Moon's Problem

(A three-body problem)

An Investigation into Intractability and Uncomputability in Theoretical Physics

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Summary

I

1 A new method for Kepler's problem

2 Closed orbits and the first sign of "chaos"

II

3 Moon's problem: a three-body problem

4 Another three-body problem: Jupiter-Saturn Couple

5 The "Four Moons": four periodicities; the Greeks

6 First order of the perturbation theory

7 Newton (d'Alembert, Clairaut, Delauney)

8 Poincare and the "weak" chaos

9 Missing integrals, instabilities, Gutzwiller and the "strong" chaos

III

10 A new route to quantizing the Hydrogen atom

IV

11 Chaos

12 Intractability, Unprovability and Undecidability

13 Uncomputability, Computational Irreducibility, Universality

in Theoretical Physics (Stephen Wolfram)

I A New Method for Kepler's Problem

$$E = m\dot{r}^2/2 + mr^2\dot{\varphi}^2/2 - \alpha/r = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r$$

$$L = mr^2\dot{\varphi}$$

(Kepler's second law)

$$U = L^2/2mr^2 - \alpha/r$$

Trajectory: $r_1 = a(1-e)$ to $r_2 = a(1+e)$, semi-major axis $a = \alpha/2\,|E|$

Eccentricity $e = \sqrt{1 - 2L^2\,|E|\,/m\alpha^2}$

U_{min} for $r_0 = L^2/m\alpha = a(1-e^2)$ and $|E| = \frac{\alpha}{2r_0}(1-e^2)$

Oscillator: Power expansion around the minimum value $r - r_0 = Au$, $A/r_0 = \varepsilon$

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\omega^2 u^3 + ...)$$

$$\omega^2 = \alpha/mr_0^3$$

(Kepler's third law)

Solutions $\varepsilon = e(1-e)$ (small eccentricities)

$$r = r_0[1 - e\cos\omega t + \frac{e^2}{2}(3 - \cos 2\omega t)]$$

$$\varphi = \omega t + 2e\sin\omega t - \frac{e^2}{2}(3\omega t - \frac{5}{2}\sin 2\omega t)]$$

$$r = r_0(1 - e\cos\varphi + e^2\cos^2\varphi + ...) = r_0/(1 + e\cos\varphi)$$

Ellipse: semi-major axis $a = r_0/(1-e^2)$, semi-minor axis $b = r_0/(1-e^2)^{1/2}$, origin displaced by $ae = r_0 e + ...$ in the focus ae (Kepler's first law)

Technical Note: Resonant (secular) terms, anharmonic corrections

Frequency shift (Poincare-Lindstedt method, 1882-1892)

$$\Omega = \omega(1 - 3e^2/2) = (\alpha/ma^3)^{1/2}$$

(second-order cubic, first-order quartic)

Automatically included by the present method

Other central-force fields

$$r = r_0[1 - e\cos\omega t + \frac{\beta e^2}{2}(3 - \cos 2\omega t)]$$

$$\varphi = \sqrt{v_1/(3v_1 + r_0 v_2)}\{\omega t + 2e\sin\omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t]\}$$

$$m\omega^2 = 3v_1/r_0 + v_2,\ \beta = (2v_1 - r_0^2 v_3/6)/(3v_1 + r_0 v_2)$$

$$r = r_0'[1 - e\cos\chi + (2 - \beta)e^2\cos^2\chi]\ ,\ r_0' = r_0[1 - 2(1 - \beta)e^2]$$

$$\varphi = \sqrt{v_1/(3v_1 + r_0 v_2)}\chi$$

Closed orbits. First sign of "chaos"

$$\sqrt{v_1/(3v_1 + r_0 v_2)} = p/q$$

Gravitational potential: $\beta = 1$, $p/q = 1$

Spatial oscillator ($v(r) = const + \alpha r^2$): $\beta = 1/2$, $p/q = 1/2$, $\chi = 2\varphi$ (ellipse centered on the origin)

Only two potentials close exactly the orbits (Bertrand's theorem, 1873)

$$p/q = (1/\pi) \int_{r_1}^{r_2} dr \cdot (L/r^2)/\sqrt{2m(E - v) - L^2/r^2}$$

(closure integral)

