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Coherent polarization driven by external electromagnetic fields

M. Apostol [a,*], M. Ganciu [b]

[a] Department of Theoretical Physics, Institute of Atomic Physics, Magurele-Bucharest, Romania
[b] Plasma Physics Laboratory, Institute of Atomic Physics, Magurele-Bucharest, Romania

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ABSTRACT

The coherent interaction of the electromagnetic radiation with an ensemble of polarizable, identical particles with two energy levels is investigated in the presence of external electromagnetic fields. The coupled non-linear equations of motion are solved in the stationary regime and in the limit of small coupling constants. It is shown that an external electromagnetic field may induce a macroscopic occupation of both the energy levels of the particles and the corresponding photon states, governed by a long-range order of the quantum phases of the internal motion (polarization) of the particles. A lasing effect is thereby obtained, controlled by the external field. Its main characteristics are estimated for typical atomic matter and atomic nuclei. For atomic matter the effect may be considerable (for usual external fields), while for atomic nuclei the effect is extremely small (practically insignificant), due to the great disparity in the coupling constants. In the absence of the external field, the solution, which is non-analytic in the coupling constant, corresponds to a second-order phase transition (super-radiance), which was previously investigated.

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1. Introduction

In a previous paper [1], the coherent interaction of the electromagnetic radiation with an ensemble of polarizable, identical particles with two energy levels has been investigated in the absence of external electromagnetic fields, and the corresponding coupled non-linear equations of motion have been solved. It was shown that the solution has a non-perturbational character (it is non-analytic in the coupling constant). The main role in this problem is played by a dimensionless coupling constant

$$\lambda = \sqrt{\frac{2\pi}{3a^3\hbar\omega_0}}\frac{J_{01}}{\omega_0}, \tag{1}$$

where J_{01} is the matrix element of the current associated with each particle, a is the mean inter-particle distance and $\hbar\omega_0 = \varepsilon_1 - \varepsilon_0$ is the energy separation between the two levels. It was shown [1] that, at zero temperature, the two levels $\varepsilon_{0,1}$ and the corresponding photon states $\hbar\omega_0$ are macroscopically occupied, provided $\lambda > 1$; at finite temperature, this coherent state sets up for $\lambda > 2$ and below a critical temperature T_c (given by $T_c \simeq \lambda^2\hbar\omega_0/8$). This second-order phase transition is usually known as a super-radiance transition [2–8]; it corresponds to a long-range order of the quantum phases (a lattice of coherence domains) [1], associated with the internal motion (polarization) of the particles.

For numerical estimates we may take $J_{01} = \omega_0 p$, where $p = el$ is the dipole momentum of the particles, l being the distance over which an electron charge e is displaced in the polarization process. For typical atomic matter we may take, for illustrative purposes, $l = a_0 = 0.53$ Å (Bohr radius), $\hbar\omega_0 = 1$ eV and $a = 3$ Å ($p = 2.4 \times 10^{-18}$ esu). We get $\lambda \simeq 0.5$, which is insufficient for setting up the coherent state. Similarly, for atomic nuclei we may take $l = 1$ fm (10^{-13} cm), $\hbar\omega_0 = 1$ MeV and $a = 3$ Å, and get $\lambda \simeq 10^{-8}$, which is an extremely small value for the coupling constant.

We turn our attention in this Letter to the presence of an external electromagnetic field, whose coherent interaction with the ensemble of particles may lead to a lasing effect. We get here the solution of the coupled non-linear equations of motion in the presence of an external field, in the stationary regime and in the limit of small values of the coupling constants. It is shown that the two levels and the corresponding photon states are macroscopically occupied, to an extent which depends on the coupling constant λ and the external field, leading thus to a lasing effect. While for atomic matter ($\lambda \simeq 0.5$) this effect may be considerable (for usual field intensities), it is extremely small (practically insignificant) for atomic nuclei ($\lambda \simeq 10^{-8}$). The problem is similar with the well-known "semi-classical theory" of the laser, which has been extensively investigated, by various approaches and from many angles [9–20]. It is worth noting that the theoretical considerations presented here pertain to a consequent field-theoretical approach to the coherent interaction of matter with electromagnetic radiation, as distinct

* Corresponding author.
E-mail address: apoma@theory.nipne.ro (M. Apostol).

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from the usual semi-classical approaches of the current theories of the laser (see, for instance, Refs. [21–23]).

