Covariance, Curved Space, Motion and Quantization

Marian Apostol

Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, PO Box MG-35, Romania

E-mail: apoma@theory.nipne.ro

Weak external forces and non-inertial motion are equivalent with the free motion in a curved space. The Hamilton-Jacobi equation is derived for such motion and the effects of the curvature upon the quantization are analyzed, starting from a generalization of the Klein-Gordon equation in curved spaces. It is shown that the quantization is actually destroyed, in general, by a non-inertial motion in the presence of external forces, in the sense that such a motion may produce quantum transitions. Examples are given for a massive scalar field and for photons.

Newton's law. We start with Newton's law

$$m\,\frac{dv_\alpha}{dt} = f_\alpha\,, \tag{1}$$

for a particle of mass m, with usual notations. I wish to show here that it is equivalent with the motion of a free particle of mass m in a curved space, *i.e.* it is equivalent with

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k = 0\,, \tag{2}$$

again with usual notations.*

Obviously, the spatial coordinates of equation (1) are euclidean, and equation (1) is a non-relativistic limit. It follows that the metric we should look for may read

$$ds^2 = (1+h)\,c^2dt^2 + 2cdt\,g_{0\alpha}dx^\alpha + g_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{3}$$

where $g_{\alpha\beta} = -\delta^\beta_\alpha (= \delta_{\alpha\beta})$, while functions h, $g_{0\alpha} \ll 1$ are determined such that equation (2) goes into equation (1) in the non-relativistic limit $\frac{v_\alpha}{c} \ll 1$ and for a correspondingly weak force f_α. Such a metric, which recovers Newton's law in the non-relativistic limit, is not unique. The metric given by equation (3) can be written as

$$g_{ij} = \begin{pmatrix} 1+h & g_{10} & g_{20} & g_{30} \\ g_{01} & -1 & 0 & 0 \\ g_{02} & 0 & -1 & 0 \\ g_{03} & 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

(where $g_{0\alpha} = g_{\alpha 0} = g_\alpha$). We perform the calculations up to the first order in h, g_α and $\frac{v_\alpha}{c}$. The distance given by (3) becomes then $ds = cdt\left(1+\frac{h}{2}\right)$ and the velocities read

$$u^0 = \frac{dx^0}{ds} = 1 - \frac{h}{2}\,, \qquad u^\alpha = \frac{dx^\alpha}{ds} = \frac{v_\alpha}{c}\,. \tag{5}$$

It is the Christoffel's symbols (affine connections)

$$\Gamma^i_{jk} = \frac{1}{2}\,g^{im}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \tag{6}$$

which require more calculations. First, the contravariant metric is $g^{00} = 1-h$, $g^{0\alpha} = g^{\alpha 0}$, $g_{0\alpha} = g_{\alpha 0}$, $g^\alpha_\beta = -\delta^\alpha_\beta$, such that $g_{im}g^{mj} = g^{jm}g_{mi} = \delta^j_i$. By making use of (6) we get

$$\left.\begin{aligned} &\Gamma^0_{00} = \frac{1}{2c}\frac{\partial h}{\partial t}\,, \qquad \Gamma^0_{0\alpha} = \Gamma^0_{\alpha 0} = \frac{1}{2}\frac{\partial h}{\partial x^\alpha} \\ &\Gamma^0_{\alpha\beta} = \Gamma^0_{\beta\alpha} = \frac{1}{2}\left(\frac{\partial g_{0\alpha}}{\partial x^\beta} + \frac{\partial g_{0\beta}}{\partial x^\alpha}\right) \\ &\Gamma^\alpha_{\beta 0} = \Gamma^\alpha_{0\beta} = \frac{1}{2}\left(\frac{\partial g_{0\beta}}{\partial x^\alpha} - \frac{\partial g_{0\alpha}}{\partial x^\beta}\right) \\ &\Gamma^\alpha_{00} = \frac{1}{2}\frac{\partial h}{\partial x^\alpha} - \frac{1}{c}\frac{\partial g_{0\alpha}}{\partial t}\,, \qquad \Gamma^\alpha_{\beta\gamma} = 0 \end{aligned}\right\}. \tag{7}$$

Now, the first equation in (2) has $\frac{du^0}{ds} = -\frac{1}{2c}\frac{\partial h}{\partial t}$ and $\Gamma^0_{jk}u^ju^k = \frac{1}{2c}\frac{\partial h}{\partial t}$, so it is satisfied identically in this approximation. The remaining equations in (2) read

$$\frac{dv_\alpha}{dt} = c^2\left(\frac{\partial g_{0\alpha}}{c\partial t} - \frac{1}{2}\frac{\partial h}{\partial x^\alpha}\right). \tag{8}$$

*The geometry of the curved spaces originates probably with Gauss (1830). It was given a sense by Riemann (*Uber die Hypothesen welche der Geometrie zugrunde liegen*, 1854), Grassman (1862), Christoffel (1869), thereafter Klein (*Erlanger Programm, Programm zum Eintritt in die philosophische Fakultät in Erlangen*, 1872), Ricci and Levi-Civita (1901). It was Einstein (1905,1916), Poincare (1905), Minkowski (1907), Sommerfeld (1910), (Kottler, 1912), Weyl (*Raum, Zeit und Materie*, 1918), Hilbert (1917) who made the connection with the physical theories. It is based on point (local) coordinate transforms, cogredient (contravariant) and contragredient (covariant) tensors and the distance element. It is an absolute calculus, as it does not depend on the point, *i.e.* the reference frame. It may be divided into the motion of a particle, the motion of the fields, the motion of the gravitational field, and their applications, especially in cosmology and cosmogony. As the curved space is universal for gravitation, so it is for the non-inertial motion, which we focus upon here. The body which creates the gravitation and the corresponding curved space is here the moving observer for the non-inertial motion, beside forces. It could be very well that the world and the motion are absolute, but they depend on subjectivity, though it could be an universal subjectivity (inter-subjectivity). See W. Pauli, *Theory of Relativity*, Teubner, Leipzig, (1921).

By comparing this with Newton's equation (1) we get the functions h and $g_{0\alpha}$ as given by

$$\frac{\partial g_{0\alpha}}{c\partial t} - \frac{1}{2}\frac{\partial h}{\partial x^\alpha} = \frac{f_\alpha}{mc^2}\,. \tag{9}$$

As it is well-known for a static gravitational potential Φ, the force is given by $f_\alpha = -\,m\,\frac{\partial\Phi}{\partial x^\alpha}$, so that $h = \frac{2\Phi}{c^2}$ and also $g_{0\alpha} = \text{const.}$*

Translations. Suppose that the force $\mathbf{f}$ is given by a static potential φ, such that $\mathbf{f} = -\frac{\partial\varphi}{\partial\mathbf{r}}$. Then $h = \frac{2\varphi}{mc^2}$ and $\mathbf{g} = \text{const.}$

Let us perform a translation

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}(t')\,, \qquad t = t'. \tag{10}$$

Then, Newton's equation $m\frac{d\mathbf{v}}{dt} = \mathbf{f}$ given by (1) becomes

$$m\,\frac{d\mathbf{v}'}{dt'} = \mathbf{f}' - m\,\frac{d\mathbf{V}}{dt'}\,, \tag{11}$$

where $\mathbf{f}'$ is the force in the new coordinates and $\mathbf{V} = \frac{d\mathbf{R}}{dt'}$ is the translation velocity. The inertial force $-\,m\frac{d\mathbf{V}}{dt'}$ appearing in (11) is accounted by the $\mathbf{g}$ in the metric of the curved space. Indeed, equation (9) gives

$$\mathbf{g} = -\frac{\mathbf{V}}{c}\,, \tag{12}$$

up to a constant. The constant reflects the principle of inertia. We may put it equal to zero. The time-dependent $\mathbf{g}$ and $\mathbf{V}$ represent a non-inertial motion. Such a non-inertial motion is therefore equivalent with a free motion in a curved space. Of course, this statement is nothing else but the principle of equivalence, or the general principle of relativity. It is however noteworthy that the non-inertial curved space depends on the observer, through the velocity $\mathbf{V}$, by virtue of the reciprocity of the motion.