Many other fields close the orbits to any finite-order of perturbation theory

But not in the limit

To the extent to which an irrational number is approximated by rational numbers

Infinitely (and densely) quasi-closed orbits: first sign of "chaos"

- Sensitive and arbitrary dependence on initial conditions (L, E)

- Change slightly $1/r$; compute closure integral; would never know whether it is rational or irrational

- Unprovability, Undecidability

II Moon's problem: a three-body problem

$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1 - \mathbf{r}_2\right|$

$m_0 \simeq 2 \times 10^{30} Kg$ (Sun), $m_1 \simeq 6 \times 10^{24} Kg$ (Earth), $m_2 \simeq 7 \times 10^{22} Kg$ (Moon)

$G \simeq 6.7 \times 10^{-11} m^3/Kg \cdot s^2$

$r_1 \simeq 150 \times 10^6 Km$ (Sun-Earth), $r \simeq 380\,000 Km$ (Earth-Moon)

$$\mathbf{L}_{tot} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2$$

Center of mass $E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2-$

$$-Gm_0m_1/\left|\mathbf{R} - m_2\mathbf{r}/M\right| - Gm_0m_2/\left|\mathbf{R} + m_1\mathbf{r}/M\right| - Gm_1m_2/r$$

Quadrupolar perturbation: $r/R \sim 3 \times 10^{-3}((r/R)^2 \sim 10^{-5})$

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{rR})^2/R^2 - r^2]/R^3$$

$$\alpha = Gm_0M,\ \beta = GmM \text{ and } \gamma = Gm_0m/2$$

Two coupled Kepler's problems $E = E_1 + E_2 + \gamma v$

$$E_1 = M\dot{\mathbf{R}}^2/2 - \alpha/R\ ,\ \ E_2 = m\dot{\mathbf{r}}^2/2 - \beta/r\ ,\ v = -r^2(3\cos^2\chi - 1)/R^3$$

Another three-body problem: Jupiter-Saturn Couple

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1 - \mathbf{r}_2\right|$$

Perturbation:

$$Gm_1m_2/\left|\mathbf{r}_1 - \mathbf{r}_2\right|$$

$m_1\,,\, m_2 \ll m_0$ and $\mathbf{r}_1$ not too close to $\mathbf{r}_2$

The "Four Moons": four periodicities; the Greeks

Sideral year: 365.25 days

Sideral Moon: 27.32 days (rotation about the fixed stars)

Synodal Moon: (29.53 days) (combining the year and the sideral Moon; Moon's phases)

Nodal Moon: 27.21 days (up and down about the ecliptic)

Anomalous Moon: 27.55 days (the acceleration toward perigee, dec to apogee)

"Second" accuracy: five decimals; The Greeks !

Polar coordinates:

$$E = E_1 + E_2 + \gamma v$$

$$E_1 = M\dot{R}^2/2 + MR^2(\dot{\Theta}^2 + \dot{\Phi}^2 \sin^2 \Theta)/2 - \alpha/R$$

$$E_2 = m\dot{r}^2/2 + mr^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta)/2 - \beta/r$$

$$v = -r^2(3\cos^2 \chi - 1)/R^3 \ , \ \cos\chi = \sin\Theta \sin\theta \cos(\Phi - \varphi) + \cos\Theta \cos\theta$$

First order of the perturbation theory

Eccentricity series: $R_0 = L^{(0)2}/M\alpha$, $\Omega^2 = \alpha/MR_0^3$ (Earth period $\sim$ 365days) and $e_1 = (1 - 2R_0 |E_1| /\alpha)^{1/2}$ (Earth eccentricity $e_1 = 0.017$)

$$R^{(0)} = R_0[1 - e_1 \cos \Omega t + \frac{e_1^2}{2}(3 - \cos 2\Omega t) + ...]$$

$$\Phi^{(0)} = \Omega t + 2e_1 \sin \Omega t - \frac{e_1^2}{2}(3\Omega t - \frac{5}{2} \sin 2\Omega t)...]$$

$$\Theta^{(0)} = \pi/2$$

Relative motion: $r_0 = l^{(0)2}/m\beta$, $\omega^2 = \beta/mr_0^3$ ($\omega \gg \Omega$; Moon's period ~ 27 days), eccentricity $e_2 = (1 - 2r_0 \left|E_2\right| /\beta)^{1/2}$ (Moon's orbit eccentricity $e_2 \simeq 0.055$)

