Physics Letters A 372 (2008) 5093–5095

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

A new approach to the quantized electrical conductance

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ARTICLE INFO

Article history:
Received 19 February 2008
Received in revised form 15 April 2008
Accepted 20 May 2008
Available online 5 June 2008
Communicated by C.R. Doering

PACS:
73.63.-b
73.63.Nm
05.60.Gg

Keywords:
Quantized electrical conductance
Nanowires
One-dimensional electron gas

ABSTRACT

The quanta of electrical conductance is derived for a one-dimensional electron gas both by making use of the quasi-classical motion of a quantum fluid and by using arguments related to the uncertainty principle. The result is extended to a nanowire of finite cross section area and to electrons in magnetic field, and the quantization of the electrical conductance is shown. An additional application is made to the two-dimensional electron gas.

Recently, there is a considerable deal of interest in the quantized electrical conductance of atomic and molecular conductors like nanowires, narrow atomic constrictions, quantum dots, carbon nanotubes, etc. [1–11]. The effect was originally predicted by Landauer [12–14]. We present here a new derivation of the quanta of electrical conductance for a one-dimensional electron gas, by making use of two procedures: the quasi-classical approach to the one-dimensional quantum electron fluid and by using arguments related to the uncertainty principle. We extend the results to the quantization of the electrical conductance in a quasi one-dimensional nanowire of finite area of the cross section, where the electron motion is confined to the transversal directions while the free longitudinal motion is subjected to the action of the electric field. An application of this result is made for the two-dimensional electron gas. Also, we apply the present approach to electrons in a magnetic field.

We consider first a one-dimensional (free) electron gas in a conductor of length l and cross section area A. We consider a purely quantum transport in such a conductor, without scattering or thermal effects. The electron density is given by $n = gk_F/\pi A$, where k_F is the Fermi wavevector and g is a degeneracy factor (e.g. $g = 2$ for spin 1/2). In the presence of an electric field E along the conductor the density is modified at the Fermi level by δn, such that, locally, higher energy levels are occupied for the electrons moving oppositely the field and Fermi energy levels are depleted for electrons moving along the field. We take the field oriented along the negative x-direction, so the net flow of electrons takes place along the positive x-direction. The electric field is sufficiently weak and slowly varying such that the electrons acquire a displacement $u(x)$ which obeys the quasi-classical equation of motion $m\ddot{u} = eE$, where m is the electron mass and $-e$ is the electron charge. The change in the electron density is given by $\delta n = -n\partial u/\partial x$, such that the density of electrons participating in the electrical flow is $-\delta n$. From these two equations we get straightforwardly

$$m\frac{d}{dt}\delta n = -enE/v_F \tag{1}$$

for a constant field, where v_F is the Fermi velocity. This is the basic equation for computing the electrical current.[1] Indeed, the electrical flow (charge per unit area of the cross section and per unit time) is given by $j = -e(-\dot{\delta n})l = -e^2nEl/mv_F$, hence the well-known electrical conductivity $\sigma = e^2nl/mv_F$. The electrical flow is negative, i.e. it is oriented along the electrical field as it should be. For the one-dimensional gas $n = gk_F/\pi A$ and $v_F = \hbar k_F/m$, so we get $\sigma = g(2e^2/h)(l/A)$, where h is Planck's constant ($\hbar = h/2\pi$). The electrical conductance is $G = \sigma A/l = g(2e^2/h)$. It can be written as $G = \sum_s G_0$, where s is the spin variable (e.g. $s = \pm 1$ for spin

[1] The quasi-classsical motion of the one-dimensional quantum electron gas was previously discussed in more detail in Ref. [15].

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0375-9601/$ – see front matter
doi:10.1016/j.physleta.2008.05.061

$1/2$) and $G_0 = 2e^2/h$. We can see that the electrical conductance can only vary by quanta G_0, according to spin degeneracy.