Rotations. A rotation of angular frequency Ω about some axis is an orthogonal transformation of coordinates defined locally by

$$d\mathbf{r}' = d\mathbf{r} + \big(\Omega\times\mathbf{r}\big)\,dt\,\,, \tag{13}$$

such that the velocity is $\mathbf{v}' = \mathbf{v} + \Omega\times\mathbf{r}$ and

$$\begin{aligned} d\mathbf{v}' &= d\mathbf{v} + \big(\dot\Omega\times\mathbf{r}\big)dt + (\Omega\times\mathbf{v})\,dt + \\ &+ \big[\Omega\times(\mathbf{v}+\Omega\times\mathbf{r})\big]\,dt = \\ &= d\mathbf{v} + \big(\dot\Omega\times\mathbf{r}\big)dt + 2\,(\Omega\times\mathbf{v})\,dt + \big[\Omega\times(\Omega\times\mathbf{r})\big]\,dt\,. \end{aligned} \tag{14}$$

It is easy to see that in Newton's law for a particle of mass m there appears a force related to the non-uniform rotation ($\dot\Omega$), the Coriolis force $\sim\Omega\times\mathbf{v}$ and the centrifugal force $\sim\Omega^2$. The lagrangian $L = \frac{1}{2}\,mv'^2 - \varphi$, where φ is a potential, leads to the hamiltonian

$$\begin{aligned} H &= \frac{mv^2}{2} - \frac{m}{2}\big(\Omega\times\mathbf{r}\big)^2 + \varphi = \\ &= \frac{1}{2m}\,p^2 - \Omega\big(\mathbf{r}\times\mathbf{p}\big) + \varphi = \frac{1}{2m}\,p^2 - \Omega\mathbf{L} + \varphi\,, \end{aligned} \tag{15}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is the angular momentum. We can see that neither the Coriolis force nor the centrifugal potential appear anymore in the hamiltonian. Instead, it contains the angular momentum.

The local coordinate transformation (13) leads to a distance given by

$$\begin{aligned} ds^2 = \left[1 + h - \frac{(\Omega\times\mathbf{r})^2}{c^2}\right](dx^0)^2 - \\ - \frac{2}{c}\big(\Omega\times\mathbf{r}\big)\,d\mathbf{r}\,dx^0 - d\mathbf{r}^2, \end{aligned} \tag{16}$$

where a static potential $\sim h$ is introduced as before, related to the potential φ in (15). It can be checked, through more laborious calculations, that the free motion in the curved space given by (16) is equivalent with the non-relativistic equations of motion given by (14).

As it is well-known, a difficulty appears however in the above metric, related to the unbounded increase with $\mathbf{r}$ of the $\Omega\times\mathbf{r}$. Therefore, we drop out the square of this term in the g_{00}-term above, and keep only the first-order contributions in $\Omega\times\mathbf{r}$ in the subsequent calculations. As one can see, this approximation does not affect the hamiltonian (15). With this approximation, the metric given by (16) is identical with the metric given by equation (4), with the identification

$$\mathbf{g} = -\frac{1}{c}\,(\Omega\times\mathbf{r})\,. \tag{17}$$

Coordinate transformations. The translation given by (10) or the rotations given by (13) correspond to local coordinate transformations. As it is well-known, we can define such transformations in general, through suitable matrices (vierbeins). They take locally the infinitesimal coordinates in a flat space into infinitesimal coordinates in a curved space. For instance, the coordinate transformation corresponding to our metric given by equation (3) is given by

$$\left.\begin{aligned} dt &= \frac{(1+h)dt' + (g+\beta\Delta)\frac{dx'}{c}}{\sqrt{(1+h)(1-\beta^2)}} \\ dx &= \frac{c\beta(1+h)\,dt' + (\beta g+\Delta)dx'}{\sqrt{(1+h)(1-\beta^2)}} \end{aligned}\right\}, \tag{18}$$

$dy = dy'$, $dz = dz'$, where $\Delta = \sqrt{1+h+g^2}$, while $\mathbf{g}$ is along $dx = dx^1$, $\beta = \frac{V}{c}$ and the velocity V is $V = \frac{dx}{dt}$ for $dx' = 0$ ($dy = dx^2$, $dz = dx^3$). The inverse of this transformation is

$$\left.\begin{aligned} dt' &= \frac{g\big(\beta dt - \frac{dx}{c}\big) + \Delta\big(dt - \frac{\beta dx}{c}\big)}{\Delta\,\sqrt{(1+h)(1-\beta^2)}} \\ dx' &= \sqrt{1+h}\,\frac{dx - c\beta\,dt}{\Delta\,\sqrt{1-\beta^2}} \end{aligned}\right\}. \tag{19}$$

*With regard to equation (3), this was for the first time when Einstein "suspected the time" (1905).

All the square roots in these equations must exist, which imposes certain restrictions upon h and β (reality conditions; in particular, $1+h>0$ and $1-\beta^2>0$).

In the local transformations given above it is assumed that there exist global transformations $x^i(x')$ and $x'^i(x)$, where x, x' stand for all x^i and, respectively, x'^i, because the coefficients in these transformations are functions of x or, respectively, x'. This restricts appreciably the derivation of metrics by means of (global) coordinate transformations, because in general, as it is well-known, the 10 elements of a metric cannot be obtained by 4 functions $x^i(x')$. Conversely, we can diagonalize the curved metric at any point, such as to reduce it to a locally flat metric (tangent space), but the flat coordinates (axes) will not, in general, be the same for all the points; they depend, in general, on the point.

One can see from (18) that in the flat limit h, $g \to 0$ the above transformations become the Lorentz transformations, as expected. Therefore, we may have corrections to the flat relativistic motion by first-order contributions of the parameters h and $\mathbf{g}$. Indeed, in this limit, the transformation (19) becomes

$$\left.\begin{aligned} dt &= \frac{\left(1+\frac{h}{2}\right)dt' + (g+\beta)\frac{dx'}{c}}{\sqrt{1-\beta^2}} \\ dx &= \frac{c\beta\left(1+\frac{h}{2}\right)dt' + (g\beta+1)\,dx'}{\sqrt{1-\beta^2}} \end{aligned}\right\} . \tag{20}$$

which include corrections to the Lorentz transformations, due to the curved space.