2. Coherent interaction

As it is well known, the electromagnetic field is described by the vector potential

$$\mathbf{A}(\mathbf{r}) = \sum_{\mu\mathbf{k}} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \left[\mathbf{e}_\mu(\mathbf{k}) a_{\mu\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} + \mathbf{e}^*_\mu(\mathbf{k}) a^*_{\mu\mathbf{k}} e^{-i\mathbf{k}\mathbf{r}}\right] \tag{2}$$

in the standard Fourier representation, with the transverse gauge $\mathrm{div}\,\mathbf{A} = 0$, where c is the velocity of light, V is the volume, $\omega_k = ck$ is the frequency and $\mathbf{e}_\mu(\mathbf{k})$ are the polarization vectors, $\mathbf{e}_\mu(\mathbf{k})\mathbf{k} = 0$, $\mathbf{e}_\mu(\mathbf{k})\mathbf{e}^*_\mu(\mathbf{k}) = \delta_{\mu\nu}$ $(\mu, \nu = \pm 1)$, $\mathbf{e}_{-\mu}(-\mathbf{k}) = \mathbf{e}^*_\mu(\mathbf{k})$. The electric and magnetic fields are given by $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$ and, respectively, $\mathbf{H} = \mathrm{curl}\,\mathbf{A}$, and three Maxwell's equations are satisfied: $\mathrm{curl}\,\mathbf{E} = -\frac{1}{c}\partial\mathbf{H}/\partial t$, $\mathrm{div}\,\mathbf{H} = 0$, $\mathrm{div}\,\mathbf{E} = 0$. The time dependence is included in the Fourier coefficients $a_{\mu\mathbf{k}}$, $a^*_{\mu\mathbf{k}}$.

We use a similar expression for the external vector potential $\mathbf{A}^0(\mathbf{r})$, the corresponding Fourier coefficients being denoted by $a^0_{\mu\mathbf{k}}$, $a^{0*}_{\mu\mathbf{k}}$, with a prescribed time-dependence.

We use also the classical lagrangian of the radiation field

$$L_f = \frac{1}{8\pi} \int d\mathbf{r}\left(E^2 - H^2\right), \tag{3}$$

which can be expressed by means of the Fourier coefficients $a_{\mu\mathbf{k}}$, $a^*_{\mu\mathbf{k}}$, and the interaction lagrangian

$$\begin{aligned} L_{int} &= \frac{1}{c} \int d\mathbf{r} \cdot \mathbf{j}(\mathbf{A} + \mathbf{A}_0) \\ &= \sum_{\mu\mathbf{k}} \sqrt{\frac{2\pi\hbar}{\omega_k}} \left[\mathbf{e}_\mu(\mathbf{k})\mathbf{j}^*(\mathbf{k})\left(a_{\mu\mathbf{k}} + a^0_{\mu\mathbf{k}}\right)\right. \\ &\quad \left. + \mathbf{e}^*_\mu(\mathbf{k})\mathbf{j}(\mathbf{k})\left(a^*_{\mu\mathbf{k}} + a^{0*}_{\mu\mathbf{k}}\right)\right], \end{aligned} \tag{4}$$

where $\mathbf{j}(\mathbf{k})$ is the Fourier transform of the current density,

$$\mathbf{j}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \mathbf{j}(\mathbf{k}) e^{i\mathbf{k}\mathbf{r}}, \tag{5}$$

with $\mathrm{div}\,\mathbf{j} = 0$ (continuity equation). The Euler–Lagrange equations for the lagrangian $L_f + L_{int}$ lead to the wave equation with sources

$$\ddot{a}_{\mu\mathbf{k}} + \ddot{a}^*_{-\mu-\mathbf{k}} + \omega_k^2\left(a_{\mu\mathbf{k}} + a^*_{-\mu-\mathbf{k}}\right) = \sqrt{\frac{8\pi\omega_k}{\hbar}} \mathbf{e}^*_\mu(\mathbf{k})\mathbf{j}(\mathbf{k}), \tag{6}$$

which is the fourth Maxwell's equation $\mathrm{curl}\,\mathbf{H} = (1/c)\partial\mathbf{E}/\partial t + 4\pi\mathbf{j}/c$.

We consider a set of N independent, non-relativistic, identical particles labelled by $i = 1, \ldots, N$ $(N \gg 1)$ and write the hamiltonian corresponding to their internal degrees of freedom as $H_s = \sum_i H_s(i)$. We introduce a set of orthonormal eigenfunctions $\varphi_n(i)$, where ε_n is the energy level of the n-state, and construct also a set of orthonormal eigenfunctions

$$\psi_n = \frac{1}{\sqrt{N}} \sum_i e^{i\theta_{ni}} \varphi_n(i), \tag{7}$$

where θ_{ni} are some undetermined phases.