Tilted frame:

$$\theta^{(0)} = \pi/2 + \theta_0 \sin \omega t + ... \, , \; \varphi^{(0)} = \omega t + ...$$

Another perturbation parameter: tilting angle

$$\theta_0 = 5^\circ = \pi/36$$

Another frequency

$$\omega' = \omega \sqrt{1 - \theta_0^2}$$

Perturbation theory: Six coordinates R, Θ, Φ, r, θ, φ

Triple series expansion: eccentricities ($e_{1,2}$), inclination (θ_0) and γ- interaction (Weierstrass, Sweden's King contest)

Zeroth order: $R^{(0)} = R_0$, $\Theta^{(0)} = \pi/2$, $\Phi^{(0)} = \Omega t$, $r^{(0)} = r_0$, $\theta^{(0)} \simeq \pi/2$, $\varphi^{(0)} = \omega t$

First-order correction:

$$\Phi = \Omega t - \gamma(3r_0^2/4M\omega^2 R_0^5)\sin 2\omega t + \ldots\,,\ \varphi = \omega t + \gamma(3/4m\omega^2 R_0^3)\sin 2\omega t + \ldots$$

The fourth frequency $\sim 3\gamma/2m\omega R_0^3 = 3Gm_0/4\omega R_0^3 = 3\Omega^2/4\omega$ appearing in $\dot{\varphi}$

$$\Omega/\omega \simeq 1/13.5$$

Newton (d'Alembert, Clairaut, Delauney)

Correction $3\gamma/2m\omega R_0^3 = 3\Omega^2/4\omega$ known to Newton!

Applied to φ leads to $4\omega/3\Omega^2 \simeq 18$ years of Moon's retrograde motion period (earthquakes?)

Sideral Moon: $\omega' = \omega\sqrt{1-\theta_0^2}$ (27.32 days)

Nodal Moon: ω (27.21 days)

Anomalous Moon: correction $3\Omega^2/4\omega^2 \sim 0.004$ applied to ω' (27.55 days; factor 2 by d'Alembert and Clairaut, ~1750)

Synodal Moon: correction $\omega - \Omega$ $(27.32 + 27 \times (1/13.5) = 29.5$ days)

Pushing up through higher orders of the perturbation theory

Delauney $\sim$1860 up to $\sim$ 500 terms! ($\sim$ 2000 print pages)

Hill $\sim$1880 (rotating frame)

Poincare $\sim$1890; Sweden's King contest; stability of the Planetary System

Pertractors of Newton

Poincare-Mittag-Leffler 4 years mistake $\sim$1900

Modern computers: 1950-1970 (aselenization)

Gutzwiller et al $\sim$1980; quasi-failure; **Slow convergence, weak accuracy, algorithms**

Poincare and the "weak" chaos

Solution as Fourier series

4 fundamental frequencies (Ω , ω' , ω and $3\gamma/2m\omega R_0^3 = 3\Omega^2/4\omega$)

Combined-frequency phenomenon (non-linearities), higher-order harmonics

Solution looks quite erratic (main pattern plus infinitely small, fine laceworking!)

Poincare "Weak" Chaos

Small contributions are important: **a small error on Earth, a big failure on Moon**!

(1km on Moon, $3''$, 10^6 accuracy!)

Missing integrals, instabilities, Gutzwiller and the "strong" chaos

6 degrees of freedom in 3-body problem and only 4 integrals (E, $\mathbf{L}_{tot}$)

Are there other integrals?

No !(Bruns-Poincare theorem) (algebraic in parameters)

Non-anayticity: $\dot{\varphi}$ very high over the pole ($\theta = 0$), sudden change of the trajectory along a longitudinal circle

Instabilities, "strong" chaos

Rather strange external perturbations

Not only Chaos (weak or strong)

In addition:

- a great deal of proliferating contributions

- time-consuming just for keeping track of them (parallel computing?)

