FOUR-FERMION CONDENSATE

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Within the mean-field approach the existence of the four-fermion condensate is proved in a translationally invariant system with four-fermion interaction.

Recently there has been a renewed interest in the four-fermion condensate in connection with the well-known question whether nuclei possess a superfluid state made of α particles (see, for example, ref. [1] and references therein). The problem has been approached [1] within the interacting boson model by means of a BCS-like treatment of two-boson condensate, each boson corresponding to a pair of nucleons. However, the underlying fermion structure of these bosons is of great importance as fas as genuine four-fermion correlations are looked for. Early attempts to account for α particles in nuclei as four-fermion condensates [2] run into difficulties which on the one hand are inherent to the complex nuclear structure and on the other hand arise from the much too general framework within the problem put forward.

Nevertheless the relevance of the four-fermion condensate is not restricted to the field of nuclear physics, the question being interesting in itself. It might be regarded as the next step in describing fermion condensates after the well-known pairing theory of superconductivity, charge density waves, excitonic insulator, etc. The aim of this paper is to put forward a simple model of translationally invariant interacting fermions which exhibits a transition towards a superfluid state made of four-fermion condensates.

The relevant one-fermion states, denoted by 1 and 2 and, respectively, 3 and 4, are restricted to the opposite ends of two arbitrary diameters of the spherical Fermi sea. Leaving aside the angular coordinates of the wavevectors as well as the spin index (although their contribution will thoroughly be taken into account by the density of states at the Fermi surface) these states are labelled by p, where $+k_F + p$ stands for the wavenumber of the states 1 and 3 and $-k_F + p$ corresponds to the states 2 and 4, k_F being the Fermi momentum. The reduced wavenumber p runs within the range $(-k_c, k_c)$, k_c being a momentum cut-off much smaller than k_F. The kinetic energy of this model is given by the well-known free-fermion hamiltonian (see, for example, ref. [3])

$$H_f^0 = v_F \sum_p p(c_{1p}^+ c_{1p} - c_{2p}^+ c_{2p}) + v_F \sum_p p(c_{3p}^+ c_{3p} - c_{4p}^+ c_{4p}), \tag{1}$$

where v_F is the Fermi velocity, c_{jp}^+ (c_{jp}), $j = 1, 2, 3, 4$, are the creation (destruction) fermion operators and the one-fermion energy (relative to the Fermi level) has been linearized in the neighbourhood of the Fermi surface. The four-fermion condensate will be thought as arising from the condensation of a hermitean field of bosons $\phi(x) = \Sigma_q \phi_q \exp(iq \cdot x)$, $\phi_q = (2\omega_q)^{-1/2}(a_q + a_{-q}^+)$, where a_q are boson operators and ω_q is the free-boson frequency. As we are interested only in the emergence of the condensate we may restrict ourselves to the boson mode $q = 0$, described by the free-boson hamiltonian

$$H_b^0 = \omega_0 a_0^+ a_0 . \tag{2}$$

Consequently, the simplest form of the coupling between fermions and bosons will be taken as

$$H_\mathrm{i}=g(2/\omega_0)^{1/2}\sum_{pp'}(a_0^+c_{1p}c_{2-p}c_{3p'}c_{4-p'}+\mathrm{h.c.}),\quad(3)$$

where g is the interaction strength. Two comments are in order here. Firstly, we should remark that a pairing correlation (opposite wavevectors) is already included in (3), between the one-fermion states 1 and 2 and, respectively, 3 and 4. More general pairing correlations can, of course, be introduced, for example those which couple the one-fermion states 1 and 3 (respectively 2 and 4) and 1 and 4 (respectively 2 and 3). However, bearing in mind the angular coordinates of the wavevectors, it might be expected that these additional correlations are much less effective as they affect one-fermion states lying on different diameters of the Fermi sea. Secondly, in order to get a four-fermion condensate a further correlation is needed between the pairs (1, 2) and (3, 4) of one-fermion states in (3). This particular correlation will be ensured by the self-consistency conditions arising from the mean-field treatment of the model hamiltonian $H=H_\mathrm{f}^0+H_\mathrm{b}^0+H_\mathrm{i}$. The hamiltonian H is invariant under the gauge transformation that corresponds to the conservation of the "charge" $B=4n_\mathrm{b}+\Sigma_j n_j$, where $n_\mathrm{b}=a_0^+a_0$ is the boson number operator and $n_j=\Sigma_p c_{jp}^+c_{jp}$ is the particle-number operator of the j-type fermions. The expectation value of the B "charge" on the non-interacting ground state is given by $\langle 0|B|0\rangle=4\rho\epsilon_\mathrm{b}$, where $\epsilon_\mathrm{b}=v_\mathrm{F}k_\mathrm{c}$ is the energy cut-off and $\rho=m(3/8\pi^4 n)^{1/3}$ is the density of states at the Fermi surface (m being the fermion mass, n the fermion density and the integration over the angular coordinates of the wavevectors being restricted to the half-space).