The field operator

$$\Psi = \sum_n b_n \psi_n, \tag{8}$$

with boson-like commutation relations $[b_n, b^*_m] = \delta_{nm}$, $[b_n, b_m] = 0$, leads to the (macroscopic) number of particles $N = \sum_n b^*_n b_n$ and to the lagrangian

$$L_s = \frac{1}{2} \sum_n i\hbar\left[b^*_n \dot{b}_n - \dot{b}^*_n b_n\right] - \sum_n \varepsilon_n b^*_n b_n, \tag{9}$$

where $H_s = \sum_n \varepsilon_n b^*_n b_n$ is the hamiltonian of the ensemble of particles. The corresponding equation of motion $i\hbar\dot{b}_n = \varepsilon_n b_n$ is Schrodinger's equation.

The current density associated with this ensemble of particles can be written as

$$\mathbf{j}(\mathbf{r}) = \sum_i \mathbf{J}(i)\delta(\mathbf{r} - \mathbf{r}_i) = \frac{1}{V} \sum_{i\mathbf{k}} \mathbf{J}(i) e^{-i\mathbf{k}\mathbf{r}_i} e^{i\mathbf{k}\mathbf{r}} = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \mathbf{j}(\mathbf{k}) e^{i\mathbf{k}\mathbf{r}}, \tag{10}$$

where $\mathbf{r}_i$ is the position of the ith particle and $\mathbf{J}(i)$ is the current associated with one particle. Now, making use of Eqs. (8) and (10), it is easy to see that the interaction lagrangian given by Eq. (4) can be written as

$$L_{int} = \sum_{nm\mu\mathbf{k}} \sqrt{\frac{2\pi\hbar}{V\omega_k}} F_{nm}(\mu\mathbf{k})\left(a_{\mu\mathbf{k}} + a^*_{-\mu-\mathbf{k}} + a^0_{\mu\mathbf{k}} + a^{0*}_{-\mu-\mathbf{k}}\right) b^*_n b_m, \tag{11}$$

where

$$F_{nm}(\mu\mathbf{k}) = \frac{1}{N} \sum_i \mathbf{e}_\mu(\mathbf{k}) \mathbf{J}_{nm}(i) e^{i\mathbf{k}\mathbf{r}_i - i(\theta_{ni} - \theta_{mi})}. \tag{12}$$

$\mathbf{J}_{nm}(i)$ being the matrix element of the current associated with the ith particle.

For any pair (n, m) of levels, the quantum phases θ_{ni} can be arranged in a periodic lattice with the shortest (generating) reciprocal vectors denoted by $\mathbf{k}_r$, $r = 1, 2, 3$. For a given pair (n, m) we take these vectors as being equal in magnitude, $k_r = k_0$ and $\omega_0 = ck_0$ [1]. Under these circumstances the phase in Eq. (12) may satisfy the condition $\mathbf{k}_r\mathbf{r}_{pi} - (\theta_{ni} - \theta_{mi}) = \mathrm{const}$, where p labels the unit cells of the phase lattice. This condition was called the coherence condition in Ref. [1]. Then, the interaction lagrangian acquires a simple form, which, limiting ourselves to only two levels, and using the coherent states operators [24] $b_{0,1}|\beta_{0,1}\rangle = \beta_{0,1}|\beta_{0,1}\rangle$, can be written as

$$L_{int} = \sqrt{\frac{2\pi\hbar}{V\omega_0}} J_{01}\left(\alpha + \alpha^* + \alpha^0 + \alpha^{0*}\right)\left(\beta^*_1\beta_0 + \beta_1\beta^*_0\right), \tag{13}$$

where we have assumed $J_{00} = J_{11} = 0$. In Eq. (13) we have also replaced the photon operators $a_{\mu\mathbf{k}_r}$, $k_r = k_0$, by c-numbers α, the same for any polarization μ and any direction of the vectors $\mathbf{k}_r$, and similarly for the external field. We note that the external field depends on time; we take $\alpha^0 + \alpha^{0*} = 2|\alpha^0|\cos\omega_0 t$. A similar replacement of the field operators by c-numbers is made in the free lagrangians of the field and particles. The summation over $\mu\mathbf{k}_r$, $k_r = k_0$, in the field lagrangian L_f gives a factor 12, for a three-dimensional lattice (three $\pm\mathbf{k}_r$'s and two polarizations). This factor can be absorbed in the photon operators, so we can write down the full "classical" lagrangian