The metric given by (3) provides the proper time

$$d\tau = \sqrt{1+h}\,dt\,, \tag{21}$$

corresponding to $dx^\alpha = 0$. The metric given by (3) can also be written as

$$\begin{aligned} ds^2 &= c^2(1+h)\left[dt + \frac{1}{c(1+h)}\,\mathbf{g}\,d\mathbf{r}\right]^2 - \\ &\qquad - \left[d\mathbf{r}^2 + \frac{1}{1+h}(\mathbf{g}\,d\mathbf{r})^2\right], \end{aligned} \tag{22}$$

hence the length given by

$$dl^2 = d\mathbf{r}^2 + \frac{1}{1+h}\,(\mathbf{g}\,d\mathbf{r})^2 \tag{23}$$

and the time

$$dt' = \sqrt{1+h}\cdot\left[dt + \frac{1}{c(1+h)}\,\mathbf{g}\,d\mathbf{r}\right], \tag{24}$$

corresponding to the length dl. The difference $\Delta t = \frac{\mathbf{g}d\mathbf{r}}{c(1+h)}$ between the two times, $dt_1 = \frac{d\tau}{\sqrt{g_{00}}} = dt$ in the proper time (21) and $dt_2 = \frac{dt'}{\sqrt{g_{00}}} = dt + \frac{\mathbf{g}d\mathbf{r}}{c(1+h)}$ in the time given by (24), gives the difference in the synchronization of two simultaneous events, infinitesimally separated. The difference in time depends on the path followed to reach a point starting from another point.

We limit ourselves to the first order in h, $\mathbf{g}$, and put $\mathbf{g} = -\frac{\mathbf{V}}{c}$, in order to investigate corrections to the motion under the action of a weak force in a flat space moving with a non-uniform velocity $\mathbf{V}$ with respect to the observer. We will do the calculations basically for translations but a similar analysis can be made for rotations, using equation (17). For the observer, such a motion is then a free motion in a curved space with metric (3). The proper time is then $d\tau = \left(1+\frac{h}{2}\right)dt$, the time given by (24) becomes $dt' = \left(1+\frac{h}{2}\right)dt + \frac{\mathbf{g}d\mathbf{r}}{c}$ and the length is given by $dl^2 = d\mathbf{r}^2$, as for a three-dimensional euclidean space.

Hamilton-Jacobi equation. Let us assume that we have a particle moving freely in a flat space. We denote its contravariant momentum by $(P_0 = \frac{E_0}{c}, \mathbf{P})$ and the corresponding covariant momentum by $(P_0 - \mathbf{P})$, such that $P_0^2 - P^2 = m^2c^2$, where E_0 is the energy of the particle, and P_0, $\mathbf{P}$ are constant.

We can use the coordinate transformation given by (19) to get the momentum of the particle in the curved space. We prefer to write it down in its covariant form, using the metric (4). We get

$$\left.\begin{aligned} p_0 &= (1+h)p^0 + gp^1 = \sqrt{1+h}\,\frac{P_0 - \beta P_1}{\sqrt{1-\beta^2}} \\ p_1 &= gp^0 - p^1 = \frac{(g+\beta\Delta)P_0 - (g\beta+\Delta)P_1}{\sqrt{(1+h)(1-\beta^2)}} \end{aligned}\right\} . \tag{25}$$

Then, it seems that we would have already an integral of motion for the motion in the curved space, by using the definition $p_i = mc\frac{du_i}{ds}$. However, this is not true, because the p_i are at point x' in the curved space, while the coefficients in the transformation (19) are at point x in the flat space. To know the global coordinate transformations $x(x')$ and $x'(x)$ would amount to solve in fact the equations of motion.

We can revert the above transformations for P_0 and P_1, and make use of $P_0^2 - P^2 = m^2c^2$, with $p_2 = -P_2$, $p_3 = -P_3$ for $g = -\beta$. We get

$$\left(p_0 + gp_1\right)^2 - \Delta^2\left(p^2 + m^2c^2\right) = 0\,, \tag{26}$$

or

$$\left(E - c\mathbf{g}\mathbf{p}\right)^2 - c^2\left(1+h+g^2\right)\left(p^2 + m^2c^2\right) = 0\,, \tag{27}$$

where E is the energy of the particle and $\mathbf{p}$ denotes its three-dimensional momentum. This is the relation between energy and momentum for the motion in the curved space. It gives the Hamilton-Jacobi equation.

Indeed, $p_i = -\frac{\partial S}{\partial x^i}$ and, obviously, for a free particle, $p_i p^i$ is a constant; we put $p_i p^i = m^2c^2$ and get $g^{ij}p_i p_j = m^2c^2$ or

$$\left(\frac{\partial S}{\partial t} + c\mathbf{g}\frac{\partial S}{\partial \mathbf{r}}\right)^2 - c^2\left(1+h+g^2\right)\left[\left(\frac{\partial S}{\partial \mathbf{r}}\right)^2 + m^2c^2\right] = 0. \tag{28}$$

In the limit $h = \frac{2\varphi}{mc^2} \to 0$ and $\mathbf{g} = -\frac{\mathbf{V}}{c} \to 0$ it describes the relativistic motion of a particle under the action of the (weak) force $\mathbf{f} = -\frac{\partial\varphi}{\partial\mathbf{r}}$ and for an observer moving with a (small) velocity $\mathbf{V}$. One can check directly that the coordinate transformations given by equation (20) takes the free Hamilton-Jacobi equation $\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left[\left(\frac{\partial S}{\partial \mathbf{r}}\right)^2 + m^2c^2\right] = 0$ into the "interacting" Hamilton-Jacobi equation (28), as expected.

The eikonal equation. Waves move through $k_i dx^i = -d\Phi$, where $k_i = -\frac{\partial\Phi}{\partial x^i} = \left(\frac{\omega}{c}, \mathbf{k}\right)$, ω is the frequency, $\mathbf{k}$ is the wavevector and Φ is called the eikonal. In a flat space k_i are constant, and the wave propagates along a straight line, such that $k_i k^i = 0$, *i.e.* $\frac{\omega^2}{c^2} - k^2 = 0$ and $\Phi = -\omega t + \mathbf{kr}$. This is a light ray. In a curved space $k_i k^i = 0$ reads $g^{ij} k_i k_j = 0$, and for g^{ij} slightly departing from the flat metric we have the geometric approximation to the wave propagation. It is governed by the eikonal equation $g^{ij}\left(\frac{\partial\Phi}{\partial x^i}\right)\left(\frac{\partial\Phi}{\partial x^j}\right) = 0$, or

$$\left(\frac{1}{c}\frac{\partial\Phi}{\partial t} + \mathbf{g}\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 - \left(1 + h + g^2\right)\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = 0\,, \tag{29}$$

which is the Hamilton-Jacobi equation (28) for $m = 0$.

We neglect the g^2-contributions to this equation and notice that the first term may not depend on the time (h is a function of the coordinates only). It follows then that the first term in the above equation can be put equal to $\frac{\omega_0}{c}$,

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \mathbf{g}\frac{\partial\Phi}{\partial\mathbf{r}} = -\frac{\omega_0}{c}\,, \tag{30}$$

where ω_0 is the frequency of the wave in the flat space, and

$$\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = k^2 = \frac{1}{1+h}\left(\frac{\omega_0}{c}\right)^2 = \frac{1}{1+h}\,k_0^2\,, \tag{31}$$

where $\mathbf{k}_0$ is the wavevector in the flat space. Within our approximation equation (30) becomes

$$\frac{\partial\Phi}{c\partial t} = -\frac{\omega_0}{c} - \mathbf{g}\mathbf{k}_0\,. \tag{32}$$

We measure the frequency ω corresponding to the proper time, *i.e.* $\frac{\omega}{c} = -\frac{\partial\Phi}{c\partial\tau}$, where $d\tau = \sqrt{1+h}\,dt$ for our metric, so the measured frequency of the wave is given by

$$\frac{\omega}{c} = -\frac{\partial\Phi}{c\partial\tau} = -\frac{1}{\sqrt{1+h}}\frac{\partial\Phi}{c\partial t} = \frac{1}{\sqrt{1+h}}\frac{\omega_0}{c} + \mathbf{g}\mathbf{k}_0\,. \tag{33}$$

There exists, therefore, a shift in frequency

$$\frac{\Delta\omega}{\omega_0} = -\frac{h}{2} + \frac{c\mathbf{g}\mathbf{k}_0}{\omega_0}\,. \tag{34}$$

The first term in equation (34) is due to the static forces (like the gravitational potential, for instance), while the second term is analogous to the (longitudinal) Doppler effect, for $\mathbf{g} = -\frac{\mathbf{V}}{c}$.

