sum_{k>0} k^{-1}\,\mathrm{e}^{-rk/2}\Big)\exp\Big(-\pi L^{-1}\sum_{k>0} k^{-1}\,\mathrm{e}^{-rk}\Big)\\ &= 2^{1/2}L/\pi r\end{aligned} \tag{4.3}$$

so we have

$$\tilde{h}_\sigma(x) = (2^{1/2}/\pi r)\psi_{1\sigma}^+(x)\psi_{2\sigma}(x)$$

where the boson representation has been used again to recover the fermion fields. The back-scattering Hamiltonian can now be diagonalised in terms of two types of free fermion with energy spectrum $\pm\lambda_\sigma(p)$, $\lambda_\sigma(p) = \operatorname{sgn} p\,(u_\sigma^2 p^2 + \Delta_\sigma^2)^{1/2}$, where the gap $\Delta_\sigma = 2^{1/2}|g_{1\perp}|/\pi r$ depends on the momentum-transfer cut-off r^{-1}. A very similar result is obtained when the Umklapp scattering term g_3 is included, the corresponding gap being $\Delta_\rho = 2^{1/2}|g_3|/\pi r$.

5. The charge-density correlation function

The contribution from the $g_{1\perp}$ term in (2.9) to the charge-density correlation function $N_2(x, t)$ given by (2.11) can be obtained straightforwardly by using the boson representation and point-splitting regularisation. As expected, the result is identical with (2.15) which corresponds to the FGM. The structure of the $g_{1\|} - g_{2\|}$ contribution from the bosonised Hamiltonian (2.9) to $N_1(x, t)$ is different from (2.14). Straightforward perturbation calculations performed with either the fermion or the boson representation of the fermion fields yield

$$N_1^1(x, t) = -2\,\mathrm{i}\, G_{11}^0(x, t) G_{21}^0(-x, -t)(g_{2\|} - g_{1\|})/2\pi v_\mathrm{F} \times \ln[r'^{-2}(x - v_\mathrm{F}t + \mathrm{i}\, r'(t))(x + v_\mathrm{F}t - \mathrm{i}\, r'(t))] \tag{5.1}$$

with $r'(t) = \frac{1}{2} r \operatorname{sgn} t$. In obtaining (5.1), use has been made of the two cut-offs; the non-interaction divergences in the free Green functions have been regularised by the band-width cut-off α^{-1} according to the point-splitting prescription while the divergences from the interaction Hamiltonian have been regularised by the momentum-transfer cut-off r^{-1} already included in this Hamiltonian according to (2.9). Exactly the same result is obtained by expanding $N(x, t)$ given by (3.3) in powers of the coupling constants if allowance is made for the $g_{1\|}$ contributions. Indeed, we obtain from (3.3)

$$N_1^1(x,t) = -2\mathrm{i}\, G_{11}(x,t) G_{21}(-x,-t)(\beta_\rho \ln g_\rho(x,t) + \beta_\sigma \ln g_\sigma(x,t))$$

which, to first order in the coupling constants and neglecting $g_{4\|}$ and $g_{4\perp}$, is exactly (5.1). The Fourier transform of (5.1) is now (see the Appendix)

$$N_1^1(\omega) \underset{\alpha\omega/v_\mathrm{F}\to 0}{\propto} [(g_{2\|} - g_{1\|})/(2\pi v_\mathrm{F})^2] \ln^2(\alpha\omega/2v_\mathrm{F}) \tag{5.2}$$