$$\begin{aligned} L_f &= \frac{\hbar}{4\omega_0}\left(\dot{\alpha}^2 + \dot{\alpha}^{*2} + 2|\dot{\alpha}|^2\right) - \frac{\hbar\omega_0}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right), \\ L_s &= \frac{1}{2} i\hbar\left(\beta^*_0\dot{\beta}_0 - \dot{\beta}^*_0\beta_0 + \beta^*_1\dot{\beta}_1 - \dot{\beta}^*_1\beta_1\right) - \left(\varepsilon_0|\beta_0|^2 + \varepsilon_1|\beta_1|^2\right), \\ L_{int} &= \frac{g}{\sqrt{N}}\left[\alpha + \alpha^* + \alpha^0 + \alpha^{0*}\right]\left(\beta_0\beta^*_1 + \beta_1\beta^*_0\right), \end{aligned} \tag{14}$$

where the coupling constant is given by

$$g = \sqrt{\frac{\pi\hbar}{6a^3\omega_0}} J_{01}; \tag{15}$$

hence, the dimensionless coupling constant $\lambda = 2g/\hbar\omega_0$ introduced in Eq. (1).

The lagrangian given by Eqs. (14) leads to the equations of motion

$$\ddot{A} + \omega_0^2 A = \frac{2\omega_0 g}{\hbar\sqrt{N}}\left(\beta_0\beta_1^* + \beta_1\beta_0^*\right),$$
$$i\hbar\dot{\beta}_0 = \varepsilon_0\beta_0 - \frac{g}{\sqrt{N}}\left[A + A^0(t)\right]\beta_1,$$
$$i\hbar\dot{\beta}_1 = \varepsilon_1\beta_1 - \frac{g}{\sqrt{N}}\left[A + A^0(t)\right]\beta_0, \tag{16}$$

where $A = \alpha + \alpha^*$ and $A^0(t) = 2|\alpha^0|\cos\omega_0 t$. It is easy to see, by using these equations of motion, that the number of particles $N = |\beta_0|^2 + |\beta_1|^2$ is conserved. Making use of Eqs. (14), we can define a total hamiltonian

$$H_f^t = \frac{\hbar}{4\omega_0}\left[\dot{A} + \dot{A}^0(t)\right]^2 + \frac{\hbar\omega_0}{4}\left[A + A^0(t)\right]^2,$$
$$H_s = \varepsilon_0|\beta_0|^2 + \varepsilon_1|\beta_1|^2,$$
$$H_{int} = -\frac{g}{\sqrt{N}}\left[A + A^0(t)\right]\left(\beta_0\beta_1^* + \beta_1\beta_0^*\right), \tag{17}$$

where the external field is included.

3. Stationary solutions

We focus on the equations of motion (16), where we put for convenience $\varepsilon_0 = 0$. In the absence of the external field ($A^0(t) = 0$) the solutions are of the form $\beta_{0,1} = B_{0,1}e^{i\Omega t}$, where $B_{0,1}$ are constant amplitudes, $B_0^2 + B_1^2 = N$, and the frequency Ω is given by $2\Omega + 1 - \lambda^2 = 0$ (and $A = \lambda\sqrt{N}(1 - 1/\lambda^4)^{1/2}$) [1]. The total energy given by Eqs. (17) (for $A^0(t) = 0$) reads

$$E = -\frac{1}{4}\hbar\omega_0\lambda^2 N\left[1 - 1/\lambda^2\right]^2 = -\hbar\Omega B_1^2 \tag{18}$$

(whence the criticality condition $\lambda > 1$ for the super-radiance transition). This energy is lower than the non-interacting ground-state energy $N\varepsilon_0 = 0$. It may be viewed as the formation enthalpy of the coherence domains. The coupled ensemble of matter and radiation is unstable for a macroscopic occupation of the particles quantum states and the associated photon states, provided $\lambda > 1$. The non-analytic character of this solution with respect to the coupling constant λ is obvious.