By (31) we have

$$\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = (1-h)\,k_0^2\,. \tag{35}$$

We assume that h depends only on the radius r, and write the above equation in spherical coordinates; Φ does not depend on θ, and we put $\theta = \frac{\pi}{2}$;

$$\left(\frac{\partial\Phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2 = (1-h)\,k_0^2\,; \tag{36}$$

the solution is of the form

$$\Phi = \Phi_r(r) + M\varphi\,, \tag{37}$$

where M is a constant and

$$\Phi_r(r) = \int_\infty^r dr\cdot\sqrt{(1-h)\,k_0^2 - \frac{M^2}{r^2}}\,; \tag{38}$$

the trajectory is given by $\frac{\partial\Phi}{\partial M} = \text{const}$,* hence

$$\varphi = -\int_\infty^r dr\cdot\frac{M}{r^2\sqrt{(1-h)\,k_0^2 - \frac{M^2}{r^2}}}\,. \tag{39}$$

For $h = 0$ we get $r\sin\varphi = \frac{M}{k_0}$, which is a straight line passing at distance $\frac{M}{k_0}$ from the centre. The deviation angle is

$$\Delta\varphi = -\frac{k_0^2}{2}\int_\infty^r dr\,\frac{h\,M}{r^2\left(k_0^2 - \frac{M^2}{r^2}\right)^{3/2}}\,. \tag{40}$$

Therefore, the light ray is bent by the static forces in a curved space.† One can also define the refractive index $\mathbf{n}$ of the curved space, by $\mathbf{k} = \mathbf{n}\frac{\omega}{c}$. Its magnitude is related to $\mathbf{g}\mathbf{k}_0$, while its direction is associated to the inhomogeneity h of the space.

It is worth noting, by (32), that the time-dependent part of the eikonal is given by

$$\Phi_t(t) = -\omega_0 t + \mathbf{k_0}\mathbf{R}(t)\,, \tag{41}$$

for $\mathbf{g} = -\frac{\mathbf{V}}{c}$, *i.e.* the eikonal corresponding to a translation, as expected. A similar solution of the Hamilton-Jacobi equation can be obtained for massive particles.

Quantization. Suppose that we have a free motion. Then we know its solution, *i.e.* the dependence of the coordinates, say some x, on some parameter, which may be called some

*Constant M is a generalized moving freely coordinate; therefore, the force acting upon it vanishes, $\frac{\partial L}{\partial M} = 0$, or $\frac{d(\partial S/\partial M)}{dt} = 0$, *i.e.* $\frac{\partial S}{\partial M} = \text{const}$.

†The metric given by (3) for $h = \frac{2\varphi}{c^2}$ differs from the metric created by a gravitational point mass m with $\varphi = \frac{Gm}{r}$; they coincide only in the non-relativistic limit. The deviation angle given by (40) for a gravitational potential is smaller by a factor of 4 than the deviation angle in the gravitational potential of a point mass.

time t. Suppose further that we have a motion under the action of some forces. Then, we know the dependence of its coordinates, say some x', on some parameter, which may be the same t as in the former case. Then, we may establish a correspondence between x and x', *i.e.* a global coordinate transformation. It follows that the motion under the action of the forces is a global coordinate transformation applied to the free motion. Similarly, two distinct motions are put in relation to each other by such global coordinate transformations.

This line of thought, due to Einstein, lies at the basis of both the special theory of relativity and the general theory of relativity.

Indeed, it has beeen noticed that the equations of the electromagnetic field are invariant under Lorentz transformations of the coordinates, which leave the distance given by $s^2 = c^2t^2 - \mathbf{r}^2$ invariant. These transformations are an expression of the principle of inertia, and this invariance is the principle of relativity. As such, the Lorentz transformations are applicable to the motion of particles, starting, for instance, from a particle at rest. Let $x = \frac{c\beta\tau}{\sqrt{1-\beta^2}}$, $t = \frac{\tau}{\sqrt{1-\beta^2}}$ be these Lorentz transformations where τ is the time of the particle at rest. We may apply these transformations to the momentum $\mathbf{p} = \frac{\partial S}{\partial \mathbf{r}}$ and $p_0 = -\frac{\partial S}{c\,\partial t} = \frac{E}{c}$, where E is the energy of the particle. Then, we get immediately $\mathbf{p} = \frac{\mathbf{v}E}{c^2}$ and $E = \frac{E_0}{\sqrt{1-\beta^2}}$. The non-relativistic limit is recovered for $E_0 = mc^2$, the "inertia of the energy". The equations of motion are $\frac{d\mathbf{p}}{dt} = \mathbf{f}$, and we can see that indeed, there appear additional, "dynamic forces", depending on relativistic $\frac{v^2}{c^2}$-terms, in comparison with Newton's law. In adition, we get the Hamilton-Jacobi equation $E^2 - c^2\left(p^2 + m^2c^2\right) = 0$. This is the whole theory of special relativity.

The situation is similar in the general theory of relativity, except for the fact that in a curved space we have not the global coordinate transformations, in general, as in a flat space. However, the Hamilton-Jacobi equation gives access to the action function, which may provide a relationship between some integrals of motion. Action S depends on some constants of integration, say M. Then, these constants can be viewed as freely-moving generalized coordinates, so $\frac{\partial S}{\partial M} = \text{const}$, because the force $\frac{\partial L}{\partial M} = \frac{d(\partial S/\partial M)}{dt}$ vanishes. Equation $\frac{\partial S}{\partial M} = \text{const}$ provides the equation of the trajectory. Of course, this is based upon the assumption that the motion is classical, *i.e.* non-quantum, in the sense that there exists a trajectory. For instance, the solution of the Hamilton-Jacobi equation for a free particle is $S = -Et + \mathbf{pr}$, where E and $\mathbf{p}$ are constants such that $E = \sqrt{m^2c^4 + c^2p^2}$. By $\frac{\partial S}{\partial E} = \text{const}$ we get $-t + \frac{E}{c^2p^2}\,\mathbf{pr} = \text{const}$, which is the trajectory of a free particle.

For a classical motion it is useless to attempt to solve the motion in a curved space produced by a non-inertial motion, like non-uniform translations, because it is much simpler to solve the motion in the absence of the non-inertial motion and then get the solution by a coordinate transformation, like a non-uniform translation for instance. For a quantum motion, however, the things change appreciably.

