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PULSE and IMPULSE of ELI

("Extreme Light Infrastructure")

Electron Pulses Accelerated by Laser Beams

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Important paper from 1979 by **Tajima and Dawson**

Wakefield acceleration of electrons (trapped, injected, longitudinal field, light pressure, ...); $10MeV$ ($\rightarrow 1GeV$), flux $\sim 10^{10}$ per pulse ($d \sim 1mm$)

High-intensity lasers developed by **Mourou et co**, ~1990s

Intensity, spot size, power, etc; $10^{18}w/cm^2$, spot $d \sim 1mm$ (pettawat!), energy $\sim 10^4 J$! $\lambda_0 \sim 1\mu$ (10^3 λ_0 in the pulse); duration $\sim 10^{-12}s$ (picosecs)

Most of the results presented here were obtained in collaboration with
M GANCIU

Electron plasma, density n, mass m, charge $-e$; neutralizing, rigid ion background

Displacement field $\mathbf{u}(\mathbf{r},t)$, volume density imbalance $\delta n = -n div\mathbf{u}$

Charge density $\rho = endiv\mathbf{u}$, current density $\mathbf{j} = -en\dot{\mathbf{u}}$

Maxwell equations

$$div\mathbf{E} = 4\pi endiv\mathbf{u}\ ,\ div\mathbf{H} = 0$$

$$curl\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}\ ,\ curl\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} - \frac{4\pi en}{c}\frac{\partial\mathbf{u}}{\partial t}$$

(non-magnetic plasma)

Wave eq with sources

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta \mathbf{E} = -4\pi en \cdot grad \cdot div \mathbf{u} + \frac{4\pi en}{c^2}\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

Non-relativistic motion, compensating polarization fields

Newton's law, external field $\mathbf{E}_0$

$$m\ddot{\mathbf{u}} = -e\mathbf{E} - e\mathbf{E}_0$$

Fourier transforms

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{(2\pi)^4}\int d\mathbf{k}d\omega \mathbf{u}(\mathbf{k}, \omega)e^{i(\mathbf{kr} - i\omega t)}$$

Solution

$$\omega^2(\omega^2 - \omega_p^2 - c^2k^2)\mathbf{u} = -\frac{e\omega_p^2 c^2}{m}\mathbf{k}\frac{\mathbf{k}\mathbf{E}_0}{\omega^2 - \omega_p^2} + \frac{e}{m}(\omega^2 - c^2k^2)\mathbf{E}_0$$

Plasma frequency

$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

Read the solutions:

Plasmons $\omega = \omega_p$, dispersionless!, longitudinal (light reflection)

Polaritons $\omega_1 = \sqrt{\omega_p^2 + c^2k^2}$, dispersive ("propagating") (light refraction)

Retain polaritons, transverse fields ($\mathbf{kE}_0 = 0$, $\mathbf{ku} = 0$, $\mathbf{kE} = 0$)

Transverse solution

$$\mathbf{u} = \frac{e}{m} \frac{\omega^2 - c^2 k^2}{\omega^2(\omega^2 - \omega_1^2)} \mathbf{E}_0$$

Dielectric function $\mathbf{E}_0 = \varepsilon \mathbf{E}_{tot} = \varepsilon(\mathbf{E}_0 + \mathbf{E})$

$$\varepsilon(\mathbf{k}, \omega) = 1 - \frac{\omega_p^2}{\omega^2 - c^2 k^2}$$

Vector potential $\mathbf{A}_0 = -\frac{ic}{\omega}\mathbf{E}_0$

Perform first the inverse Fourier transform with respect to frequency (ω_1-contribution)

The full inverse Fourier transform

$$\mathbf{u}(\mathbf{r},t) = -\frac{e\omega_p^2}{4mc}\frac{1}{(2\pi)^3}\int d\mathbf{k}\frac{1}{\omega_1^2}\mathbf{A}_0(\mathbf{k},\omega_1)e^{i(\mathbf{kr}-\omega_1 t)}$$

Focus on a certain wavevector $\mathbf{k}_0$, make a series expansion of ω_1 in powers of $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$, $0 < q < q_c$, cutoff $q_c \ll k_0$ (isotropic)

Get an isotropic **wave packet** extending over $d = 2\pi/q_c \gg \lambda_{10}$

λ_{10} is the wavelength of $\omega_{10} = \sqrt{\omega_p^2 + c^2 k_0^2}$

The pulse propagates with the group velocity $\mathbf{v} = \partial\omega_1/\partial\mathbf{k}$ for $\mathbf{k} = \mathbf{k}_0$

$$v = \frac{c^2 k_0}{\sqrt{\omega_p^2 + c^2 k_0^2}}$$

The pulse

$$\mathbf{u}(\mathbf{r}, t) \simeq -\frac{e\omega_p^2}{4mc}\frac{1}{\omega_{10}^2}\mathbf{A}_0(\mathbf{k}_0, \omega_{10})\delta(\mathbf{r} - \mathbf{v}t)$$