We assume now $A^0(t) \neq 0$. It is convenient to put the problem in more general terms. First, we introduce the notation $\varepsilon_1 = \hbar\omega_1$, where, in general, ω_1 may differ from ω_0. Second, we introduce the total field $A_t(t) = A + A^0(t)$ and define the parameter

$$x(t) = \frac{2g}{\hbar\omega_1\sqrt{N}}A_t(t) = \frac{\lambda}{\sqrt{N}}A_t(t), \tag{19}$$

where $\lambda = 2g/\hbar\omega_1$. We look for solutions of the form $\beta_{0,1} = B_{0,1}e^{i\theta}$ for the system of the last two equations (16). We get immediately $\dot{B}_{0,1} = 0$ and

$$\beta_0 = B_0e^{i\theta_0} - fB_1e^{i\theta_1}, \qquad \beta_1 = fB_0e^{i\theta_0} + B_1e^{i\theta_1},$$
$$\dot{\theta}_{0,1} = \frac{1}{2}\omega_1\left(-1 \pm \sqrt{x^2(t) + 1}\right), \tag{20}$$

where

$$f(t) = \frac{x(t)}{\sqrt{x^2(t) + 1} + 1}. \tag{21}$$

The coefficients $B_{0,1}$ are determined by requiring the initial values of the occupancy numbers $|\beta_{0,1}(t = 0)|^2$ be equal with $N_{0,1}$ ($N_0 + N_1 = N$). We get the amplitudes

$$B_{0,1} = \frac{1}{1 + f^2(t)}\left[\sqrt{N_{0,1}} \pm f(t)\sqrt{N_{1,0}}\right] \tag{22}$$

and the occupancy numbers

$$|\beta_{0,1}|^2 = N_{0,1} \pm \frac{1}{2}\frac{x(t)}{x^2(t) + 1}\left[2\sqrt{N_0N_1} - x(t)(N_0 - N_1)\right]$$
$$\times\left[1 - \cos(\theta_0 - \theta_1)\right], \tag{23}$$

where the phase difference $\theta_0 - \theta_1$ is given by

$$\Delta\theta = \theta_0 - \theta_1 = \omega_1\int_0^t dt\,\sqrt{x^2(t) + 1}. \tag{24}$$

The oscillations in the occupancies given by Eq. (23) are reminiscent of the well-known Rabi oscillations in the Jaynes–Cummings model (see, for instance, Refs. [25–27]). We take the time averages of all the relevant quantities given above. We can see, by Eqs. (20), that the energy levels $\varepsilon_{0,1}$ are changed by interaction into the mean values of $\hbar\dot{\theta}_{0,1}$, and, in addition, the interaction mixes up the two states, as expected. We can see also that the mean values of the coefficients $B_{0,1}$, as well as the mean values of the coefficients $fB_{0,1}$ entering Eqs. (20), are constants, as it is required for a stationary solution; it becomes apparent that $N_{0,1}$ are constants of integration.

4. Polarization field

We turn now to the first equation (16) for the polarization field A. It is worth noting that the r.h.s. of this equation is proportional to the polarization of the ensemble of particles. Indeed, making use of Eqs. (7) and (8), the polarization

$$\mathbf{P} = \frac{1}{V}\sum_i \mathbf{p}(i) \tag{25}$$

acquires the form

$$\mathbf{P} = \frac{1}{NV}\sum_i\left[\mathbf{p}_{01}(i)e^{-i(\theta_{0i} - \theta_{1i})}\beta_0^*\beta_1 + \text{c.c.}\right], \tag{26}$$

where $\mathbf{p}_{01}(i) = \mathbf{p}_{10}^*(i)$ are the matrix elements of the dipole momentum $\mathbf{p}(i)$ of the ith particle. The ensemble of particles is polarized by the field, so these dipole momenta are oriented along the field and have the same spatial dependence as the field, corresponding to the reciprocal vectors $\mathbf{k}_r$ of the coherence domains lattice ($k_r = k_0 = \omega_0/c$). Then, it is easy to see that the coherence condition used before ($\mathbf{k}_r\mathbf{r}_{pi} - (\theta_{ni} - \theta_{mi}) = \text{const}$) gives a non-vanishing polarization, involving the Fourier coefficients $p_{01}(\mathbf{k}_r)$ of the components along the field of the dipole momenta. There is no particular reason to have different dipole momenta $p_{01}(\mathbf{k}_r)$ for different vectors $\mathbf{k}_r$, so we may put $p_{01}(\mathbf{k}_r) = p_{10}(\mathbf{k}_r) = p$. The polarization becomes

$$P = \frac{p}{V}\left(\beta_0\beta_1^* + \beta_1\beta_0^*\right), \tag{27}$$

which is proportional to the r.h.s. of the first equation (16), as expected.