It is worth noting that the same result can be obtained by applying the uncertainty principle. Indeed, the equation of motion $m\,d\dot{u}/dt = eE$ can also be written as $m\dot{u} = eE\tau$, where τ is the time of motion. The electrical flow can be written as $j = -e\dot{u}\delta n$, where $\delta n = g\delta k_F/\pi A$ is the density of electrons participating in conduction. Combining these two equations we get $j = -e^2E\tau\delta n/m$ and $\sigma = e^2\tau\delta n/m = ge^2\tau\delta k_F/\pi Am$, which is another representation for the electrical conductivity. Now we use the uncertainty principle in the form $\tau = \delta n_F(h/\delta\mathcal{E})$, where $\delta n_F = l\delta k_F/2\pi$ and the change in energy is given by

$$\delta\mathcal{E} = \frac{\hbar^2}{2m}(k_F + \delta k_F)^2 + \frac{\hbar^2}{2m}(k_F - \delta k_F)^2 - \frac{\hbar^2}{m}k_F^2 = \frac{\hbar^2}{m}(\delta k_F)^2. \tag{2}$$

The motion time given by the uncertainty principle corresponds to δn_F cycles of quanta of action h. The change in energy given by (2) is also $\delta\mathcal{E} = -eV$, where V is the voltage drop, which shows that the voltage is also quantized. We consider the energy levels sufficiently dense as to allow a continuous change in the electrical potential. Combining all these formula given above we arrive again at the conductance $G = gG_0$.

Within the quasi-classical description by means of the displacement field u the electrical field is given by $E = d\varphi/du$, where φ is the electrical potential, such that the equation of motion $m\ddot{u} = eE$ ensures the conservation of energy. This equation of motion can also be written as $\dot{\Pi} = eE = e\,d\varphi/du$, or $\dot{u} = e\,d\varphi/d\Pi$, where $\Pi = m\dot{u}$ is the momentum associated to the field u. One may check indeed that $\Pi = \hbar\delta k_F$ by making use of Eq. (2), so we may also write $\dot{u} = e\,d\varphi/dp_F = e\,d\varphi/\hbar\,dk_F$. Now the electrical flow $j = -e\dot{u}\delta n$ becomes $j = -e^2(d\varphi/\hbar\,dk_F)\delta n = -e^2(dn/\hbar\,dk_F)V$ (since $\delta n = (dn/d\varphi)V$), hence the electrical current $I = jA = -gG_0V$. This was, in essence, the original argument of Landauer [14]. One can see that the conductance is proportional to the density of states.[2]

We consider next a nanowire of thickness d ($A = d^2$) and a confined transversal motion of the electrons, such that the energy levels are given by

$$\varepsilon = \frac{\hbar^2k^2}{2m} + \frac{\pi^2\hbar^2}{2md^2}\left(n_1^2 + n_2^2\right), \tag{3}$$

where $n_{1,2}$ are positive integers.[3] We have now multiple branches of one-dimensional electron gas and the Fermi wavevector depends on the duplex (n_1, n_2). Therefore, the electron density is given by

$$n = (g/\pi A)\sum_{(n_1,n_2)} k_F(n_1, n_2) \tag{4}$$

and the electrical conductivity $\sigma = e^2nl/mv_F$ becomes

$$\sigma = \left(e^2l/\pi Am\right)\sum_{(n_1,n_2),s} k_F(n_1, n_2)/v_F(n_1, n_2). \tag{5}$$

By (3), the Fermi velocity is $v_F = \hbar k_F/m$, such that the above electrical conductivity becomes $\sigma = G_0(l/A)M$ and the electrical conductance is quantized according to $G = G_0M$, where

$$M = \sum_{(n_1,n_2),s} 1 \tag{6}$$

is the number of branches in the electron spectrum (number of channels), spin included. One can see that M is the number of channel electron states, so the conductance can only change in steps of quanta G_0.

