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## Journal of Modern Optics

Publication details, including instructions for authors and subscription information:

<http://www.informaworld.com/smpp/title~content=t713191304>

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First published on: 25 February 2011

**To cite this Article** Apostol, M.(2011) 'Dynamics of electron-positron pairs in a vacuum polarized by an external radiation field', Journal of Modern Optics, 58: 7, 611 – 618, First published on: 25 February 2011 (iFirst)

**To link to this Article:** DOI: 10.1080/09500340.2011.558221

**URL:** <http://dx.doi.org/10.1080/09500340.2011.558221>

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## Dynamics of electron–positron pairs in a vacuum polarized by an external radiation field

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(Received 4 December 2010; final version received 19 January 2011)

The polarization of the vacuum under the action of an external classical field of electromagnetic radiation is investigated in the stationary regime. The electron–positron pairs interact both with the external field and with their own polarization field. For a macroscopic piece of vacuum the pairs are condensed on the low-momenta states and tend to form a quasi-localized electron–positron plasma of pairs, with single-particle states labeled by the position vector. In the polarization process under the action of a classical field of radiation the electron–positron and photon dynamics can be treated by means of classical fields. Under these circumstances, the corresponding coupled non-linear equations of motion are solved. It is shown that the pair dynamics consists of quasi-stationary single-particle states, while the polarization field reduces to a static magnetic field. The single-particle ‘energy’ (temporal phase) due to a monochromatic external field exhibits a spatial distribution characteristic of a stationary wave. Both the pair energy and the polarization energy are computed. Their values are extremely small, even for highly focused, reasonably high, external fields. The number of pairs is determined by the external energy. Under the action of a classical field the polarized vacuum is magnetized, and the corresponding (very low) magnetic susceptibility (the refractive index of the vacuum) is computed.

**Keywords:** electron–positron pairs; vacuum polarization; stationary regime; vacuum refractive index; high external fields

Creation of electron–positron pairs from vacuum under the action of high external electromagnetic fields is a classical and still very attractive subject [1–16]. A great deal of work has been devoted recently to various aspects of this problem [17–24]. With the recent development of high-power laser pulses focused on very small spatial regions [25–36] the vacuum polarization could become soon a matter of experimental testing and routine measurements.

The annihilation of the electron–positron pairs in the polarized vacuum suggests a possible stationary regime, which may resemble a plasma of electron–positron pairs. This paper deals with the stationary dynamics of the polarized vacuum under the action of an external classical field of electromagnetic radiation (the thermalization of a non-equilibrium electron–positron plasma has been investigated in [37]). It is shown that the interaction of the electron–positron pairs with their own polarization field in a macroscopic piece of vacuum is mainly governed by the condensation of these pairs on low-momenta states. The pairs are described by single-particle states labeled by the position vector. In the polarization process under the action of a classical radiation field the electron–positron and photon dynamics can be treated by means of classical fields. The corresponding coupled non-linear equations of motion are solved.

The solutions exhibit a quasi-stationary regime, and the single-particle energies are computed. The corresponding polarization field is static, in the sense that the electric field is vanishing and only a static magnetic field is present. The polarized vacuum gets magnetized. Under the action of an external field (monochromatic plane wave), the single-particle ‘energies’ (temporal phase) acquire the shape of a stationary wave driven by the external field. The number of pairs, the pairs energy and the polarization energy are computed. The number of pairs is determined by the external field energy. The resulting values are extremely small, even for reasonably high external fields and energy densities. This is due, mainly, to the Compton wavelength of the electrons which is much smaller than the size of the space region over which the external energy is focused. The magnetic susceptibility is also evaluated (the refractive index) for the polarized vacuum, and, similarly, it is found to acquire very low values.

As is well known, the electromagnetic radiation field is described by the vector potential

$$A(\mathbf{r}) = \sum_{\mu k} \left( \frac{2\pi\hbar c^2}{V\omega_k} \right)^{1/2} [e_{\mu}(\mathbf{k})a_{\mu k} \exp(i\mathbf{k}\mathbf{r}) + e_{\mu}^*(\mathbf{k})a_{\mu k}^* \exp(-i\mathbf{k}\mathbf{r})] \quad (1)$$

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in the standard Fourier representation, with the transverse gauge  $\text{div } \mathbf{A} = 0$ , where  $c$  is the velocity of light,  $V$  is the volume,  $\omega_k = ck$  is the frequency and  $\mathbf{e}_\mu(\mathbf{k})$  are the polarization vectors,  $\mathbf{e}_\mu(\mathbf{k})\mathbf{k} = 0$ ,  $\mathbf{e}_\mu(\mathbf{k})\mathbf{e}_\nu^*(\mathbf{k}) = \delta_{\mu\nu}$  ( $\mu, \nu = \pm 1$ ),  $\mathbf{e}_{-\mu}(-\mathbf{k}) = \mathbf{e}_\mu^*(\mathbf{k})$ . The electric and magnetic field are given by  $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t$  and, respectively,  $\mathbf{H} = \text{curl } \mathbf{A}$ , and three Maxwell's equations are satisfied:  $\text{curl } \mathbf{E} = -(1/c)\partial\mathbf{H}/\partial t$ ,  $\text{div } \mathbf{H} = 0$ ,  $\text{div } \mathbf{E} = 0$ . The time dependence is included in the Fourier coefficients  $a_{\mu k}$ ,  $a_{\mu k}^*$  (photon annihilation and, respectively, creation operators).

The Lagrangian of the radiation field

$$\begin{aligned} L_f &= \frac{1}{8\pi} \int d\mathbf{r} (E^2 - H^2) \\ &= \sum_{\mu k} \frac{\hbar}{4\omega_k} (\dot{a}_{\mu k} + \dot{a}_{-\mu-k}^*) (\dot{a}_{-\mu-k} + \dot{a}_{\mu k}^*) \\ &\quad - \sum_{\mu k} \frac{\hbar\omega_k}{4} (a_{\mu k} + a_{-\mu-k}^*) (a_{-\mu-k} + a_{\mu k}^*) \end{aligned} \quad (2)$$

leads to the equation of motion

$$\ddot{a}_{\mu k} + \ddot{a}_{-\mu-k}^* + \omega_k^2 (a_{\mu k} + a_{-\mu-k}^*) = 0, \quad (3)$$

which is the fourth Maxwell's equation  $\text{curl } \mathbf{H} = (1/c)\partial\mathbf{E}/\partial t$ .