The ground state of the condensate has broken symmetry with respect to the aforementioned transformation and is defined by a non-vanishing expectation value of the boson field $\langle\phi_0\rangle=\varphi$ and the pairing correlations

$$\sum_p\langle c_{1p}c_{2-p}\rangle=\chi_{12},\quad\sum_p\langle c_{3p}c_{4-p}\rangle=\chi_{34}.\qquad(4)$$

The conditions $\langle\phi_0\rangle=\varphi$ and $\langle\pi_0\rangle=0$, where $\pi_0=\mathrm{i}(\omega_0/2)^{1/2}(a_0^+-a_0)$ is the canonical-conjugate momentum, lead to $\langle a_0\rangle=\langle a_0^+\rangle=(\omega_0/2)^{1/2}\varphi$. The mean-field hamiltonian $\mathcal{H}$ is straightforwardly obtained then as

$$\mathcal{H}=\tfrac{1}{2}\omega_0^2\varphi^2+h_{12}+h_{34}-g\varphi(\chi_{12}\chi_{34}+\mathrm{h.c.}),\qquad(5)$$

where

$$h_{12}=v_\mathrm{F}\sum_p p(c_{1p}^+c_{1p}-c_{2p}^+c_{2p})+g\varphi\Big(\chi_{34}\sum_p c_{1p}c_{2-p}+\mathrm{h.c.}\Big),\qquad(6)$$

and h_{34} is obtained from h_{12} by changing the labels 1 and 2 into 3 and, respectively, 4. The one-fermion hamiltonians h_{12} and h_{34} can straightforwardly be diagonalized by means of a Bogoljubov-type transform. As a matter of fact this one-fermion problem is of the same type as those encountered in the superconductivity [4] and charge density waves [5] theories. The new point arises here from the fact that the ground states of h_{12} and h_{34} will depend on χ_{34} and, respectively, χ_{12}, so that (4) turn out to be self-consistency conditions between the parameters χ_{12} and χ_{34}: they will ensure the aforementioned correlations between the fermion paris (1, 2) and (3, 4). Indeed, the diagonalized form of h_{12} is acquired by means of the transform $c_{1p}=u_p\tilde{c}_{1p}+v_p\tilde{c}^+_{2-p}$, $c_{2p}=u_p\tilde{c}_{2p}+v_p\tilde{c}^+_{1-p}$, where $u_p=\cos\theta_p$, $v_p=\mathrm{sgn}(p)\sin\theta_p\exp[\mathrm{i}(\arg g+\arg\varphi-\arg\chi_{34})]$, $\tan 2\theta_p=\Delta_{12}/v_\mathrm{F}|p|$, $2\Delta_{12}=2|g\varphi\chi_{34}|$ being the magnitude of the gap opened up at $p=0$ and arg is standing for the phases of the corresponding quantities ($\arg g$, $\arg\varphi=0,\pi$). The hamiltonian h_{34} can similarly be diagonalized, so that the values in (4) can be worked out. Providing $\arg\chi_{12}+\arg\chi_{34}=\pi+\arg g+\arg\varphi$ we get from (4) the self-consistent equations

$$\Delta_{12}=-\rho|g\varphi|\Delta_{34}\ln\left(\frac{\Delta_{34}/\epsilon_\mathrm{b}}{1+(1+\Delta_{34}^2/\epsilon_\mathrm{b}^2)^{1/2}}\right),$$

$$\Delta_{34}=-\rho|g\varphi|\Delta_{12}\ln\left(\frac{\Delta_{12}/\epsilon_\mathrm{b}}{1+(1+\Delta_{12}^2/\epsilon_\mathrm{b}^2)^{1/2}}\right),\qquad(7)$$