Now we want to compute the number of channels M for this model. We assume a dense distribution of spectrum branches and write $n_1^2 + n_2^2 = \rho^2$ in Eq. (3). The chemical potential μ is established by the equalities

$$\mu = \frac{\hbar^2k_F^2}{2m} + \frac{\pi^2\hbar^2}{2md^2}\rho^2, \tag{7}$$

hence the Fermi wavevector k_F which is used in Eq. (4). The number of channels is then given by $M = \pi gN_t^2$, where N_t is the highest integer ρ satisfying equation (7). It is given approximately by $N_t^2 = 2md^2\mu/\pi^2\hbar^2$. Eq. (4) can then be written as

$$n = (2g/A)\int_0^{N_t} d\rho\,\rho\sqrt{2m\mu/\hbar^2 - \pi^2\rho^2/d^2}. \tag{8}$$

This equation gives a relationship between N_t and μ, which, together with the equation $N_t^2 = 2m\,d^2\mu/\pi^2\hbar^2$ written above, serve to determine both the chemical potential μ and the number N_t of transverse channels, hence the total number of channels M, as a function of the density of the electron gas. The integral in Eq. (8) can be performed straightforwardly. We get $N_t = (3Nd/\pi gl)^{1/3}$ and $M = (\pi g)^{1/3}(3Nd/l)^{2/3}$, where N is the total number of electrons and l is the length of the sample. The electrical conductance reads $G = G_0(\pi g)^{1/3}(3Nd/l)^{2/3}$. This is the main result for the overall continuous behaviour of the quantized conductance of an ideal nanowire of finite cross sectional area.

A simple application of the above result pertains to a two-dimensional electron gas with point contacts [10]. In this case we may consider that the energy spectrum given by Eq. (3) contains only one quantum number, say n_1, as corresponding to the quantized motion along one transverse direction. The number of channels is now $M = gN_t$, where N_t is given by $\mu = \pi^2\hbar^2N_t^2/2md^2$. The chemical potential μ can be obtained straightforwardly from Eq. (4) as $\mu = 2\pi\hbar^2An/gmd$, which leads to $M = gN_t = 2\sqrt{gNd/\pi l}$. If we take, as usually, $N = gW^2k_F^2/4\pi$, where W is the width of the contacts, we get $M = gk_FW(d/l)^{1/2}/\pi$, which is a slight generalization of the classical result verified experimentally in the quantized conductance $G = G_0M$ of a GaAs–AlGaAs heterostructure with ballistic point contacts [10].

As it is well known, for electrons in a magnetic field H we can write the energy levels as

$$\varepsilon = \frac{\hbar^2k^2}{2m} + \hbar\omega_c(n + 1/2) + \mu_BHs, \tag{9}$$

where $\omega_c = eH/mc$ is the cyclotron frequency, $\mu_B = e\hbar/2mc$ is the Bohr magneton and $s = \pm 1$. Eq. (5) gives now $G = G_0M$, where

$$M = \sum_{ns} 1. \tag{10}$$

It is approximately $M = 2n_l$, which is in fact twice the number of spectrum branches. We have also $\mu = \hbar\omega_cn_l$, where μ is the chemical potential given by

$$n = \frac{4eH}{\pi ch}\int_0^{n_l} dn\sqrt{2m\mu/\hbar^2 - 2m\omega_cn/\hbar}, \tag{11}$$

which is similar to Eq. (8). Here it is worth noting the well-known transversal degeneracy $2eHA/ch$ ($\gg 1$) of the energy levels in the magnetic field. We get $M = (3\sqrt{\pi}/4\sqrt{2})^{2/3}n^{2/3}ch/eH$ (which should be much larger than unity). This main result allows the

[2] See in this respect Ref. [3].

[3] We impose fixed-ends boundary conditions for the transversal motion as for electrons confined in an infinite potential well (see Refs. [1–3]).

calculation of the quantized conductance $G = G_0 M$. It is worth emphasizing that the electrical conductance (or magnetoresistance) can be varied in quantum steps by varying the magnetic field, as it is well known.

The inclusion in such a treatment of interaction, scattering or thermal effects (or finite-size boundary effects), as well as other particularities,[4] renders the problem a bit more complicated. Generally speaking, the starting point in such a treatment is the notion of elementary excitations and their lifetime. Particularly interesting is this problem for multi-wall carbon nanotubes, due to their specific electron energy structure [6].

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[4] E.g. for specular reflection in a cylindrical conductor in magnetic field see Ref. [8].