The standard Dirac field for electrons and positrons is written as

$$\begin{aligned} \psi(\mathbf{r}) &= \sum_{\sigma p} \frac{1}{(2\varepsilon V)^{1/2}} \left[ u_{p\sigma} b_{p\sigma} \exp\left(\frac{i}{\hbar} \mathbf{p}\mathbf{r}\right) \right. \\ &\quad \left. + v_{p\sigma} c_{p\sigma}^* \exp\left(-\frac{i}{\hbar} \mathbf{p}\mathbf{r}\right) \right], \end{aligned} \quad (4)$$

where  $\varepsilon = (c^2 p^2 + m^2 c^4)^{1/2}$ ,  $m$  is the electron mass,  $\sigma = \pm 1$  is the spin label and the bispinors  $u_{p\sigma}$ ,  $v_{p\sigma}$  are given by

$$\begin{aligned} u_{p\sigma} &= \begin{pmatrix} (\varepsilon + mc^2)^{1/2} w_\sigma \\ (\varepsilon - mc^2)^{1/2} (\mathbf{n}\boldsymbol{\sigma}) w_\sigma \end{pmatrix}, \\ v_{p\sigma} &= \begin{pmatrix} (\varepsilon - mc^2)^{1/2} (\mathbf{n}\boldsymbol{\sigma}) w'_\sigma \\ (\varepsilon + mc^2)^{1/2} w'_\sigma \end{pmatrix}; \end{aligned} \quad (5)$$

here  $\mathbf{n} = \mathbf{p}/p$  is the unit vector along the momentum  $\mathbf{p}$ ,  $\boldsymbol{\sigma}$  denote the Pauli matrices and  $w_\sigma$ ,  $w'_\sigma = -\sigma_y w_{-\sigma}$  are normalized spinors,  $w_\sigma^* w_{\sigma'} = \delta_{\sigma\sigma'}$ ,  $w_\sigma'^* w_{\sigma'}' = \delta_{\sigma\sigma'}$  (otherwise arbitrary). The notation  $*$  means transposition together with complex conjugation. In general, we use the notations and conventions from [38,39]. As is well known, the free hamiltonian of the fermions reads

$$H_0 = \sum_{\sigma p} \varepsilon (b_{p\sigma}^* b_{p\sigma} - c_{p\sigma} c_{p\sigma}^*), \quad (6)$$

which leads to the equations of motion  $i\hbar \dot{b}_{p\sigma} = \varepsilon b_{p\sigma}$ ,  $i\hbar \dot{c}_{p\sigma} = \varepsilon c_{p\sigma}$  for the electron and positron destruction operators  $b_{p\sigma}$ ,  $c_{p\sigma}$ .

Making use of Equations (1) and (4) we compute the interaction Hamiltonian

$$H_{\text{int}} = -\frac{e}{c} \int d\mathbf{r} \psi^*(\mathbf{r}) \mathbf{j} \psi(\mathbf{r}) \mathbf{A}(\mathbf{r}), \quad (7)$$

where  $-e$  is the electron charge and  $\mathbf{j} = c\boldsymbol{\alpha}$  is the particle current,

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad (8)$$

being the Dirac  $\alpha$ -matrices. The computation of the matrix elements of the current  $\mathbf{j}$  between different electron-positron states involved in Equation (7) is lengthy but straightforward. It is worth noting that the current density for interacting electrons (positrons) differs from the group velocity  $c^2 \mathbf{p}/\varepsilon$  of the free electrons (positrons). The general form of the interaction matrix elements can be represented as  $M_{\sigma\sigma'}^\mu(\mathbf{p}, \mathbf{p}')$ . They contain the matrix elements  $(\bar{\boldsymbol{\sigma}})_{\sigma\sigma'} = w_\sigma^* \boldsymbol{\sigma} w_{\sigma'}$  of the Pauli matrices  $\boldsymbol{\sigma}$ . In general, the spinors may depend on the momenta  $\mathbf{p}$  (as for helicities), such that  $(\bar{\boldsymbol{\sigma}})_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}') = w_\sigma^*(\mathbf{p}) \boldsymbol{\sigma} w_{\sigma'}(\mathbf{p}')$ . It is important to note that there is an arbitrariness in these matrix elements, due to the arbitrariness in the spinors  $w_\sigma$ . The matrix  $\bar{\boldsymbol{\sigma}}$  is related to the polarization matrix of each elementary act of interaction, but the spinors do not reduce necessarily to the well-defined spin states in the rest frame, nor the vector  $\bar{\boldsymbol{\sigma}}$  reduces to the polarization vector measured usually in scattering experiments. In the interaction process neither the spin, nor the helicities are conserved, i.e. both are undetermined. There is no reason to have a 'spin' dependence in the interaction, so we omit the spin label in the electron-positron operators and the polarization label in the photon operators. The interaction matrix elements can then be summed over the spin and polarization labels,  $M(\mathbf{p}, \mathbf{p}') = \sum_{\mu\sigma\sigma'} M_{\sigma\sigma'}^\mu(\mathbf{p}, \mathbf{p}')$ . The general structure of the interaction hamiltonian is then given by

$$\begin{aligned} H_{\text{int}} &= -ec \sum_{pk} \left( \frac{2\pi\hbar}{V\omega_k} \right)^{1/2} \left[ A(\mathbf{p}, \mathbf{p} - \mathbf{k}) b_p^* b_{p-k} \right. \\ &\quad \left. + B(\mathbf{p}, -\mathbf{p} + \mathbf{k}) b_p^* c_{-p+k}^* + B^*(-\mathbf{p} - \mathbf{k}, \mathbf{p}) c_p b_{-p-k} \right. \\ &\quad \left. + C(\mathbf{p}, \mathbf{p} + \mathbf{k}) c_p c_{p+k}^* \right] (a_k + a_k^*), \end{aligned} \quad (9)$$

where the coefficients  $A$ ,  $B$  and  $C$  (the matrix elements  $M(\mathbf{p}, \mathbf{p}')$ ) are given in Appendix 1. The most general structure of the vector  $\bar{\boldsymbol{\sigma}} = \sum_{\sigma\sigma'} w_\sigma^* \boldsymbol{\sigma} w_{\sigma'}$  is given in Appendix 2. In accordance with our assumption that the interaction matrix elements should not depend on the 'spin' orientation we take the mean value of this vector over all possible polarizations, and get  $(\bar{\boldsymbol{\sigma}})_{\text{av}} = 0$ .