Assume $\omega_p \ll \omega_0 = ck_0$; then, the group velocity

$$v \simeq c\left(1 - \frac{\omega_p^2}{2\omega_0^2}\right) \simeq c$$

The First Great Equation: Electron energy

$$E_{el} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq \frac{\omega_0}{\omega_p} mc^2$$

For realistic values $\hbar\omega_0 = 1eV$ ($\lambda_0 \simeq 1\mu$), electron density $n = 10^{18} cm^{-3}$, $\hbar\omega_p = 3 \times 10^{-2} eV$

$$E_{el} = \frac{\omega_0}{\omega_p} mc^2 \gg mc^2 \simeq 17 MeV$$

Displacement in the pulse

$$\mathbf{u}_0 = -\frac{e\omega_p^2}{4mc\omega_0^2 d^3}\mathbf{A}_0(\mathbf{k}_0, \omega_0)$$

Similar pulse for the vector potential $\mathbf{A}_0$

$$\mathbf{A}_0(\mathbf{r}, t) = \mathbf{A}_0 d^3 \delta(\mathbf{r} - \mathbf{v}t)$$

Superposition of frequencies in the range $\Delta\omega = cq_c = 2\pi c/d$, so $\mathbf{A}_0(\mathbf{k}_0, \omega_0) = \mathbf{A}_0 d^4/c$ and get finally

$$\mathbf{u}_0 = -\frac{e\omega_p^2 d}{4mc^2\omega_0^2}\mathbf{A}_0$$

Transverse displacement $\mathbf{u}_0$ ($\mathbf{k}_0\mathbf{u}_0 = 0$)

No volume charge density in the pulse

Charge distributed over the **pulse surface** over a region of thickness $\sim \lambda_0$

Approximately $\delta n_0 = n u_0/\lambda_0$

Total number of electrons in the pulse

$$\delta N = \pi n d^3 \frac{e\omega_p^2}{4mc^2\omega_0^2} A_0$$

Express the vector potential A_0 by the density of the field energy $w_0 = k_0^2 A_0^2/4\pi$

Introduce the notations $\varepsilon_p = \hbar\omega_p$, $\varepsilon_0 = \hbar\omega_0$ and $\varepsilon_{el} = e^2/d$, the later being the Coulomb energy of an electron localized in the pulse

Get the **Second Great Equation** (electron flux)

$$\delta N = nd^2\lambda_0 \frac{\varepsilon_p^2}{4mc^2\varepsilon_0^2}\sqrt{\pi\varepsilon_{el}W_0}$$

where W_0 is the total amount of field energy in the pulse

$$W_0 = I_0 d^3/c$$

where I_0 is the laser intensity ($10^{18} w/cm^2$)

Numerics:

Typical values $I_0 = 10^{18} w/cm^2$, $d = 1mm$ ($W_0 = 10^{23} eV$ and $\varepsilon_{el} = 10^{-6} eV$)

$n = 10^{18} cm^{-3}$ ($\varepsilon_p = 3 \times 10^{-2} eV$), $\varepsilon_0 = 1eV$ ($\lambda_0 \simeq 1\mu$) and $mc^2 = 0.5MeV$

Get $\delta N \simeq 10^{11}$ electrons in the pulse, accelerated at the energy $\simeq 17MeV$

Their total energy is $W_{el} \simeq 10^{18} eV$, the remaining energy (up to $W_0 = 10^{23} eV$) being left in the laser pulse

Recent experimental measurements seem to be in good agreement with these equations (Giulietti et al, 2010, etc, etc)

Two Big Conclusions

Electron energy

$$E_{el} \sim \frac{\omega_0}{\omega_p} mc^2$$

Electron flux

$$\delta N \sim n d^{3/2} \frac{\omega_p^2}{mc^2 \omega_0^2} \sqrt{I_0}$$

Some Comments

- **Efficiency coefficient**, $W_{el} = E_{el}\delta N$

$$\eta = \frac{W_{el}}{W_0} = nd^2\lambda_0 \frac{\varepsilon_p}{4\varepsilon_0}\sqrt{\frac{\pi\varepsilon_{el}}{W_0}} \ll 1\,(10^{-5})\ \ (\varepsilon_{el} \sim e^2/d)$$

($\eta = 1$ limit).