The Hamilton-Jacobi equation admits another kind of motion too, the quantum motion. Obviously, for a free particle, the classical action given above is the phase of a wave. Then, it is natural to introduce a wavefunction ψ through $S = -i\hbar \ln\psi$, where $\hbar$ turns out to be Planck's constant. The classical motion is recovered in the limit $\hbar \to 0$, $\mathrm{Re}\psi = \text{finite}$ and $\mathrm{Im}\psi \to \infty$, such that $S = \text{finite}$. With this transformation we have $\mathbf{p} = -i\hbar\frac{\partial\psi/\partial\mathbf{r}}{\psi}$ and $E = i\hbar\frac{\partial\psi/\partial t}{\psi}$, which means that momentum and energy are eigenvalues of their corresponding operators, $-i\hbar\frac{\partial}{\partial\mathbf{r}}$ and $i\hbar\frac{\partial}{\partial t}$, respectively.* It follows that the physical quantities have not well-defined values anymore, in contrast to the classical motion. In particular, there is no trajectory of the motion. Instead, they have mean values and deviations, *i.e.* they have a statistical meaning, and the measurement process has to be defined in such terms. It turns out that the wavefunction squared is just the density of probability for the motion to be in some quantum state, and for a defined motion this probability must be conserved.

Klein-Gordon equation. With the substitution $E \to i\hbar\frac{\partial}{\partial t}$ and $\mathbf{p} \to -i\hbar\frac{\partial}{\partial\mathbf{r}}$ in the Hamilton-Jacobi equation in the flat space we get the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\,\frac{\partial^2\psi}{\partial\mathbf{r}^2} + \frac{m^2c^4}{\hbar^2}\,\psi = 0\,. \tag{42}$$

A similar quantization for the Hamilton-Jacobi equation given by (28) encounters difficulties, since the operators $1 + h + g^2$ and $p^2 + m^2c^2$ do not commute with each other, nor with the operator $E - c\mathbf{g}\mathbf{p}$.† We may neglect the g^2-term in $1 + h + g^2$, and write the Hamilton-Jacobi equation (28) as

$$\frac{1}{1+h}\left(E - c\mathbf{g}\mathbf{p}\right)^2 = c^2\left(p^2 + m^2c^2\right), \tag{43}$$

where the two operators in the left side of this equation commute now, up to quantities of the order of hg (or higher), which we neglect. With these approximations, the quantization rules can now be applied, and we get an equation which can be written as

$$\left(\frac{\partial}{\partial t} + c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)^2\psi - c^2(1+h)\left[\frac{\partial^2\psi}{\partial\mathbf{r}^2} - \frac{m^2c^2}{\hbar^2}\,\psi\right] = 0. \tag{44}$$

It can be viewed as describing the quantum motion of a particle under the action of a weak force $-\frac{mc^2}{2}\frac{\partial h(\mathbf{r})}{\partial\mathbf{r}}$, as seen by an observer moving with the small velocity $-c\mathbf{g}(t)$. It can

*Einstein's (1905) quantization of energy and de Broglie's (1923) quantization of momentum follow immediately by this assumption, which gives a meaning to the Bohr-Sommerfeld quantization rules (Bohr, 1913, Sommerfeld, 1915). The quantum operators was first seen as matrices by Heisenberg, Born, Jordan, Pauli (1925-1926).

†We recall that h is a function of the coordinates only, $h(\mathbf{r})$, and $\mathbf{g}$ is a function of the time only, $\mathbf{g}(t)$.

be derived directly from (42) by the coordinate transformations (20), in the limit $h, \mathbf{g} \to 0$.* It is worth noting, however, that there is still a slight inaccuracy in deriving this equation, arising from the fact that the operator $(1+h)(p^2+m^2c^2)$ is not hermitean. It reflects the indefineteness in writing $(1+h)(p^2+m^2c^2)$ or $(p^2+m^2c^2)(1+h)$ when passing from (43) to (44). This indicates the ambiguities in quantizing the relativistic motion, and they are remedied by the theory of the quantal fields, as it is shown below.

The above equation can be written more conveniently as

$$\left(i\hbar\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)^2\psi-c^2(1+h)\left(p^2+m^2c^2\right)\psi=0\,, \quad (45)$$

where $\mathbf{p}=-i\hbar\frac{\partial}{\partial\mathbf{r}}$ and $i\hbar\frac{\partial}{\partial t}$ stands for the energy E.

We introduce the operator

$$\begin{aligned} H^2 &= c^2(1+h)\left(p^2+m^2c^2\right) = \\ &= c^2\left(p^2+m^2c^2\right)+c^2h\left(p^2+m^2c^2\right), \end{aligned} \quad (46)$$

which is time-independent, and treat the h-term as a small perturbation. It is easy to see, in the first-order of the perturbation theory, that the wavefunctions are labelled by momentum $\mathbf{p}$, and are plane waves with a weak admixture of plane waves of the order of h; we denote them by $\varphi(\mathbf{p})$. Similarly, in the first-order of the perturbation theory, the eigenvalues of H^2 can be written as $E^2(p)=c^2(1+\bar{h})(p^2+m^2c^2)$, where $\bar{h}=\frac{1}{V}\int d\mathbf{r}\cdot h$, V being the volume of normalization. We have, therefore, $H^2\varphi(\mathbf{p})=E^2(p)\,\varphi(\mathbf{p})$. Now, we look for a time-dependent solution of equation (45) $\left(i\hbar\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)^2\psi= = H^2\psi$, which can also be written as $\left(i\hbar\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)\psi=H\psi$, where ψ is a superposition of eigenfunctions

$$\psi=\sum_{\mathbf{p}}c_{\mathbf{p}}(t)\;e^{-iE(p)t/\hbar}\;\varphi(\mathbf{p})\,. \quad (47)$$

We get

$$\dot{c}_{\mathbf{p}'}=-\frac{i}{\hbar}\sum_{\mathbf{p}}c_{\mathbf{p}}\,e^{-i[E(p)-E(p')]/\hbar}\,c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}}\,, \quad (48)$$

where $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ is the matrix element of the momentum $\mathbf{p}$ betwen the states $\varphi(\mathbf{p}')$ and $\varphi(\mathbf{p})$. We assume $c_{\mathbf{p}}=c^0_{\mathbf{p}}+c^1_{\mathbf{p}}$, such as $c^0_{\mathbf{p}'}=0$ for all $\mathbf{p}'\neq\mathbf{p}$ and $c^0_{\mathbf{p}}=1$, and get

$$\dot{c}^1_{\mathbf{p}'}=-\frac{i}{\hbar}\,e^{-i[E(p)-E(p')]/\hbar}\,c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}}\,, \quad (49)$$

which can be integrated straightforwardly. The square $|c^1_{\mathbf{p}'}|^2$ gives the transition probability from state $\varphi(\mathbf{p})$ in state $\varphi(\mathbf{p}')$.

It follows that an observer in a non-uniform translation might see quantum transitions between the states of a relativistic particle, providing the frequencies in the Fourier expansion of $\mathbf{g}(t)$ match the difference in the energy levels. In the zeroth-order of the perturbation theory the eigenfunctions $\varphi(\mathbf{p})$ are plane waves, and the matrix elements $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ of the momentum vanish, so there are no such transitions to this order. In general, if the total momentum is conserved, as for free or interacting particles, these transitions do not occur. In the first order of the perturbation theory for the external force represented by h the matrix elements of the momentum do not vanish, in general, and we may have transitions, as an effect of a non-uniform translation. Within this order of the perturbation theory the matrix elements of the momentum are of the order of h, and the transition amplitudes given by (49) are of the order of gh. We can see that the time-dependent term of the order of gh neglected in deriving equation (45) produces corrections to the transition amplitides of the order of gh^2, so its neglect is justified.