This amounts to a statistical (uniform) average of the interaction hamiltonian over ‘spin’ states. The coefficients  $A$ ,  $B$  and  $C$  simplify then appreciably, and the interaction Hamiltonian becomes

$$H_{\text{int}} = -ec \sum_{\mathbf{p}\mathbf{k}} \left( \frac{2\pi\hbar}{V\omega_k} \right)^{1/2} [A(\mathbf{p}, \mathbf{p} - \mathbf{k})b_{\mathbf{p}}^*b_{\mathbf{p}-\mathbf{k}} + B(\mathbf{p}, -\mathbf{p} + \mathbf{k})b_{\mathbf{p}}^*c_{-\mathbf{p}+\mathbf{k}}^* + B(\mathbf{p}, -\mathbf{p} - \mathbf{k})c_{\mathbf{p}}b_{-\mathbf{p}-\mathbf{k}} + A(\mathbf{p}, \mathbf{p} + \mathbf{k})c_{\mathbf{p}}c_{\mathbf{p}+\mathbf{k}}^*](a_{\mathbf{k}} + a_{-\mathbf{k}}^*), \quad (10)$$

where

$$A(\mathbf{p}, \mathbf{p}') = \frac{1}{(\varepsilon\varepsilon')^{1/2}} [\mathbf{e}\mathbf{n}[(\varepsilon - mc^2)(\varepsilon' + mc^2)]^{1/2} + \mathbf{e}\mathbf{n}'[(\varepsilon + mc^2)(\varepsilon' - mc^2)]^{1/2}],$$

$$B(\mathbf{p}, \mathbf{p}') = -\frac{1}{(\varepsilon\varepsilon')^{1/2}} \{e_y[(\varepsilon + mc^2)(\varepsilon' + mc^2)]^{1/2} + [(\mathbf{e}\mathbf{n})n'_y - (\mathbf{n}\mathbf{n}')e_y + (\mathbf{e}\mathbf{n}')n_y] \times [(\varepsilon - mc^2)(\varepsilon' - mc^2)]^{1/2}\} \quad (11)$$

and  $\varepsilon' = (c^2p'^2 + m^2c^4)^{1/2}$ ,  $\mathbf{e} = \sum_{\mu} e_{\mu}(\mathbf{k})$ . We emphasize the dependence on  $\mathbf{k}$  of the coefficients  $A(\mathbf{p}, \mathbf{p}')$  and  $B(\mathbf{p}, \mathbf{p}')$ , through the polarization vector  $\mathbf{e}$ . For brevity, we use notations like  $\mathbf{p} \pm \mathbf{k}$  for  $\mathbf{p} \pm \hbar\mathbf{k}$ . The interaction Hamiltonian given by Equation (10) contains electron–electron, positron–positron interactions (the terms with the coefficients  $A(\mathbf{p}, \mathbf{p} \mp \mathbf{k})$ ) and the creation and annihilation of pairs (the terms with the coefficients  $B(\mathbf{p}, -\mathbf{p} \pm \mathbf{k})$ ). The corresponding equations of motion read

$$\ddot{a}_{\mathbf{k}} + \ddot{a}_{-\mathbf{k}}^* + \omega_k^2(\ddot{a}_{\mathbf{k}} + \ddot{a}_{-\mathbf{k}}^*) = 2ec \sum_{\mathbf{p}} \left( \frac{2\pi\omega_k}{\hbar V} \right)^{1/2} [A(\mathbf{p}, \mathbf{p} + \mathbf{k})b_{\mathbf{p}}^*b_{\mathbf{p}+\mathbf{k}} + B(\mathbf{p}, -\mathbf{p} - \mathbf{k})b_{\mathbf{p}}^*c_{-\mathbf{p}-\mathbf{k}}^* + B(\mathbf{p}, -\mathbf{p} + \mathbf{k})c_{\mathbf{p}}b_{-\mathbf{p}+\mathbf{k}} + A(\mathbf{p}, \mathbf{p} - \mathbf{k})c_{\mathbf{p}}c_{\mathbf{p}-\mathbf{k}}^*] \quad (12)$$

and

$$i\hbar\dot{b}_{\mathbf{p}} = \varepsilon_{\mathbf{p}}b_{\mathbf{p}} - ec \sum_{\mathbf{k}} \left( \frac{2\pi\hbar}{V\omega_k} \right)^{1/2} \times [A(\mathbf{p}, \mathbf{p} - \mathbf{k})b_{\mathbf{p}-\mathbf{k}} + B(\mathbf{p}, -\mathbf{p} + \mathbf{k})c_{-\mathbf{p}+\mathbf{k}}^*](a_{\mathbf{k}} + a_{-\mathbf{k}}^*),$$

$$i\hbar\dot{c}_{\mathbf{p}} = \varepsilon_{\mathbf{p}}c_{\mathbf{p}} + ec \sum_{\mathbf{k}} \left( \frac{2\pi\hbar}{V\omega_k} \right)^{1/2} \times [A(\mathbf{p} - \mathbf{k}, \mathbf{p})c_{\mathbf{p}-\mathbf{k}} + B(-\mathbf{p} + \mathbf{k}, \mathbf{p})b_{-\mathbf{p}+\mathbf{k}}^*](a_{\mathbf{k}} + a_{-\mathbf{k}}^*). \quad (13)$$

It is easy to check that the interaction Hamiltonian given by Equation (10) conserves the charge  $Q = \sum_{\mathbf{p}} (b_{\mathbf{p}}^*b_{\mathbf{p}} - c_{\mathbf{p}}^*c_{\mathbf{p}})$ . This is the standard framework (not manifestly covariant) provided by the quantum electrodynamics for an ensemble of interacting electrons, positrons and photons. For the creation and annihilation of electron–positron pairs in the process of polarization of a piece of macroscopic vacuum we adopt here a special route.

Before passing to solving the equations of motion given above, it is worth noting that our procedure of averaging over spin states, used here in order to simplify the interaction matrix elements, is not applicable anymore if a polarized external field is present in the interaction (e.g. a circularly polarized field). Then, it is natural to refer the spinors  $w_{\sigma}$  to the external polarized field, thus removing the arbitrariness in their definition. The interaction matrix elements will depend now on the polarization direction of the external field, and the average over the angular directions of the momenta in the effective coupling constant  $b$  given below (Equation (18)) must take into account this circumstance. The result is a corresponding change in  $b$ , without affecting qualitatively the general conclusions (as long as we are not interested in the spin dynamics, which, in general, is affected by the polarization of the external field).