which differs from both the $g_{1\perp}$ contribution (2.15) of the bosonised TFM and the $g_{1\|} - g_{2\|}$ contributions (2.16) and (2.17) of the corresponding FGM. The discrepancy between (5.2) and (2.15) results from the two different ways in which the band-width cut-off α^{-1} applies to the $g_{1\perp}$ and the $g_{1\|} - g_{2\|}$ contributions. This discrepancy has previously been pointed out by Grest (1976) for the ultraviolet cut-off α^{-1} in the Luther and Peschel (1974) boson representation. Consequently, it is not related to the bosonisation technique but, instead, it arises from the different forms that the $g_{1\|} - g_{2\|}$ and $g_{1\perp}$ terms take in the bosonised TFM (2.9) (in particular, they are not spin rotationally invariant). The difference between (5.2) and (2.16) is that the former is controlled by the band-width cut-off α^{-1} while the latter contains the momentum-transfer cut-off r^{-1}. The band-width cut-off α^{-1} appears in (2.17) although the opposite limiting process ($r \to 0$, α finite) has been used in deriving (2.17) as compared with that employed in determining (5.2) ($\alpha \to 0$, r finite). As has been emphasised in § 2, the difference between (5.2) and (2.16) arises from the different forms that the $g_{1\|} - g_{2\|}$ terms take in the bosonised TFM (2.9) and the corresponding FGM (2.7). In the bosonised TFM the $g_{1\|} - g_{2\|}$ term is expressed by means of the B_{js} and $\rho_{js}(\mp k)$ operators which come from the quantities $\psi_{js}^+(x)\psi_{js}(x)$ via a definite point-splitting regularisation according to (2.8). This regularisation is not pre-included in the FGM (2.7); it appears during the perturbation calculations with every free Green function (according to (2.10)), e.g. (2.14). The discrepancy between (5.2) and (2.16) compared with the identical results (2.15) for the $g_{1\perp}$ contributions to the charge-density correlation function shows that the bosonised TFM (2.9) and the corresponding

FGM (2.7) are not equivalent. These are two distinct models, each of them requiring its own regularisation procedure.

6. Conclusions

The consistent way of using the Jordan boson representation of fermion fields in one dimension has been shown by applying it to the bosonised TFM with the momentum-transfer cut-off and zero-mode terms included. It involves the band-width cut-off α^{-1} as required by the point-splitting regularisation of the divergencies arising from the infinite particle density of the model and the momentum-transfer cut-off r^{-1} which regularises the remaining divergences from the interaction. The consistency of the calculations is ensured by letting α go to zero whenever possible while r is kept finite (Apostol 1983). It has been shown that, in contrast with the use of the ultraviolet cut-off in current practice, the present regularisation procedure leads to the same mathematical expressions for the correlation functions of the Tomonaga–Luttinger model as those obtained by diagram summation. In addition, the energy gaps in the back-scattering and Umklapp scattering models acquire the desired dependence on r^{-1} instead of on α^{-1}. Consequently, the method used by Emery *et al* (1976) to obtain the scaling equations of the coupling constants for these models can no longer be applied; instead one has to resort to another approach which scales the coupling constants with the momentum-transfer cut-off r^{-1} instead of with α^{-1} (Apostol *et al* 1984, 1985). Further elaboration on this point, which will be given elsewhere, will provide a firm basis on which a direct comparison can be made between the scaling results of the bosonised TFM and FGM (Grest *et al* 1976, Grest 1976, Emery 1979, Rezayi *et al* 1979). As regards the latter, it has been shown that the divergences due to the quantities $\psi^{+}_{js}(x)\psi_{js}(x)$ are regularised by the band-width cut-off α^{-1} via the free Green functions appearing in the perturbation calculations and, consequently, there is almost no need for a momentum-transfer cut-off. The situation is quite different from that encountered with the bosonised TFM where these quantities are regularised directly into the Hamiltonian which, in turn, requires the presence of the momentum-transfer cut-off. A two-cut-off problem is obtained therefore as in the original formulations of the one-dimensional many-fermion system, although the true back-scattering character of the interaction is lost. The FGM discussed in the present paper includes the momentum-transfer cut-off in the ($g_{1\parallel} - g_{2\parallel}$ term as in the model proposed by Rezayi *et al* (1979) and Grinstein *et al* (1979); however, in contrast with this latter model a contact (zero-range) interaction has been allowed for in the $g_{1\perp}$ term. First-order perturbation calculations yield the same $g_{1\perp}$ contribution to the charge-density correlation function for both the FGM and the bosonised TFM. The expression for this is controlled by the band-width cut-off α^{-1}. In contrast, the $g_{1\parallel} - g_{2\parallel}$ contribution (2.16) of the FGM contains the momentum-transfer cut-off r^{-1} (in agreement with the results obtained by Rezayi *et al* 1979) while the corresponding contribution (5.2) of the bosonised TFM is controlled by the band-width cut-off α^{-1}. This indicates that the two models are not equivalent; this arises because the two distinct Hamiltonians (2.7) and (2.9) require distinct regularisation techniques. In fact, this non-equivalence has already been suggested in the literature. It is well known that the FGM (2.7) with the momentum-transfer cut-off r^{-1} included in the $g_{1\perp}$ term (Rezayi *et al* 1979) separates the charge-density degrees of freedom from the spin-density degrees of freedom (in the scaling equations of the coupling constants) as the bosonised TFM does. However, this separation is achieved in the latter model only by letting r go to zero in the $g_{1\perp}$ term (Luther and