In general, the solution of the second-order differential equation (44) can be approached by using the Fourier transform. Then, it reduces to a homogeneous matricial equation, where labels are the frequency and the wavevector $(\omega,\mathbf{k})$, conveniently ordered. The condition of a non-trivial solution is the vanishing of the determinant of such an equation. This gives a set of conditions for the ordered points $(\omega,\mathbf{k})$ in the $(\omega,\mathbf{k})$-space, but these conditions do not provide anymore an algebraic connection between the frequency ω and the wavevector $\mathbf{k}$. This amounts to saying that for a given ω the wavevectors are not determined, and, conversely, for a given wavevector $\mathbf{k}$ the frequencies are not determined, *i.e.* the quantum states do not exist in fact, anymore. The particle exhibits quantum transitions, which make its quantum state undetermined. The same conclusion can also be seen by introducing a non-uniform translation in the phase of a plane wave, expanding the plane wave with respect to this translation, under certain restrictions, and then using the time Fourier expansion of the translation. The frequency of the original plane wave changes correspondingly, which indicates indeed that there are quantum transitions. One may say that for a curved space as the one represented by the metric given here, the quantization question has no meaning anymore, or it has the meaning given here.

In the non-relativistic limit, the above Klein-Gordon equation becomes

$$i\hbar\frac{\partial\psi}{\partial t}=H\psi=\left(mc^2+\frac{p^2}{2m}+\varphi\right)\psi+c\mathbf{g}\mathbf{p}\psi\,, \quad (50)$$

which is Schrödinger's equation up to the rest energy mc^2, and one can see more directly the perturbation $c\mathbf{g}\mathbf{p}=-\mathbf{V}\mathbf{p}$. It is worth noting that the derivation of Schrödinger's equation holds irrespectively of the ambiguities related to the quantization of the Hamilton-Jacobi equation. It follows, that under the conditions mentioned above, *i.e.* in the presence of a (non-trivial) external field φ, an observer in a non-uniform translation may observe quantum transitions in the non-

*It has to be compared with the Klein-Gordon equation written as $\left(i\hbar\frac{\partial}{\partial t}-e\varphi\right)^2\psi-c^2\left[\left(i\hbar\frac{\partial}{\partial\mathbf{r}}+\frac{e\mathbf{A}}{c}\right)^2+m^2c^2\right]=0$ for a particle with charge e in the electromagnetic field $(\varphi,\mathbf{A})$, which, hystorically, was first considered for the Hydrogen atom (Schrödinger, Klein, Gordon, 1926). There, the forces come by the electromagnetic gauge field.

relativistic limit, due to the non-inertial motion.* Obviously, the frequency of this motion must match the quantum energy gaps, for such transitions to be observed.

Similar considerations hold for the metric corresponding to rotations. It is the hamiltonian (15) which is subjected to quantization in that case, so we may have quantum transitions between the states of the particle, providing these states do not conserve the angular momentum. This requires a force, as the one given by a potential φ. The $\Omega\times\mathbf{r}$ is exactly the rotation velocity $\mathbf{V}$, so we can apply directly the formalism developed above for a non-uniform translation to a non-uniform rotation. The only difference is that the $\mathbf{g}$ for rotations depends on the spatial coordinates too, beside its time dependence. The $\mathbf{g}$-interaction gives rise to terms of the type $\Omega\mathbf{L}$, and the evaluation of the matrix elements in the interacting terms becomes more cumbersome. It is worth keeping in mind the condition $\Omega r \ll c$ in such evaluations.

The difficulties encountered above with the quantization of the Klein-Gordon equation in curved spaces remain for a corresponding Dirac equation. It is impossible, in general, to get a Dirac equation for equation (44), because the operators $\left(1-\frac{h}{2}\right)(E-c\mathbf{g}\mathbf{p})$ and $\alpha c\mathbf{p}+\beta mc^2$ (with α and β the Dirac matrices), which represent the square roots of the two sides of equation (43), do not commute anymore. Nevertheless, if we limit ourselves to the first order of the perturbation theory, we can see that the operator H^2 defined above reduces to $c^2(p^2+m^2)$ providing we redefine the energy levels such as to include the factor $1+\bar{h}$. Within this approximation, we get the Dirac equation

$$\left(i\hbar\,\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)\psi=(\alpha c\mathbf{p}+\beta mc^2)\,\psi\,, \tag{51}$$

where ψ contains now a weak admixture of plane waves, of the order of h. It is worth noting that this equation is the Dirac equation corresponding to (42), subjected to the translation $\mathbf{r}=\mathbf{r}'+\mathbf{R}$, and $t=t'$. The non-uniform translation in the left side of equation (51) gives now quantum transitions.

As it is well-known, there remain problems with the quantization of the Klein-Gordon equation, which are not solved by the Dirac equation. These problems find for themselves a natural solution with the quantum fields.

A scalar field in a curved space. Let

$$S=\int dx^0 d\mathbf{r}\,\sqrt{-g}\,\left[(\partial_i\psi)(\partial^i\psi)+\frac{m^2c^2}{\hbar^2}\,\psi^2\right] \tag{52}$$

be the lagrangian for the (real) scalar field ψ, where $g=-\Delta^2=-(1+h+g^2)$ is the determinant of the metric given by (4).† It is easy to see that the principle of least action for ψ in a flat space leads to the Klein-Gordon equation (42). For the metric given by (4), and neglecting g^2-terms, we get a generalized Klein-Gordon equation

$$\begin{gathered}\left(\frac{\partial}{\partial t}+c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)\frac{1}{\sqrt{1+h}}\left(\frac{\partial}{\partial t}+c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)\psi- \\ -c^2\,\frac{\partial}{\partial\mathbf{r}}\sqrt{1+h}\,\frac{\partial}{\partial\mathbf{r}}\,\psi+\sqrt{1+h}\,\frac{m^2c^4}{\hbar^2}\,\psi=0\,.\end{gathered} \tag{53}$$

We can apply the same perturbation approach to this equation as we did for equation (43). Doing so, we get equation (45) and an additional term $i\,\frac{c^2\hbar}{2}\frac{\partial h}{\partial\mathbf{r}}\,\mathbf{p}$, which yields no difficulties in the perturbation approach. The resulting equation reads

$$\begin{gathered}\left(i\hbar\,\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)^2\psi-c^2(1+h)(p^2+m^2c^2)\psi+ \\ +\frac{i\,c^2\hbar}{2}\left(\frac{\partial h}{\partial\mathbf{r}}\right)\mathbf{p}\psi=0\,.\end{gathered} \tag{54}$$

It is worth noting that in the limit $\mathbf{g}\to 0$ this is an exact equation. The qualitative conclusions derived above for equation (45), as regards the quantum transitions produced by the non-uniform translation, remain valid, though, we have now a language of fields. It follows that a quantum particle, either relativistic or non-relativistic, in a curved space of the form analyzed herein becomes a wave packet from a plane wave (or even forms a bound state), as a consequence of the forces, and, at the same time, it may suffer quantum transitions, due to the time-dependent metric (as if in a non-inertial translation for instance). This gives no meaning to the problem of the quantization in curved spaces, or it gives the meaning discussed here.