In general, the states of interacting fermions with spin one-half are admixtures of empty ( $|0\rangle$ ) and occupied ( $|1\rangle$ ) states. The creation and destruction operators can then be equivalently represented by one  $c$ -number. For instance, let  $|s\rangle = \alpha|0\rangle + \beta|1\rangle$  be such a state, with coefficients  $\alpha, \beta$ . The destruction operator  $b$  has only one non-vanishing matrix element,  $\langle 0|b|s\rangle = \beta$ , or  $\langle s|b|1\rangle = \alpha^*$ . For definiteness, we choose  $\langle 0|b|s\rangle = \beta$ . The occupation number is given by  $\langle s|b^*b|s\rangle = |\beta|^2$ . Since the states  $|s\rangle$  for an ensemble of interacting fermions are not, in general, well-defined single-particle states,  $|\beta|^2$  is not subjected to the restriction  $|\beta|^2 \leq 1$ . Instead, the summation of the occupation numbers over all the states must equal the total number of fermions. Consequently, we can take such matrix elements in the first Equation (13), which amounts to work with fermionic amplitudes which are  $c$ -numbers, instead of operators. These amplitudes can be viewed as classical fields. The charge conservation  $Q=0$  for pairs suggests the replacement

$$b_{\mathbf{p}} \rightarrow \beta_{\mathbf{p}}, \quad c_{-\mathbf{p}}^* \rightarrow \beta_{\mathbf{p}}, \quad (14)$$

in accordance with the particle–hole symmetry. This procedure holds whenever the starting states, i.e. the particular states we use to formulate the problem (in our case momentum states), or finite linear combinations of them, are not exact states (which, as we see

below, in our case are states labeled by the position vector  $\mathbf{r}$ ). Obviously, this implies a macroscopic number of fermions. Similarly, we replace the photon operators by  $c$ -numbers,

$$a_{\mathbf{k}} + a_{-\mathbf{k}}^* \rightarrow A_{\mathbf{k}}, \quad (15)$$

where  $A_{\mathbf{k}}$  are viewed as classical fields. As is well known, this amounts to employing coherent states and a coherent interaction of matter and radiation [40]. The (macroscopic) number of photons is not defined anymore in such classical fields, in contrast with the field phase, which is well defined. It is worth noting that Equation (15) makes no distinction anymore between absorption or creation of a photon, in agreement with the fact that the number of photons is not a well-defined number. The interaction hamiltonian becomes

$$H_{\text{int}} = -2ec \sum_{\mathbf{p}\mathbf{k}} \left( \frac{2\pi\hbar}{V\omega_{\mathbf{k}}} \right)^{1/2} B(\mathbf{p}, -\mathbf{p} + \mathbf{k}) \beta_{\mathbf{p}}^* \beta_{\mathbf{p}-\mathbf{k}} A_{\mathbf{k}}, \quad (16)$$

and the equations of motion read

$$\begin{aligned} \ddot{A}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 A_{\mathbf{k}} &= 4ec \sum_{\mathbf{p}} \left( \frac{2\pi\omega_{\mathbf{k}}}{\hbar V} \right)^{1/2} B(\mathbf{p}, -\mathbf{p} - \mathbf{k}) \beta_{\mathbf{p}}^* \beta_{\mathbf{p}+\mathbf{k}}, \\ i\hbar \dot{\beta}_{\mathbf{p}} &= \varepsilon_{\mathbf{p}} \beta_{\mathbf{p}} - 2ec \sum_{\mathbf{k}} \left( \frac{2\pi\hbar}{V\omega_{\mathbf{k}}} \right)^{1/2} \\ &\times B(\mathbf{p}, -\mathbf{p} + \mathbf{k}) \beta_{\mathbf{p}-\mathbf{k}} A_{\mathbf{k}}. \end{aligned} \quad (17)$$

We can see that the scattering of individual electrons (or positrons) disappears from the interaction Hamiltonian (the terms with the  $A$ -coefficients in Equation (10)), and the interaction is determined by the vacuum polarization (creation and annihilation of pairs), as expected. The product  $\beta_{\mathbf{p}}^* \beta_{\mathbf{p}-\mathbf{k}}$  in the interaction Hamiltonian can also be viewed as corresponding to the excitation (and dis-excitation) of an ensemble of particles, each with two energy levels (labeled by  $\mathbf{p}$  and  $-\mathbf{p} + \mathbf{k}$ ), the levels corresponding to positive and, respectively, negative energy states. This latter feature is incorporated in the structure of the  $B$ -coefficients. As is well known, such an ensemble of particles can be excited (polarized) in a stationary regime by an external classical field of radiation, which pumps energy in the ensemble, resembling to some extent the laser effect [41]. We note also that the product  $\beta_{\mathbf{p}}^* \beta_{\mathbf{p}+\mathbf{k}}$  appearing in the  $rhs$  of the first Equation (17) for the electromagnetic field is related to the medium polarization (more exactly to the polarization current).

For reasonable energies we may limit ourselves to  $p, \hbar k < p_0 \ll mc$ , where  $p_0$  is a momentum cutoff, and expand the coupling coefficients  $B(\mathbf{p}, \mathbf{p} \pm \mathbf{k})$  in powers of  $\mathbf{p}$  and  $\mathbf{k}$ . Similarly, we approximate

$\varepsilon_{\mathbf{p}}$  in Equations (17) by  $\varepsilon_0 = mc^2$ . For such small values of the momenta the angular dependence of the coupling function  $B(\mathbf{p}, -\mathbf{p} + \mathbf{k})$  is practically irrelevant for the qualitative behavior of the solutions of the system of Equations (17). The structure of this system shows that the relevant coupling function is the product  $B(\mathbf{p}, -\mathbf{p} - \mathbf{k})B(\mathbf{p}, -\mathbf{p} + \mathbf{k})$ . Averaging over  $\mathbf{p}$  we get

$$b = \left[ \overline{B(\mathbf{p}, -\mathbf{p} - \mathbf{k})B(\mathbf{p}, -\mathbf{p} + \mathbf{k})} \right]^{1/2} = \frac{p_0^2}{35^{1/2} m \varepsilon_0}. \quad (18)$$