where $\Delta_{34}=|g\varphi\chi_{12}|$. The energy δE_{12} gained by the ground state of h_{12} as a result of the interaction which lowers the one-fermion levels can straightforwardly be obtained as

$$\delta E_{12} = -\rho\epsilon_b^2\left[(1+\Delta_{12}^2/\epsilon_b^2)^{1/2} - 1 - (\Delta_{12}^2/\epsilon_b^2)\ln\left(\frac{\Delta_{12}/\epsilon_b}{1+(1+\Delta_{12}^2/\epsilon_b^2)^{1/2}}\right)\right]. \quad (8)$$

A similar expression holds for δE_{34}, so that the difference in energy between the condensed ground state of $\mathcal{H}$ given by (5) and the uncondensed one ($\varphi = 0$) is

$$\delta E = \tfrac{1}{2}\omega_0^2\varphi^2 + \delta E_{12} + \delta E_{34} + 2\Delta_{12}\Delta_{34}/|g\varphi|. \quad (9)$$

It is noteworthy here that the self-consistent equations (7) ensure the minimum of the energy δE with respect to the fermion parameters Δ_{12} and Δ_{34}. Eqs. (7) admit the non-trivial solution

$$\Delta_{12} = \Delta_{34} = \epsilon_b[\sinh(x/2)]^{-1}, \quad (10)$$

where $x = 2(\rho|g\varphi|)^{-1}$, so that the energy difference (9) becomes

$$\delta E = (2\rho\epsilon_b^2/r)f(x),$$
$$f(x) = 1/2x^2 - r\,e^{-x/2}[\sinh(x/2)]^{-1},$$
$$r = 2\rho(\rho|g|\epsilon_b/2\omega_0)^2. \quad (11)$$

The effective-potential function $f(x)$ is plotted in fig. 1 for various values of the dimensionless parameter r. For small values of r the curves fall smoothly down to zero as x goes to infinity. With increasing values of r the curves start to exhibit a maximum and a minimum, the latter vanishing at $x_0 \approx 1.6$ for $r_0 \approx 0.39$. The most remarkable fact is that for r larger than r_0 the function $f(x)$ develops a negative minimum located at x_m, $0 < x_m < x_0$. Therefore, one may conclude that for $r > r_0$ the four-fermion condensed ground state is stable and energetically favoured with respect to the uncondensed one.

In order to achieve a complete description of the condensed ground state one should resort to the physical interpretation of the boson field. Since each boson corresponds to a four-fermion bound state it follows that the boson number in the new ground state should equal one-quarter of the total fermion number, $\langle n_b\rangle = \omega_0\varphi_m^2/2 = \rho\epsilon_b$, where $|\varphi_m| = 2(\rho|g|x_m)^{-1}$. This condition yields $r = 2\rho(\rho^2|g|x_m^2)^{-2}$ which combined with $f'(x_m) = 0$ leads to

$$\rho^{3/2}|g| = [\sqrt{x_m}\,\sinh(x_m/2)]^{-1}. \quad (12)$$

Eq. (12) provides the critical condition for the emergence of the condensed state

$$\rho^{3/2}|g| \geqslant (\rho^{3/2}|g|)_{cr} = [\sqrt{x_0}\,\sinh(x_0/2)]^{-1} \approx 0.9, \quad (13)$$

where $x_0 \approx 1.6$ is the largest permissible value of x_m which corresponds to the setting up of the negative minimum of the function $f(x)$. It follows that the condensed state exists only for sufficiently large coupling constants or sufficiently high fermion densities (one should bear in mind that $\rho \sim n^{1/3}$). When the same picture is applied to the non-interacting ground state one can obtain the free-boson frequency

$$\omega_0 = E_0/\langle n_b\rangle = 4\mu - 2\epsilon_b = 4\mu\left(1 + \frac{4\sinh^2(x_m/2)}{x_m}\right)^{-1} \quad (14)$$

and the cut-off energy

$$\epsilon_b = 2\mu\left[1 - \left(1 + \frac{4\sinh^2(x_m/2)}{x_m}\right)^{-1}\right], \quad (15)$$