The quantity $\beta_0\beta_1^* + \text{c.c.}$ entering the r.h.s. of the first equation (16) can be computed by using Eqs. (20). We get

$$\beta_0\beta_1^* + \beta_1\beta_0^* = \frac{1}{x^2+1}\big\{x[2x\sqrt{N_0N_1} + N_0 - N_1] + \big[2\sqrt{N_0N_1} - x(N_0 - N_1)\big]\cos\Delta\theta\big\}. \tag{28}$$

The external field A^0, which satisfies the wave equation $\ddot{A}^0 + \omega_0^2 A^0 = 0$, may be added to the polarization field A in the first equation (16); this equation becomes

$$\ddot{x} + \omega_0^2 x = \omega_0\omega_1 \frac{\lambda^2}{N}\frac{1}{x^2+1}\big\{x[2x\sqrt{N_0N_1} + N_0 - N_1] + \big[2\sqrt{N_0N_1} - x(N_0 - N_1)\big]\cos\Delta\theta\big\}. \tag{29}$$

This is a non-linear (integro-differential) equation. We assume $\lambda \ll 1$ and $A^0(t)/\sqrt{N}$, $A(t)/\sqrt{N}$ finite, so that we can seek the solution as a power series in λ, $x = \lambda x_0 + \lambda^2 x_1 + \lambda^3 x_2 + \cdots$, where $x_0 = \frac{2|\alpha^0|}{\sqrt{N}}\cos\omega t$. The frequency ω will be determined by requiring the absence of the ω-resonating terms. The leading contribution to the phase difference $\Delta\theta$ can then be written as $\widetilde{\omega}_1 t$, where the frequency $\widetilde{\omega}_1$ remains to be determined. Eq. (29) becomes

$$\ddot{x} + \omega_0^2 x = 2\omega_0\omega_1\frac{\sqrt{N_0N_1}}{N}\lambda^2\cos\widetilde{\omega}_1 t + \omega_0\omega_1\frac{N_0 - N_1}{N}\lambda^3 x_0(1 - \cos\widetilde{\omega}_1 t) + \cdots. \tag{30}$$

A similar series expansion $\omega = \omega_0 + \lambda^2\Omega + \cdots$ is used for the frequency ω. We get

$$x_1 = \frac{\sqrt{N_0N_1}}{N}\frac{2\omega_0\omega_1}{\omega_0^2 - \omega_1^2}\cos\widetilde{\omega}_1 t,$$
$$x_2 = \frac{|\alpha^0|}{\sqrt{N}}\frac{N_0 - N_1}{N}\omega_0\left[\frac{1}{2\omega_0 + \omega_1}\cos(\omega + \widetilde{\omega}_1)t - \frac{1}{2\omega_0 - \omega_1}\cos(\omega - \widetilde{\omega}_1)t\right] \tag{31}$$

and

$$\omega = \omega_0 - \lambda^2\omega_1\frac{N_0 - N_1}{2N}, \qquad \widetilde{\omega}_1 = \omega_1 + \lambda^2\omega_0\frac{|\alpha^0|^2}{N}. \tag{32}$$

($\Omega = -\omega_1(N_0 - N_1)/2N$) for $\omega_1 \neq \omega_0, \pm 2\omega_0$. We note that these resonances can be related to the parametric resonances $2\omega_0 \simeq n\omega_1$ (n positive integer) of a Mathieu equation [28], which, for $N_1 = 0$, may be viewed as a linearized, approximate form of Eq. (29). As a matter of fact, except for the resonances, the solutions given above for $N_{0,1} = 0$ are close to the leading contributions to the (non-periodic) solutions of Mathieu's equation. In the particular case $\omega = \widetilde{\omega}_1$ (or other similar cases of the approximate form $2\omega = n\widetilde{\omega}_1$) they are very close to the leading contributions to the (periodic) Mathieu function $ce_2(\widetilde{\omega}_1 t/2)$ (or, in general, $ce_n(\widetilde{\omega}_1 t/2)$). However, we must note that the linearized form of Eq. (29), which is a Mathieu equation, is not a satisfactory approximation for the non-linear Eq. (29), because of the apparition of the x-term in the r.h.s. of this equation, instead of the correct x_0-term, as in Eq. (30). In other words, a consequent expansion in powers of the parameter λ makes the leading contributions to Eq. (29) to acquire a form which is different, in fact, from a Mathieu equation. Leaving aside the (weak) frequency renormalization, the resonances exhibited by Eqs. (31) are in fact what we may expect from a non-linear oscillator with the basic frequency ω_0 subjected to an external force of frequency ω_1. As it is well known, such an oscillator exhibits the combined-frequency phenomenon, as reflected in the occurrence of frequencies of the form $\omega_0 \pm \omega_1$ and denominators $2\omega_0 \pm \omega_1$, etc. (arising from terms like $\omega_0^2 - (\omega_0 \pm \omega_1)^2$).