The constant  $b$  plays the role of an effective coupling coefficient. Introducing the coupling constant  $g_{\mathbf{k}} = 2ecb(2\pi/V\hbar\omega_{\mathbf{k}})^{1/2}$ , the system of Equations (17) becomes

$$\begin{aligned} \ddot{A}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 A_{\mathbf{k}} &= 2\omega_{\mathbf{k}} g_{\mathbf{k}} \sum_{\mathbf{p}} \beta_{\mathbf{p}}^* \beta_{\mathbf{p}+\mathbf{k}}, \\ i\dot{\beta}_{\mathbf{p}} &= \Omega \beta_{\mathbf{p}} - \sum_{\mathbf{k}} g_{\mathbf{k}} \beta_{\mathbf{p}-\mathbf{k}} A_{\mathbf{k}}, \end{aligned} \quad (19)$$

where  $\Omega = \varepsilon_0/\hbar$ . It is easy to see that Equations (19) are solved by the Fourier transforms

$$\beta_{\mathbf{p}} = \frac{1}{V} \int d\mathbf{r} \beta(\mathbf{r}) \exp\left(-\frac{i}{\hbar} \mathbf{p}\mathbf{r}\right), \quad \beta(\mathbf{r}) = \sum_{\mathbf{p}} \beta_{\mathbf{p}} \exp\left(\frac{i}{\hbar} \mathbf{p}\mathbf{r}\right). \quad (20)$$

The number of pairs is given by

$$N = 4 \sum_{\mathbf{p}} |\beta_{\mathbf{p}}|^2 = \frac{4}{V} \int d\mathbf{r} |\beta(\mathbf{r})|^2; \quad (21)$$

we have also

$$\sum_{\mathbf{p}} \beta_{\mathbf{p}}^* \beta_{\mathbf{p}+\mathbf{k}} = \frac{1}{V} \int d\mathbf{r} |\beta(\mathbf{r})|^2 \exp(-i\mathbf{k}\mathbf{r}). \quad (22)$$

The solution is immediately given by

$$\beta(\mathbf{r}) = B(\mathbf{r}) \exp\left[-i\Omega t + i \int^t dt' \lambda(\mathbf{r}, t')\right], \quad (23)$$

where  $\lambda(\mathbf{r}, t) = \sum_{\mathbf{k}} g_{\mathbf{k}} A_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r})$ ;  $B(\mathbf{r})$  is a 'constant' of integration, given by

$$N = \frac{4}{V} \int d\mathbf{r} |B(\mathbf{r})|^2. \quad (24)$$

From Equation (23) we can see that the pair dynamics is governed by single-particle states labelled by the position vector  $\mathbf{r}$ . This dynamics is quasi-stationary, in the sense that it conserves the 'occupation' number  $|\beta(\mathbf{r})|^2 = |B(\mathbf{r})|^2$ . Similarly, from the first Equation (19) and Equation (22), we can see that the polarization field does not depend on the time, so we get

$$A_{\mathbf{k}} = \frac{2g_{\mathbf{k}}}{\omega_{\mathbf{k}}} \frac{1}{V} \int d\mathbf{r} |B(\mathbf{r})|^2 \exp(-i\mathbf{k}\mathbf{r}); \quad (25)$$

it follows

$$\begin{aligned}\lambda(\mathbf{r}, t) &= \lambda(\mathbf{r}) = \sum_k g_k A_k \exp(i\mathbf{k}\mathbf{r}) \\ &= \sum_k \frac{2g_k^2}{\omega_k} \frac{1}{V} \int d\mathbf{r}' |B(\mathbf{r}')|^2 \exp[-i\mathbf{k}(\mathbf{r}' - \mathbf{r})] \quad (26)\end{aligned}$$

and

$$\beta(\mathbf{r}) = B(\mathbf{r}) \exp[-i\Omega t + i\lambda(\mathbf{r})t]. \quad (27)$$

The single-particle energy  $\hbar\Omega - \hbar\lambda(\mathbf{r})$  has a spatial dependence, reflecting the local force exerted on the pairs by the polarization field. If one assumes the pairs confined to a spatial region of finite extent, we can see that this force tends to localize the pairs in that region, as expected.

It is reasonable to assume that the pairs are distributed uniformly in space, i.e.  $B(\mathbf{r}) = B = N^{1/2}/2$ . This amounts to a condensation of the fermions on the  $\mathbf{p} = 0$  state; in fact, the pairs are distributed ('condensed') over the low-momenta fermionic states. The field  $A_k$  (Equation (25)), and the single-particle energy  $\lambda$  (Equation (26)) exhibit a singularity for  $k \rightarrow 0$ , as expected for such an infinite uniform distribution. In practice, the pairs are distributed quasi-uniformly in space over a region of finite linear size  $d$ , so we may take  $A_k \simeq g_k N/2\omega_k$  for  $k < k_0 = 1/d$ . The single-particle energy becomes

$$-\hbar\lambda \simeq -\frac{2e^2 b^2}{\pi d} N, \quad (28)$$

and Equation (20) gives  $\beta_p = B \exp(-i\Omega t + i\lambda t)$  for  $\mathbf{p} \rightarrow 0$ . From the conservation of the number of particles

$$N = 4 \sum_p |\beta_p|^2 = 4B^2 \frac{V}{(2\pi)^3 \hbar^3} \frac{4\pi p_0^3}{3} = 4B^2 \quad (29)$$

we get the momentum cutoff  $p_0/\hbar = (6\pi^2)^{1/3}/d$ , which is of the order of  $1/d$ , as expected. It is worth noting to see now the coupling coefficient  $b$  given by Equation (18),

$$b = \frac{p_0^2}{35^{1/2} m \epsilon_0} = \frac{(6\pi^2)^{2/3}}{35^{1/2}} \left(\frac{\lambda_c}{d}\right)^2, \quad (30)$$

where  $\lambda_c = \hbar/mc$  is the electron Compton wavelength. Since  $\lambda_c \simeq 0.3 \times 10^{-10}$  cm, we can see that the coupling coefficient  $b$  acquires an extremely small value. The single-particle energy given by Equation (28) can be written as

$$-\hbar\lambda \simeq -\frac{2e^2 b^2}{\pi d} N = -\frac{12}{35} (6\pi^2)^{1/3} \frac{e^2}{d} \left(\frac{\lambda_c}{d}\right)^4 N, \quad (31)$$

which is extremely small. It is worth noting the occurrence of the Coulomb energy  $e^2/d$  of an electron localized in a spatial region of linear size  $d$ . The assumption  $B(\mathbf{r}) = B = \text{const}$  used here is not a restriction, it is just a simplifying (and reasonable) condition, corresponding to a quasi-uniform distribution of pairs in the spatial region of interest. For pairs interacting with their own polarization field it seems natural to adopt such an assumption. It corresponds also to an external field focused quasi-uniformly in the region of interest. For other experimental situations, depending on the spatial distribution of the external field, the amplitude  $B(\mathbf{r})$  may have a prescribed spatial dependence, as we can see from Equation (25). In general,  $B(\mathbf{r})$  has the status of a 'constant' of integration for the equations of motion.