- Displacement $u_0 \simeq \lambda_0$, as expected

-Quasi-static pulse ($e^{i(\omega_0 t - \mathbf{k}_0 \mathbf{r})}$), frequency $\sim \omega_p^2/2\omega_0$, (wavelength $\sim 10^3\lambda_0$), electron velocity in the pulse $\sim c(\omega_p/\omega_0)^2 = c/10^3$, non-relativistic approximation

-It is their trapped motion carried along by the pulse that made them acquire relativisic velocities

-But this motion is decoupled from the displacement $\mathbf{u}$, it pertains to the pulse coordinate $\mathbf{r}$

-**Dielectric function**, $k \simeq k_0$ and $\omega \simeq \omega_0 = ck_0$, which makes an infinite dielectric function

-Therefore, highly effective polarization in the pulse, total field inside almost vanishing, which justifies again the use of the non-relativistic equation of motion for the internal motion of the electrons inside the pulse

-Macroscopic pulse dielectric fnction $\varepsilon = 4\omega_0 c/\omega_p^2 d$, which, for our numerical values given above is of the order of unity (the pulse is transparent!)

-This pulse dielectric function is effective in the motion of an external electron affected by the pulse, which experiences a high field, of the order of the external field E_0

-**External electric field** is $E_0 \simeq 10^{12} V/m$, external magnetic field is $H_0 \simeq 10^3 Ts$

-**Pulse dispersion**; high-order contribtutions in the q-expansion of the frequency around the wavevector k_0; it flatens gradually the pulse

-**Fluctuations** in the plasma density (Maxwellian), which are of the order of n; induce a corresponding dispersion in the plasma frequency, group velocity, in fact a set of pulses, propagating with various velocities; dispersion in the electron energy of the order of E_{el}

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Addenda

$$L_e = \frac{m}{2a^3}\int d\mathbf{r}\left[\dot{\mathbf{u}}^2 - \frac{m^2c^4}{\hbar^2}\mathbf{u}^2 - mc^2(div\mathbf{u})^2\right]$$

Real, vectorial (spin one) **Proca field**

(mixed components of an antisymmetric tensor of rank two

or the space components of a four-vector $u_\mu = (u_0, -\mathbf{u})$, with transversality condition $p^\mu u_\mu = i\hbar\partial^\mu u_\mu = 0$)

$\mathbf{u}$ describes relativistic electron excitations in plasma, spin one, Bose-Einstein statistics

Equation of motion

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{u}}{\partial t^2} - \Delta \mathbf{u} + \frac{m^2c^2}{\hbar^2}\mathbf{u} = 0$$

gives the frequency $\omega = \sqrt{m^2c^4/\hbar^2 + c^2k^2}$, $\hbar\omega = \sqrt{m^2c^4 + c^2p^2}$

Density of electric charge $\rho = -e\delta n = endiv\mathbf{u}$, density of electric current $\mathbf{j} = -en\dot{\mathbf{u}}$

$$L_{int} = -\int d\mathbf{r}\rho\mathbf{\Phi} + \frac{1}{c}\int d\mathbf{r}\mathbf{j}\left(\mathbf{A} + \mathbf{A_0}\right) = -en\int d\mathbf{r}\cdot div\mathbf{u}\cdot\mathbf{\Phi} - \frac{en}{c}\int d\mathbf{r}\cdot\dot{\mathbf{u}}\left(\mathbf{A} + \mathbf{A_0}\right)$$

Maxwell equations

$$\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = 4\pi\rho = 4\pi en div\mathbf{u}$$

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\mathbf{j} = -\frac{4\pi en}{c}\dot{\mathbf{u}}$$

(Lorenz gauge)

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + div\mathbf{A} = 0$$

and

$$\frac{1}{c^2}\frac{\partial^2\mathbf{u}}{\partial t^2} - \Delta\mathbf{u} + \frac{m^2c^2}{\hbar^2}\mathbf{u} = \frac{e}{mc^2}grad\Phi + \frac{e}{mc^3}\left(\dot{\mathbf{A}} + \dot{\mathbf{A}}_0\right)$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \simeq \frac{\Omega}{\omega_p} E_0 \ , \ \hbar\Omega = E_0 = mc^2$$

$E_0 = 0.5 MeV$, $\omega_p = 10 eV$, $E \simeq 25 GeV$

The total number of accelerated electrons

$$\delta N \simeq nd^2 \lambda_0 \frac{\varepsilon_p^2 \varepsilon_0^2}{E_0^5} \sqrt{\varepsilon_{el} W_0}$$

Energy flux $I_0 = 10^{24} w/cm^2$ ($W_0 = I_0 d^3/c = 10^{29} eV$) in a pulse of size $d = 1mm$ (electric field $\simeq 10^{15} V/m$, magnetic field $\simeq 10^7 Ts$)

Coulomb energy is $\varepsilon_{el} = 10^{-6} eV$

Typical values $n = 10^{22} cm^{-3}$, $\varepsilon_0 = 1eV$ ($\lambda_0 = 1\mu$) (and $\varepsilon_p = 10eV$, $E_0 = 0.5MeV$)

$$\delta N \simeq 10$$

High energy ($25GeV$), low electron flux