We note that the term x_1 in Eqs. (31) represents the oscillations of the ensemble of particles (for $N_{0,1} \neq 0$), and the effect of the external field appears only in the next order (the term x_2), with combined frequencies $\omega \pm \widetilde{\omega}_1$. For $N_{1,0} = 0$, the polarization process is governed entirely by the external field, as expected (and the constraint $\omega_0 \neq \omega_1$ is removed). We note also that the interaction shifts both the frequency of the external field and the energy levels of the ensemble of particles, according to Eq. (32).

Having known the parameter $x(t)$, we can determine the phase difference $\Delta\theta$ (and $\cos\Delta\theta$) according to Eq. (24), and the mean values (averages over the time) of all the relevant quantities can be computed, as given by Eqs. (20)–(24). We get, for instance, the frequencies

$$\Omega_0 = \langle\dot{\theta}_0\rangle = \lambda^2\omega_1\frac{|\alpha^0|^2}{2N},$$
$$\Omega_1 = \langle\dot{\theta}_1\rangle = -\omega_1 - \lambda^2\omega_1\frac{|\alpha^0|^2}{2N} \tag{33}$$

and the mean occupancies

$$\langle|\beta_{0,1}|^2\rangle = N_{0,1} \mp \lambda^2\frac{N_0N_1}{N}\frac{\omega_0\omega_1}{\omega_0^2 - \omega_1^2} \mp \lambda^2\frac{N_0 - N_1}{N}|\alpha^0|^2. \tag{34}$$

One can see that the external field can pump, or deplete, the upper level, depending on the parameters $N_{0,1}$ and $\omega_{0,1}$. Particularly interesting is the case $N_1 = 0$ (corresponding to an upper level which is empty at the initial moment $t = 0$). In this case, the occupancy of the upper level is given by

$$\langle|\beta_1|^2\rangle = \lambda^2|\alpha^0|^2, \tag{35}$$

the external field leads to a macroscopic occupation of this level. The release of the corresponding energy $E_s = \hbar\omega_1\langle|\beta_1|^2\rangle$ is a lasing effect, driven by the external field.

The polarization can be computed from Eqs. (27) and (28), by using the solution $x(t)$ given by Eqs. (31). Within this approximation, the polarization contains many oscillating terms, including both a quadratic dependence on the external field and frequency doubling, as expected for such non-linear equations. We collect here a few relevant contributions:

$$\begin{aligned}\beta_0^*\beta_1 + \beta_1^*\beta_0 = {} & 2\sqrt{N_0N_1}\cos\widetilde{\omega}_1 t \\ & + 2\lambda\frac{|\alpha^0|}{\sqrt{N}}(N_0 - N_1)(1 - \cos\widetilde{\omega}_1 t)\cos\omega t \\ & + 2\lambda^2\frac{\sqrt{N_0N_1}}{N}\left[4|\alpha^0|^2\cos^2\widetilde{\omega}_0 t - (N_0 - N_1)\frac{\omega_0\omega_1}{\omega_0^2 - \omega_1^2}\cos^2\widetilde{\omega}_1 t\right] \\ & + \lambda^3\frac{\omega_1|\alpha^0|^2}{4\omega_0 N}(N_0 - N_1)\sin\widetilde{\omega}_0 t\sin\widetilde{\omega}_1 t.\end{aligned} \tag{36}$$

The mean value of the polarization is given by

$$\langle\beta_0\beta_1^* + \beta_1\beta_0^*\rangle = \lambda^2\frac{\sqrt{N_0N_1}}{N}\left[4|\alpha^0|^2 - (N_0 - N_1)\frac{\omega_0\omega_1}{\omega_0^2 - \omega_1^2}\right], \tag{37}$$

where the quadratic dependence on the external field is to be noted. It is also worth noting that it vanishes for $N_{0,1} = 0$. Making use of Eq. (2) we can compute the electric field $E_t = -(1/c)\partial A_t/\partial t$, while the polarization is given by Eq. (28). The permittivity, defined as $P = \kappa E_t$ (for the Fourier components), is $\kappa =$

$(2p^2/\hbar\omega_1 a^3)(N_0 - N_1)/N$ for the ω-component. We can see that the particle polarizability is $\alpha = \kappa a^3 = 2p^2/\hbar\omega_1$ (for $N_1 = 0$), so that we can also represent the coupling constant as $\lambda = \sqrt{\pi\alpha/3a^3}$ (for $\omega_0 = \omega_1$). It follows that we are justified in assuming $\lambda \ll 1$, as long as the polarizability per unit volume of the ensemble of particles is sufficiently small. Similarly, introducing the electric field, Eq. (35) can be transformed into

$$\langle|\beta_1|^2\rangle = N\left(\frac{pE_0}{\hbar\omega_0}\right)^2, \tag{38}$$

where E_0 is the strength of the external electric field. One can recognize in Eq. (38) the well-known Rabi frequency $pE_0/\hbar$.