The density L of lagrangian in the action $S=\int dt\,d\mathbf{r}\cdot L$ given by (52) gives the momentum $\Pi=\frac{\partial L}{\partial(\partial\psi/\partial t)}$ and the hamiltonian density $\Pi\,\frac{\partial\psi}{\partial t}-L$. The quantized field reads

$$\psi=\sum_{\mathbf{p}}\frac{c\hbar}{2\sqrt{\varepsilon}}\left(a_{\mathbf{p}}\,e^{-i\varepsilon t/\hbar+i\mathbf{p}\mathbf{r}/\hbar}+a^+_{\mathbf{p}}\,e^{i\varepsilon t/\hbar-i\mathbf{p}\mathbf{r}/\hbar}\right), \tag{55}$$

and

$$\Pi=-i\sum_{\mathbf{p}}\frac{\sqrt{\varepsilon}}{c}\left(a_{\mathbf{p}}\,e^{-i\varepsilon t/\hbar+i\mathbf{p}\mathbf{r}/\hbar}-a^+_{\mathbf{p}}\,e^{i\varepsilon t/\hbar-i\mathbf{p}\mathbf{r}/\hbar}\right), \tag{56}$$

where $\varepsilon=c\sqrt{m^2c^2+p^2}$ and $\left[\psi(t,\mathbf{r}),\Pi(t,\mathbf{r}')\right]=i\hbar\,\delta(\mathbf{r}-\mathbf{r}')$ with usual commutation relations for the bosonic operators $a_{\mathbf{p}}$, $a^+_{\mathbf{p}}$ and a normalization of one $\mathbf{p}$-state in a unit volume. The hamiltonian is obtained by integrating its density given

*A suitable unitary transformation of the wavefunction — for instance, $\exp\left(-i\,\frac{\mathbf{R}\mathbf{p}}{\hbar}\right)$ — can produce such an interaction in the time-dependent left side of the Schrödinger equation, but, at the same time, it produces an equivalent interaction in the hamiltonian, such that the Schrödinger equation is left unchanged. Such unitary transformations are related to symmetries (Wigner's theorem, 1931) and they are different from a change of coordinates.

†In general, the action for fields must be written by replacing the flat metric η_{ij} by the curved metric g_{ij} (including $\sqrt{-g}$ in the elementary volume of integration) and replacing the derivatives ∂_i by covariant derivatives D_i. The latter requirement can produce technical difficulties, in general. However, for a scalar field or for the electromagnetic field the D_i has the same effect as ∂_i, so the former are superfluous.

above over the whole space. It can be written as $H = H_0 + H_{1h} + H_{1g}$, where

$$H_0 = \int d\mathbf{r}\cdot\left[\frac{1}{4}\,c^2\Pi^2 + \left(\frac{\partial\psi}{\partial\mathbf{r}}\right)^2 + \frac{m^2c^2}{\hbar^2}\,\psi^2\right] = \sum_{\mathbf{p}}\frac{\varepsilon}{2}\left(a_{\mathbf{p}}a^+_{\mathbf{p}} + a^+_{\mathbf{p}}a_{\mathbf{p}}\right) \tag{57}$$

is the free hamiltonian,

$$H_{1h} = \int d\mathbf{r}\times(\sqrt{1+h}-1)\left[\frac{1}{4}\,c^2\Pi^2 + \left(\frac{\partial\psi}{\partial\mathbf{r}}\right)^2 + \frac{m^2c^2}{\hbar^2}\,\psi^2\right] \tag{58}$$

is the interacting part due to the external field h and

$$H_{1g} = -\frac{c}{2}\int d\mathbf{r}\cdot\left[\Pi\left(\mathbf{g}\,\frac{\partial\psi}{\partial\mathbf{r}}\right) + \left(\mathbf{g}\,\frac{\partial\psi}{\partial\mathbf{r}}\right)\Pi\right] = -\frac{c}{2}\sum_{\mathbf{p}}(\mathbf{g}\mathbf{p})(a_{\mathbf{p}}a^+_{\mathbf{p}} + a^+_{\mathbf{p}}a_{\mathbf{p}}) \tag{59}$$

is the time-dependent interaction. Perturbation theory can now be applied systematically to the first-order of $\mathbf{g}$ and all the orders of h, with the same results as those described above: the quanta will scatter both their wavevectors and their energy. Similar field theories can be set up for charged particles, or for particles with spin $\frac{1}{2}$ and for photons, moving in a curved space given by the metric (4).

Electromagnetic field in curved spaces. Photons. The action for the electromagnetic field is

$$S = -\frac{1}{16\pi c}\int dx^0 d\mathbf{r}\cdot\sqrt{-g}\,F_{ij}F^{ij}\,, \tag{60}$$

where the electromagnetic fields F_{ij} are given by the potentials A_i through $F_{ij} = \partial_i A_j - \partial_j A_i$. This leads immediately to the first pair of Maxwell equations (the free equations) $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ and the principle of least action gives the second pair of Maxwell equations

$$\partial_j(\sqrt{-g}\,F^{ij}) = 0\,. \tag{61}$$

In the presence of charges and currents the right side of equation (61) contains the current, conveniently defined. The antisymmetric tensor F_{ij} consists of a vector and a three-tensor in spatial components, the latter being representable by another vector, its dual. Let these vectors be denoted by $\mathbf{E}$ and $\mathbf{B}$. Similarly, by raising or lowering the suffixes we can define other two vectors, related to the former pair of vectors, and denoted by $\mathbf{D}$ and $\mathbf{H}$. Then, the Maxwell equations obtained above take the usual form of Maxwell equations in matter, namely $\mathrm{curl}\,\mathbf{E} = -\frac{1}{c\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\mathbf{B})}{\partial t}$, $\mathrm{div}\,\mathbf{B} = 0$ (the free equations) and $\mathrm{div}\,\mathbf{D} = 4\pi\rho$, $\mathrm{curl}\,\mathbf{H} = \frac{1}{c\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\mathbf{D})}{\partial t} + \frac{4\pi}{c}\rho\mathbf{v}$, where ρ is the density of charge divided by $\sqrt{\gamma}$ and $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}$ is the spatial metric (div and curl are conveniently defined in the curved space). For our metric, and neglecting g^2, the matrix γ reduces to the euclidean metric of the space ($\gamma = 1$).

We use $A_0 = 0$, $F_{0\alpha} = \partial_0 A_\alpha$ and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We define an electric field $\mathbf{E} = \mathrm{grad}\,\mathbf{A}$ and a magnetization field $\mathbf{B} = -\mathrm{curl}\,\mathbf{A}$. Then, neglecting g^2, equation (61) can be written as

$$\mathrm{div}\left[\frac{1}{\Delta}(\mathbf{E} + g\times\mathbf{B})\right] = 0 \tag{62}$$

and

$$\frac{\partial}{c\,\partial t}\left[\frac{1}{\Delta}(\mathbf{E} + \mathbf{g}\times\mathbf{B})\right] = \mathrm{curl}\left[\Delta\mathbf{B} + \frac{1}{\Delta}\,\mathbf{g}\times\mathbf{E}\right], \tag{63}$$

where $\Delta = \sqrt{1+h}$. One can see that we may have a displacement field $\mathbf{D} = \frac{\mathbf{E}+\mathbf{g}\times\mathbf{B}}{\Delta}$ and a magnetic field $\mathbf{H} = \Delta\mathbf{B} + \frac{\mathbf{g}\times\mathbf{E}}{\Delta}$, and the Maxwell equations $\mathrm{div}\,\mathbf{D} = 0$, $\frac{\partial\mathbf{D}}{c\,\partial t} = \mathrm{curl}\,\mathbf{H}$ without charges.