- more computing time than the real, natural process (more program bites than number of bits produced - Chaitin)

- Uncomputability, Computational Irreducibility, Intractability

III A new route to quantizing the Hydrogen atom

A special classical motion: passing through the centre

$$E = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r$$

$L = 0$, $E = -\alpha/r_0$

Oscillations between 0 and r_0 around $r_0/2$

$\dot{u}^2 + \omega^2(u-1)/(u+1) = 0$, $r = r_0(1+u)/2$, $\omega^2 = 8\alpha/mr_0^3 = 8\,|E|^3\,/m\alpha^2$

Solution $2\,\text{arcsin}\sqrt{(1-u)/2} + \sqrt{1-u^2} = \omega t$, $-1 < u < 1$, periodicity, $\omega/2$

Quantization

$$m\dot{\rho}^2/2 + m\omega^2\rho^2/2 + ... - \alpha/r_0 = 0$$

Harmonic oscillator

$$\hbar\omega(n + 1/2)/2 = \alpha/r_0 \ , \ \hbar\omega\delta n/2 = |E|_q \ , \ \delta n = n$$

$$|E|_q = \frac{m\alpha^2}{2\hbar^2 n^2}$$

Variation equation, anharmonic corrections, etc

Case $L \neq 0$

$$r_0 = L^2/m\alpha \ , \ m\omega^2 = \alpha/r_0^3$$

$$E = m\dot{r}^2/2 + (\alpha/2r_0^3)(r - r_0)^2 + ... - \alpha/2r_0 \ , \ \hbar\omega(n + 1/2)/2 = \alpha/2r_0 + E$$

$$L^2/2I = L^2/2mr_0^2 = \alpha/2r_0 = Energy$$

$$\hbar\omega(Energy)\delta n/2 = Energy$$

$$Energy = \frac{m\alpha^2}{2\hbar^2 n^2}$$

A general central-field potential $v(r)$

$$-L^2/mr_0^3 + v_1 = 0\ ,\ Energy = L^2/2mr_0^2 = v_1 r_0/2$$

$$r_0(Energy)$$

$$\omega^2 = 3v_1/mr_0 + v_2/m$$

$$\hbar\omega(Energy)\delta n/2 = Energy$$

$$\hbar^2[3v_1(Energy)/mr_0(Energy) + v_2(Energy)/m]n^2/4 = Energy^2$$

A few comments on Chaos

1 Endless orbits never repeating

2 Sensitive dependence on initial conditions (Lyapunov exponents)

3 Non-linearities

4 Logistic maps, bifurcation and Feigenbaum number (~ 4.16; $x_{n+1} = Lx_n(a - x_n)$; ratio of two successive L's)

5 Fractal dimension ($r = 3^n$, $N = 4^n$: $N = r^D$, $D = \ln 4/\ln 3 = 1.26$; Koch's curve)

6 Cellular automata

What have we seen?

Chaos (open orbits, great variability in small things with huge consequences)

Unprovability, undecidability (closing orbits)

Intractability, uncomputability and computational irreduciblility (3-body problem)

Universality (Nature is a Universal Machine, emulating everything)

Questions

1 What about friction? (Earth, loses 1 second per century)

2 What about the effect of the other Planets on Moons' motion? (time-space stochastic perturbation)?

3 What about the couple Jupiter-Saturn? (Laplace $2:5$ resonance, Jupiter 12 years, Saturn 30 years)?

4 Another resonance, Moon's ever-staring side (Lagrange, distorted Moon, the coupling, pin-down the motion; Gauss and the blocking of the phase)

5 The great analogy with CDW's (or SDW's); pinning, commensurate, incommensurate, order parameter, symmetry breaking, Goldstone modes, etc, etc

6 Dipolar (and multipolar) coupling

7 What about relativistic corrections? ($v/c \sim 10^{-5}$)

....

A New Research Program

Investigations into Intractability and Uncomputability of the 3-Body Problem