Emery 1974), as we did in the present paper. This point has been emphasised by Grinstein *et al* (1979) who showed that the partition function of the bosonised TFM (corresponding to a two-dimensional Coulomb gas) is obtained only by letting r go to zero in the $g_{1\perp}$ term. This is a characteristic feature of the regularisation required by the bosonisation technique (the functional integration method, of Fogedby (1976), employed by Grinstein *et al* (1979), is, in fact, simply the bosonisation technique) in contrast with the regularisation (via the free Green functions) required by the FGM.

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Appendix. The Fourier transforms of the various contributions to the charge-density correlation function $N(x, t)$

The Fourier transform (2.13) of the zeroth-order contribution $N_0(x, t)$ to the charge-density correlation function can easily be obtained by performing the integration over x over the upper half-plane:

$$N_0(\omega) = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{1}{-2v_\mathrm{F}t + 2\mathrm{i}\alpha} + (\omega \to -\omega).$$

Changing the t-integration path to the lower imaginary semi-axis we get

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{1}{-2v_\mathrm{F}t + 2\mathrm{i}\alpha} = \frac{1}{2\pi v_\mathrm{F}} \mathrm{e}^{\alpha\omega|v_\mathrm{F}} \,\mathrm{Ei}(-\alpha\omega|v_\mathrm{F}) \underset{\alpha\omega|v_\mathrm{F}\to 0}{\propto} \frac{1}{2\pi v_\mathrm{F}} \ln(\alpha\omega/v_\mathrm{F})$$

where Ei is the exponential integral (Erdélýi 1953). Equation (2.13) can then be readily obtained.

The Fourier transform of $N_1^1(x, t)$ given by (5.1) can be written as

$$N_1^1(\omega) = \frac{g_{2\parallel} - g_{1\parallel}}{2\pi v_\mathrm{F}}(-2\ln r' N_0(\omega) + I_1(\omega) + I_1(-\omega) + I_2(\omega) + I_2(-\omega))$$

where

$$I_1(\omega)\frac{1}{\pi}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \frac{\ln(-2v_\mathrm{F}t + \mathrm{i}r' + \mathrm{i}\alpha)}{-2v_\mathrm{F}t + 2\mathrm{i}\alpha}$$

$$I_2(\omega) = -\frac{\mathrm{i}}{2\pi^2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathrm{i}\omega t} \int \mathrm{d}x \frac{\ln(x + v_\mathrm{F}t - \mathrm{i}r')}{(x + v_\mathrm{F}t - \mathrm{i}\alpha)(x - v_\mathrm{F}t + \mathrm{i}\alpha)}.$$

The integral $I_1(\omega)$ can straightforwardly be performed in the limit $\alpha \to 0$ and holding r finite, as required by the regularisation procedure. Following a technique similar to that used above for obtaining $N_0(\omega)$ we get

$$I_1(\omega) \underset{\alpha\omega/v_\mathrm{F}\to 0}{\propto} \frac{1}{2\pi v_\mathrm{F}} \ln(\mathrm{i}r') \ln(\alpha\omega/v_\mathrm{F}).$$

The contour of integration over x in $I_2(\omega)$ should encircle the $-v_F t + i\alpha$ pole and the slit $(0, r')$ and should close over the upper half-plane. In the limit $\alpha \to 0$ the leading contributions (real part) to $I_2(\omega)$ are

$$I_2(\omega) \underset{\alpha\omega/v_F \to 0}{\propto} \frac{1}{2\pi v_F} \ln(-i\, r') \ln\left(\frac{\alpha\omega}{v_F}\right) + \frac{1}{4\pi v_F} \ln^2\left(\frac{\alpha\omega}{2v_F}\right)$$

the latter coming from the integral (Bateman 1953)

$$\int_1^\infty dz \exp[-(\alpha\omega/v_F)z] \frac{\ln(2z-1)}{z}.$$

Putting the results together we get

$$I_1(\omega) + I_2(\omega) + (\omega \to -\omega) \underset{\alpha\omega/v_F \to 0}{\propto} 2 \ln r' N_0(\omega) + \frac{1}{2\pi v_F} \ln^2(\alpha\omega/2v_F).$$

Hence (5.2) follows at once.

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