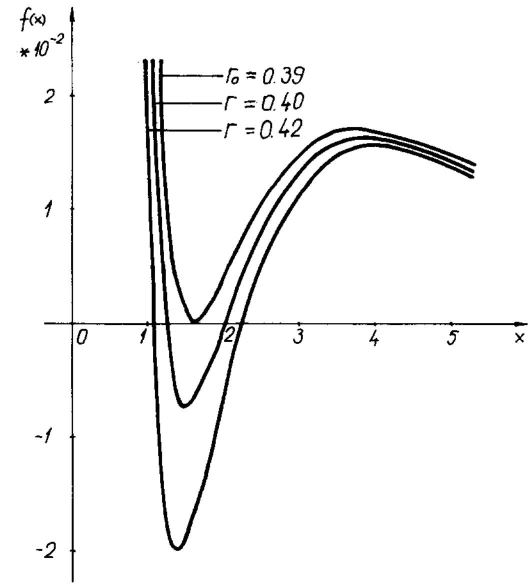

Fig. 1. Effective-potential function $f(x)$ given by (11) for various values of the parameter r. A negative minimum occurs for $r > r_0 \approx 0.39$ corresponding to the transition towards the four-fermion condensed state.

where E_0 is the energy of the non-interacting ground state and $\mu = (3\pi^2 n)^{2/3}/2m$ is the Fermi level. One can see from (14) and (15) that for large values of the quantity $\rho^{3/2}|g|$, i.e. for small values of x_m, ω_0 approaches 4μ and ϵ_b/μ vanishes. Consequently, for sufficiently strong coupling the four-fermion condensates occur within a very thin shell around the Fermi level. It is a straightforward matter now to calculate other quantities of interest, such as the binding energy of a boson

$$b = \delta E_m/\langle n_b\rangle = \omega_0[1 - 2(1 - e^{-x_m})/x_m] , \tag{16}$$

where δE_m is given by (11) for $x = x_m$, the latter being obtained from (12) for each given value of $\rho^{3/2}|g|$ larger than the critical one. The critical temperature T_c below which the four-fermion transition might be expected could also be estimated as $k_B T_c \approx 1.14\epsilon_b \exp(-x_m/2)$, k_B being the Boltzmann constant.

In summary, one can say that the translationally invariant model with four-fermion interaction discussed in this paper exhibits, within the mean-field theory, a transition towards a superfluid state made of four-fermion condensates. The relevance of this simple model for the realistic fermion systems might, admittedly, be questionable. The treatment of the correlations between the one-fermion states lying on different diameters of the Fermi sea or arbitrarily located in the neighbourhood of the Fermi surface would be interesting generalizations. However, in the opinion of the author, the most interesting development of the present model would consist in the investigation of the existence of aggregates larger than four fermions.

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Note added. The hamiltonian H given by (1)–(3) is equivalent to the four-particle interaction hamiltonian

$$\tilde{H} = e^S H e^{-S} = H_f^0 + H_b^0 + GP^+P ,$$

where $G = -2g^2/\omega_0^2$ and $P = \Sigma_{p,p'} c_{1p}c_{2-p}c_{3p'}c_{4-p'}$ with p and p' near zero (wavevectors in the neighbourhood of the Fermi surface) and constant number of fermions. The unitary transformation has the well-known expression

$$S = g(2/\omega_0)^{1/2} \sum_{pp'} \frac{1}{\omega_0 - 2v_F(p + p')} \times (a_0^+ c_{1p}c_{2-p}c_{3p'}c_{4-p'} - \text{h.c.}) .$$

References

[1] Y.K. Gambhir, P. Ring and P. Schuck, Phys. Rev. Lett. 51 (1983) 1235.
[2] B.H. Flowers and M. Vujicić, Nucl. Phys. 49 (1963) 586; J. Eichler and M. Yamamura, Nucl. Phys. A182 (1972) 33; M. Kamimura, T. Marumori and K. Takada, Prog. Theor. Phys. Suppl. 52 (1972) 282.
[3] J. Sólyom, Adv. Phys. 28 (1979) 201.
[4] J.R. Schrieffer, Theory of superconductivity (Benjamin, New York, 1964).
[5] H. Fröhlich, Proc. R. Soc. A233 (1954) 296; M.J. Rice and S. Strässler, Solid State Commun. 13 (1973) 125.