Making use of Equation (1), we can compute the magnetic field  $\mathbf{H} = \text{curl } \mathbf{A}$  and the electromagnetic energy  $E_{\text{em}}$  stored by the polarization field  $A_k \simeq g_k N/2\omega_k$  for  $k < k_0 = 1/d$ . We get

$$E_{\text{em}} = \frac{2e^2 b^2}{\pi d} N^2 = \frac{12}{35} (6\pi^2)^{1/3} \frac{e^2}{d} \left(\frac{\lambda_c}{d}\right)^4 N^2. \quad (32)$$

The number of pairs can be obtained from the conservation of energy

$$E_{\text{em}} + 2mc^2 N = \frac{12}{35} (6\pi^2)^{1/3} \frac{e^2}{d} \left(\frac{\lambda_c}{d}\right)^4 N^2 + 2mc^2 N = W, \quad (33)$$

where  $W$  is the total energy and we have neglected the single-particle energy  $-\hbar\lambda$ . It is easy to see that the  $N^2$ -term brings an extremely small contribution (due to the fourth power of the ratio  $\lambda_c/d \ll 1$ ), so the number of pairs is given by  $N \simeq W/2mc^2$ . For a numerical reference, we can take  $W = 1$  J and get  $N \simeq 10^{13}$  pairs. We can see that the pairs number does not depend practically on the size of the spot where the energy is concentrated.

It is also worth commenting upon the solutions

$$\beta(\mathbf{r}) = B \exp(-i\Omega t + i\lambda t) \quad (34)$$

(for  $r < d$ ). According to Equation (23) they represent single-particle eigenstates. We can see that they correspond to electrons (positrons) quasi-localized in space, in the sense that these field amplitudes are labelled by the position vector  $\mathbf{r}$ . We can divide the space into small, identical cells of volume  $v \ll V$ , and write the number of particles (Equation (24)) as

$$N = \frac{4}{V} \int d\mathbf{r} |B(\mathbf{r})|^2 = \frac{4v}{V} \sum_r B^2 = \frac{vN}{V} \sum_r 1, \quad (35)$$

whence one can see that the 'occupation' number in each cell of volume  $v$  is unity, as for fermions. This is

a check of consistency for our assumption  $B = \text{const}$ , and substantiates the picture of electrons (positrons) quasi-localized in space and represented by fields given by Equation (34).

The quasi-localization concept used here can be seen straightforwardly by the following general argument. Making use of the Fourier transforms given by Equations (20) we can write a general fermion field as

$$\begin{aligned}\psi(\mathbf{r}) &= \sum_{\mathbf{p}} \beta_{\mathbf{p}} \exp\left(\frac{i}{\hbar} \mathbf{p} \mathbf{r}\right) \\ &= \int d\mathbf{r}' \beta(\mathbf{r}') \frac{1}{V} \sum_{\mathbf{p}} \exp\left(\frac{i}{\hbar} \mathbf{p}(\mathbf{r} - \mathbf{r}')\right); \quad (36)\end{aligned}$$

here  $\beta(\mathbf{r}')$  are field amplitudes and the summation over  $\mathbf{p}$  plays the role of single-particle wavefunctions. We can see that these wavefunctions (of variable  $\mathbf{r}$ ) are labelled by  $\mathbf{r}'$ , i.e. the single-particle states are labelled by the position vector  $\mathbf{r}'$ . If the momentum  $\mathbf{p}$  in the above summation is allowed to run over the whole space, the summation gives the wavefunctions  $\delta(\mathbf{r} - \mathbf{r}')$ , which corresponds to a perfect localization (and the field  $\psi(\mathbf{r})$  reduces to one amplitude  $\beta(\mathbf{r})$ ). In our case, the limitation imposed upon the variation domain of the momentum  $\mathbf{p}$  leads to wavefunctions quasi-localized over the spatial region of finite linear size  $d$ .

We can see that the vacuum can be polarized with electron–positron pairs, which create a polarization field and acquire an additional  $-\hbar\lambda$  energy for each electron (positron). Even for very high energy densities the number of pairs, the polarization energy and the single-particle energies are extremely small. Comparing the first Equation (19) with the classical wave equation  $\partial^2 A / \partial t^2 - c^2 \Delta A = 4\pi c \mathbf{j}$ , where  $\mathbf{j}$  is the density of the polarization current, we get

$$\begin{aligned}j(\mathbf{k}) &= \frac{ecb}{V} \sum_{\mathbf{p}} \beta_{\mathbf{p}}^* \beta_{\mathbf{p}+\mathbf{k}} = \frac{4ecb}{V^2} \int d\mathbf{r} |B(\mathbf{r})|^2 \exp(-i\mathbf{k}\mathbf{r}) \\ &\simeq \frac{ecb}{V} N = \frac{1}{35^{1/2}} (6\pi^2)^{2/3} \frac{ec}{V} \left(\frac{\lambda_c}{d}\right)^2 N, \quad k < 1/d, \quad (37)\end{aligned}$$

for the Fourier transform of the current density (for one polarization). This is a very small current density.