5. Concluding remarks

Making use of the parameter $x(t)$ derived above and averaging over time in the hamiltonian given by Eqs. (17), we get the leading contributions to the energy:

$$\begin{aligned} E_f^t &= E_f^0 + \frac{1}{2}\lambda^2\left[\frac{\hbar(\omega_0^2+\omega_1^2)}{\omega_0}\frac{N_0N_1}{N}\left(\frac{\omega_0\omega_1}{\omega_0^2-\omega_1^2}\right)^2 \right. \\ &\quad \left. - \hbar\omega_1\frac{N_0-N_1}{N}\left|\alpha^0\right|^2\right], \\ E_s &= \hbar\omega_1\left[N_1 + \lambda^2\left(\frac{N_0N_1}{N}\frac{\omega_0\omega_1}{\omega_0^2-\omega_1^2} + \frac{N_0-N_1}{N}\left|\alpha^0\right|^2\right)\right], \\ E_{int} &= -\hbar\omega_1\lambda^2\left(\frac{N_0N_1}{N}\frac{\omega_0\omega_1}{\omega_0^2-\omega_1^2} + \frac{N_0-N_1}{N}\left|\alpha^0\right|^2\right), \end{aligned} \tag{39}$$

where $E_f^0 = \hbar\omega_0|\alpha^0|^2$ is the energy of the (bare) external field. The total field energy can also be written as

$$E_f^t = \hbar\omega\left|\alpha^0\right|^2 + \lambda^2\frac{\hbar(\omega_0^2+\omega_1^2)}{2\omega_0}\frac{N_0N_1}{N}\left(\frac{\omega_0\omega_1}{\omega^2-\omega_1^2}\right)^2. \tag{40}$$

For $N_1 = 0$ the above equations become

$$\begin{aligned} E_f^t &= \hbar\omega\left|\alpha^0\right|^2 = E_f^0 - \frac{1}{2}\hbar\omega_1\lambda^2\left|\alpha^0\right|^2, \\ E_s &= -E_{int} = \hbar\omega_1\lambda^2\left|\alpha^0\right|^2 = \frac{\omega_1}{\omega_0}\lambda^2 E_f^0. \end{aligned} \tag{41}$$

One can see that the total energy $E_t = E_f^t + E_s + E_{int}$ reduces to the total field energy E_f^t, the polarization energy (E_s) being entirely compensated by the interaction energy, as expected. The efficiency quotient of this lasing process is $\lambda^2(\omega_1/\omega_0)$. It may appear that it is favorable to diminish ω_0 with respect to ω_1, but one must avoid the resonance occurring at $2\omega_0 = \omega_1$, on one hand, and, on the other, one must be aware that a decreasing ω_0 is limited by $\lambda = 2g/\hbar\omega_1 \ll 1$ (according to Eq. (15)) (and by $E_f^t > 0$).

For $N_1 = 0$ we take for convenience $\omega_0 = \omega_1$. As discussed in Section 1, for a typical sample of atomic matter the coupling constant is $\lambda = 0.5$ ($\hbar\omega_1 = 1$ eV, $a = 3$ Å, $p = 2.4 \times 10^{-18}$ esu). For reasonable values $E_f^0 = 10^3$ J, $N = 6 \times 10^{23}$ (Avogadro's number) we get $E_s = 250$ J, which may be viewed as a considerable effect. For atomic nuclei $\lambda = 10^{-8}$ ($\hbar\omega_1 \simeq \hbar\omega_0 = 1$ MeV, $a = 3$ Å, $p = 5 \times 10^{-23}$ esu), and we can see that the released energy is extremely small.

In conclusion, we may say that we have solved the coupled non-linear equations of motion, in the stationary regime and for small coupling constants, for an ensemble of polarizable, identical particles with two energy levels interacting coherently with their own polarization field and with an external electromagnetic field. It was shown that a lasing effect is possible, driven by the external field. For typical atomic matter the effect may be considerable, while for an ensemble of atomic nuclei the effect is extremely small. The difference originates in the great disparity between the corresponding coupling constants.

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