Equations (62) and (63) can be solved by the perturbation theory, for small values of h and $\mathbf{g}$, starting with free electromagnetic waves as the unperturbed solution. Doing so, we arrive immediately at the result that the solution must be a wave packet, and the frequencies are not determined anymore, in the sense that either for a given wavevector we have many frequencies or for a given frequency we have many wavevectors. This can be most conveniently expressed in terms of photons which suffer quantum transitions.

The quantization of the electromagnetic field in a curved space proceeds in the usual way. The action given by (60) can be written as

$$S = \frac{1}{8\pi}\int dt\,d\mathbf{r}\cdot\Delta\big(\mathbf{D}^2 - \mathbf{B}^2\big) = \frac{1}{8\pi}\int dt\,d\mathbf{r}\cdot\frac{1}{\Delta}\left[\mathbf{E}^2 + 2\mathbf{E}(\mathbf{g}\times\mathbf{B}) - \Delta^2\mathbf{B}^2\right], \tag{64}$$

which exhibits the well-known density of lagrangian in the limit $h, \mathbf{g}\to 0$. We change now to the covariant vector potential $\mathbf{A}\to -\mathbf{A}$, such that $\mathbf{E} = -\frac{\partial\mathbf{A}}{c\,\partial t}$ and $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$. Leaving aside the factor $\frac{1}{8\pi}$, the momentum is given by $\Pi = \frac{\partial L}{\partial(\partial\mathbf{A}/\partial t)} = \frac{2}{\Delta c^2}\left(\frac{\partial\mathbf{A}}{\partial t} - \mathbf{g}\times\mathbf{B}\right)$. The vector potential is represented as

$$\mathbf{A}_\alpha = \sum_{\alpha\mathbf{p}}\frac{c\hbar}{2\sqrt{\varepsilon}}\left[a_{\alpha\mathbf{p}}\,\mathbf{e}^\alpha\,e^{-i\varepsilon t/\hbar + i\mathbf{p}\mathbf{r}/\hbar} + h\,c\right] \tag{65}$$

and the momentum by

$$\Pi_\alpha = -i\sum_{\alpha\mathbf{p}}\frac{\sqrt{\varepsilon}}{c}\left[a_{\alpha\mathbf{p}}\,\mathbf{e}^\alpha\,e^{-i\varepsilon t/\hbar + i\mathbf{p}\mathbf{r}/\hbar} - h\,c\right], \tag{66}$$

where $\mathbf{e}^\alpha$ is the polarization vector along the direction α, perpendicular to $\mathbf{p} = \hbar\mathbf{k}$ (we assume the transversality condition $\mathrm{div}\mathbf{A} = 0$), $\varepsilon = \hbar\omega = cp$, while ω is the frequency and $\mathbf{k}$ is the wavevector. The commutation relations are the usual bosonic ones, and we get the hamiltonian $H = H_0 + H_{1h} + H_{1g}$, given by

$$\left.\begin{aligned} H_0 &= \int d\mathbf{r}\cdot\left(\frac{1}{4}\,c^2\Pi^2 + B^2\right) = \\ &\qquad = \sum_{\alpha\mathbf{p}} \frac{\varepsilon}{2}\left(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}\right) \\ H_{1h} &= \int d\mathbf{r}\cdot\left(\sqrt{1+h}-1\right)\left(\frac{1}{4}\,c^2\Pi^2 + \mathbf{B}^2\right) \\ H_{1g} &= -\frac{1}{2}\sum_{\alpha\mathbf{p}} \mathbf{gp}\left(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}\right) \end{aligned}\right\}. \tag{67}$$

Systematic calculations can now be performed within the perturbation theory, and we can see that quantum transitions between the photonic states may appear, starting with the hg-order of the perturbation theory. Therefore, an observer moving with a non-uniform velocity is able to see a "blue shift" in the frequency of the photons "acted" by a force like the gravitational one.* The shift occurs obviously at the expense of the energy of the observer's motion.†

Other fields. A similar approach can be used for other fields in a curved space. In particular, it can be applied to spin-1/2 Dirac fields, with similar conclusions, though, technically, it is more cumbersome to write down the action for spinors in curved spaces. It can be speculated upon the question of quantizing the gravitational field in a similar manner. Indeed, weak perturbations of the flat metric can be represented as gravitational waves, which can be quantized by using the gravitational action $\int dx^0 d\mathbf{r}\cdot\sqrt{-g}\,R$, where R is the curvature of the space.‡ Now, we may suppose that these gravitons move in a curved space with the metric g. We may use the same gravitational action as before, where g is now the metric of the space and R contains the graviton field. Or, alternately, we expand $g = g_0 + \delta g$, where g_0 is the background part and δg is the graviton part. We get a field theory of gravitons interacting with the underlying curved space, and we get quantum transitions of the gravitons, which gives a meaning to the quantization of the gravity, in the sense that either it is not possible or the gravitons suffer quantum transitions. The space and time (the gravitons) are then scattered statistically by matter (which in turn suffers a similar process) or by the non-inertial motion.

Conclusion. The quantum motion implies, basically, delocalized waves, like plane wave, both in space and time. The general theory of relativity, gravitation or curved space as the one discussed here, arising from weak static forces and non-inertial motion, imply localized field, both in space and time. Consequently, the quantization is destroyed in those situations involved by the latter case, in the sense that quanta are scattered both in energy and the wavevector, and we have to deal there with transition amplitudes and probabilities, *i.e.* with a statistical perspective. The basic equations for the classical motion in these cases become meaningful only with scattered quanta. This shows indeed that the quantization is both necessary and illusory. The basic aspect of the natural world is its statistical character in terms of quanta.

Submitted on December 12, 2007
Accepted on December 21, 2007

References

1. Pauli W. Theory of relativity. Teubner, Leipzig, 1921.
2. Dirac P. A. M. General Theory of Relativity. Princeton University Press, Princeton, 1975.
3. Weinberg S. Gravitation and cosmology. Wiley and Son, New York, 1972.
4. Kaku M. Quantum field theory: a modern introduction. Oxford University Press, Oxford, 1993.
5. Wald R. M. Quantum field theory in curved spacetime and black hole thermodynamics (Chicago lectures in physics). The University of Chicago Press, Chicago, 1994.

*This is similar with the Unruh effect (1976).

†It is worth investigating the change in the equilibrium distribution of the black-body radiation as a consequence of the non-uniform translation in a gravitational field. The frequency shift amounts to a change of temperature, which increases, most likely, by $\frac{\Delta T}{T} \sim \left(g\bar{h}\right)^2$, with temporal and spatial averages (for the quantization of the black-body radiation see Fermi, 1932). In this respect, the effect discussed here, though related to the Unruh effect, is different. The Unruh effect assumes rather that the external non-uniform translation, as a macroscopic motion, consists of a coherent vacuum, so equilibrium photons can be created; the related increase in temperature is rather the measurement made by the observer of its own motion.

‡Though there are difficulties in establishing a relativistically-invariant quantum theory for particles with helicity 2, like the gravitons. Another related difficulty is the general non-localizability of the gravitational energy.