We introduce now an external field  $A_{\mathbf{k}_0}^{\text{ext}} = A_{-\mathbf{k}_0}^{\text{ext}} = A_0 = 2a_0 \cos \omega_0 t$  (monochromatic wave), with the frequency  $\omega_0 = ck_0$ . The second Equation (19) becomes

$$i\dot{\beta}_{\mathbf{p}} = \Omega \beta_{\mathbf{p}} - \sum_{\mathbf{k}} g_{\mathbf{k}} \beta_{\mathbf{p}-\mathbf{k}} A_{\mathbf{k}} - g_0 (\beta_{\mathbf{p}-\mathbf{k}_0} + \beta_{\mathbf{p}+\mathbf{k}_0}) A_0, \quad (38)$$

where  $g_0 = 2ecb(2\pi/V\hbar\omega_0)^{1/2}$ . The solution is given by

$$\beta(\mathbf{r}) = B(\mathbf{r}) \exp[-i\Omega t + i\lambda t + i\varphi(\mathbf{r}, t)], \quad (39)$$

where

$$\varphi(\mathbf{r}, t) = \frac{4g_0 a_0}{\omega_0} \sin \omega_0 t \cos \mathbf{k}_0 \mathbf{r}. \quad (40)$$

This phase implies an ‘energy’

$$\begin{aligned}\delta\varepsilon(\mathbf{r}, t) &= -4\hbar g_0 a_0 \cos \omega_0 t \cos \mathbf{k}_0 \mathbf{r} \\ &= -2\hbar g_0 a_0 [\cos(\mathbf{k}_0 \mathbf{r} - \omega_0 t) + \cos(\mathbf{k}_0 \mathbf{r} + \omega_0 t)], \quad (41)\end{aligned}$$

for the electron–positron pairs (it depends on position and time), which appears as a stationary wave driven by the external field. It is worth noting that, in contrast with the polarization energy  $-\hbar\lambda$  given by Equation (31), which is quadratic in the coupling coefficient  $b$ , the energy caused by the external field is linear in  $b$ , as expected. It is convenient to estimate the mean value of this ‘energy’ by making use of the external field energy  $W_0 = 2\hbar\omega_0 |a_0|^2$ . We get straightforwardly

$$\overline{\delta\varepsilon} = \frac{4}{35^{1/2}} (6\pi^2)^{2/3} \frac{c}{\omega_0 d} \left(\frac{\lambda_c}{d}\right)^2 \left(\frac{e^2 W_0}{d}\right)^{1/2}, \quad (42)$$

which, even for reasonably high energy densities, is still a very low energy. The energy of the external field is distributed over the energy of the polarization field (which is very low) and the energy of the pairs, according to Equation (33). It is worth noting that the above results are sensitive to decreasing  $d$  (except for the number of pairs), so we can enhance the relevant values by the focalization of the energy in very small volumes. However, for usually available energies this enhancement is still insufficient for getting any appreciable result.

The external field induces a polarization field which is stationary (the vector potential does not depend on the time), as a consequence of the stationary dynamics of the electrons and positrons. Therefore, the polarization electric field is vanishing, and we are left only with a static magnetic field. Under the action of an external field the vacuum gets magnetized. The corresponding vector potential of the polarization field is given by  $A_0^{\text{pol}} = g_0 N / 2\omega_0$ , according to the discussion made above. This polarization field depends on the strength  $A_0$  of the external field through the field energy  $W_0$  which generates the number of pairs  $N$ . Consequently, we can define a static magnetic susceptibility of the polarized vacuum. We get straightforwardly the magnetic permeability

$$\mu = 1 + \frac{eb}{4mc\omega_0} H_0 = 1 + \frac{(6\pi^2)^{2/3}}{4(35^{1/2})} \left(\frac{\lambda_c}{d}\right)^2 \frac{eH_0}{mc\omega_0}, \quad (43)$$

where it is worth noting the linear dependence on the strength  $H_0$  of the external magnetic field. As expected, the vacuum polarized under the action of an external field, acquires a (very small, static) magnetic

susceptibility, and, consequently, a refractive index  $n = \mu^{1/2}$  (slightly greater than unity). It is worth noting in Equation (43) the ratio of the magnetic energy (Bohr magneton in the magnetic field  $H_0$ ) to the energy quanta  $\hbar\omega_0$  of the external field.

In conclusion, we may say that the vacuum gets polarized with electron–positron pairs under the action of an external classical field of electromagnetic radiation. The polarization field is static, i.e. the electric field is vanishing and the vacuum sustains only a static magnetic field. The corresponding magnetic permeability (the refractive index of the vacuum) has been computed for an external monochromatic wave. The electron–positron pairs are condensed on low-momenta states and exhibit a quasi-stationary dynamics. They acquire a single-particle ‘energy’ (temporal phase), which is a stationary wave driven by the monochromatic external field. The number of pairs are determined by the external energy, while the single-particle energies and the energy of the polarization field depend on the energy density of the external field. All these numerical results are extremely small, even for reasonably high external energies and energy densities. An important role in the magnitude of these effects is played by the Compton wavelength of the electron, which is very small in comparison with the extent of the spatial region over which we can concentrate the energy of the external field. The results presented here have been derived by treating the electron–positron and photon dynamics by means of classical fields, a procedure justified by the polarization process, which implies continuous creation and annihilation of electron–positron pairs under the action of a classical field of radiation, resembling a (macroscopic) plasma of electron–positron pairs. The coupled non-linear equations of motion have been solved for these fields, and the solution led to the results described above.

### Acknowledgements

The author is indebted to the members of the Seminar of the Institute of Atomic Physics, Magurele-Bucharest, especially to M. Ganciu and F.D. Buzatu, for many useful discussions, the Institute of Physics and Nuclear Engineering and the Institute for Lasers, Plasma and Radiation, Magurele-Bucharest, for the generous support offered within the framework of the European–Romanian Research Project ‘Extreme Light Infrastructure – Nuclear Physics’ (ELI-NP).

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### Appendix 1. The coefficients A, B and C in Equation (9)

Making use of the bi-spinor definition (Equations (5)) and the algebra of the Pauli spin matrices (in particular  $\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k$ , where  $\varepsilon_{ijk}$  is the totally antisymmetric tensor of rank 3), the coefficients  $A$ ,  $B$  and  $C$  appearing in the interaction hamiltonian (Equation (9)) can be computed straightforwardly (leaving aside the spin dependence in the electron–positron operators). They are given by

$$A(\mathbf{p}, \mathbf{p}') = \frac{1}{2(\varepsilon\varepsilon')^{1/2}} \left\{ [(\varepsilon - mc^2)(\varepsilon' + mc^2)]^{1/2} [2\mathbf{en} - i\bar{\boldsymbol{\sigma}}(\mathbf{e} \times \mathbf{n})] \right. \\ \left. + [(\varepsilon + mc^2)(\varepsilon' - mc^2)]^{1/2} [2\mathbf{en}' + i\bar{\boldsymbol{\sigma}}(\mathbf{e} \times \mathbf{n}')] \right\}$$

$$B(\mathbf{p}, \mathbf{p}') = -\frac{1}{2(\varepsilon\varepsilon')^{1/2}} \left\{ [(\varepsilon + mc^2)(\varepsilon' + mc^2)]^{1/2} [2e_y + i(\mathbf{e} \times \bar{\boldsymbol{\sigma}})_y] \right. \\ \left. + [(\varepsilon - mc^2)(\varepsilon' - mc^2)]^{1/2} [2(\mathbf{en})_y - 2(\mathbf{nn}')_y \right. \\ \left. + 2(\mathbf{en}')_y - i(\mathbf{en})(\mathbf{n}' \times \bar{\boldsymbol{\sigma}})_y + i\mathbf{n}'_y(\mathbf{e} \times \bar{\boldsymbol{\sigma}}) \right. \\ \left. + i(\mathbf{e}\bar{\boldsymbol{\sigma}})(\mathbf{n}' \times \mathbf{n})_y + i(\mathbf{n}\bar{\boldsymbol{\sigma}})(\mathbf{e} \times \mathbf{n}')_y \right\},$$

$$C(\mathbf{p}, \mathbf{p}') = \frac{1}{2(\varepsilon\varepsilon')^{1/2}} \left\{ [(\varepsilon - mc^2)(\varepsilon' + mc^2)]^{1/2} [2\mathbf{en} - i\bar{\boldsymbol{\sigma}}_y(\mathbf{e} \times \mathbf{n}) \right. \\ \left. - i\mathbf{n}_y(\mathbf{e} \times \bar{\boldsymbol{\sigma}})_y + i\mathbf{e}_y(\mathbf{n} \times \bar{\boldsymbol{\sigma}})_y] \right. \\ \left. + [(\varepsilon + mc^2)(\varepsilon' - mc^2)]^{1/2} [2(\mathbf{n}'\mathbf{e}) + i\bar{\boldsymbol{\sigma}}_y(\mathbf{e} \times \mathbf{n}')_y \right. \\ \left. + i\mathbf{n}'_y(\mathbf{e} \times \bar{\boldsymbol{\sigma}})_y - i\mathbf{e}_y(\mathbf{n}' \times \bar{\boldsymbol{\sigma}})_y \right\},$$

where  $\varepsilon = (c^2 p^2 + m^2 c^4)^{1/2}$ ,  $\varepsilon' = (c^2 p'^2 + m^2 c^4)^{1/2}$ ,  $\mathbf{e} = \sum_{\mu} e_{\mu}(\mathbf{k})$  and  $\bar{\boldsymbol{\sigma}} = \sum_{\sigma\sigma'} w_{\sigma}^* \boldsymbol{\sigma} w_{\sigma'}$ .

### Appendix 2. The vector $\bar{\boldsymbol{\sigma}}$

The most general form of the spinor  $w_{\sigma}(\mathbf{p})$  is

$$w_{+1} = \exp(-i\varphi) \cos \theta u + \exp(i\varphi) \sin \theta v, \\ w_{-1} = -\exp(-i\varphi) \sin \theta u + \exp(i\varphi) \cos \theta v,$$

where  $u$ ,  $v$  are the eigenvectors of the Pauli matrix  $\sigma_z$  ( $\sigma_z u = u$ ,  $\sigma_z v = -v$ ) and the angles  $\theta$ ,  $\varphi$ , corresponding to the wavevector  $\mathbf{p}$ , are arbitrary; for  $\mathbf{p}'$  (in  $w_{\sigma}(\mathbf{p}')$ ) we denote these angles by  $\theta'$ ,  $\varphi'$ . It is worth noting that, in spite of some resemblance, these spinors are not those related to the helicities. The calculations of the vector  $\bar{\boldsymbol{\sigma}} = \sum_{\sigma\sigma'} w_{\sigma}^*(\mathbf{p}) \boldsymbol{\sigma} w_{\sigma'}(\mathbf{p}')$  is then straightforward. We get

$$\bar{\sigma}_x = 2i \sin(\varphi - \varphi') \sin(\theta - \theta') + 2 \cos(\varphi - \varphi') \cos(\theta + \theta'), \\ \bar{\sigma}_y = 2i \cos(\varphi - \varphi') \sin(\theta - \theta') + 2 \sin(\varphi - \varphi') \cos(\theta + \theta'), \\ \bar{\sigma}_z = -2i \sin(\varphi + \varphi') \cos(\theta - \theta') - 2 \cos(\varphi + \varphi') \sin(\theta + \theta').$$

It is worth noting that  $\bar{\boldsymbol{\sigma}}$  is a complex vector, which depends on four parameters (the angles  $\varphi$ ,  $\varphi'$ ,  $\theta$ ,  $\theta'$ ), as expected for the polarization of an ensemble of two fermions of spin 1/2. Indeed, we have a polarization vector for one fermion, relative to the direction of the polarization vector of the other fermion, i.e. three parameters, and another parameter for the magnitude of the former polarization vector. We can have various choices for  $\bar{\boldsymbol{\sigma}}$ , for instance we may take it perpendicular to the two wavevectors  $\mathbf{p}$ ,  $\mathbf{p}'$  (which amounts to four equations with four unknowns). We can also write  $\bar{\boldsymbol{\sigma}}$  as  $\bar{\boldsymbol{\sigma}} = 2i\mathbf{s}_1 + 2\mathbf{s}_2$ , where  $\mathbf{s}_{1,2}$  are two real, linearly independent vectors, and take  $\mathbf{s}_1$  parallel with  $\mathbf{p}$  (in which case we are left with only one free parameter). None of such choices brings an appreciable simplification in the interaction matrix elements, and, in fact, any particular choice is arbitrary. The only meaningful procedure is the averaging over angles, which gives  $(\bar{\boldsymbol{\sigma}})_{\text{av}} = 0$ , as for unpolarized elementary interaction acts. This assumption is equivalent to a uniform statistical average of the interaction with respect to the ‘spin’ polarization, as noted in the main text. We note that such an average is consistent with the classical treatment of the interaction given here. As noted also in the main text, the arbitrariness in the spinors  $w_{\sigma}$  is lifted if they are referred to a polarized external field, and an average over the spin directions is not needed, nor appropriate, in this situation.