Curved space, Covariance, Motion and Quantization

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PART I : The MESSAGE

Fields Today

1) Gauge&Symmetries —-Imitation of QElectrodyn

2) Geometrization

The Third Way

Principle of Equivalence, Non-Inertial Motion

A "New Deal" in Fields

PART II : MOTION DEPENDS ON OBSERVER

Curved Space

Newton's Law

$$m\frac{dv_\alpha}{dt} = f_\alpha$$

Equivalent with a free motion in a curved space

$$Du^i/ds = \frac{du^i}{ds} + \Gamma^i_{jk}u^j u^k = 0$$

Metric tensor

$$ds^2 = (1+h)c^2dt^2 + 2cdtg_\alpha dx^\alpha - dx^\alpha dx^\alpha$$

$$g_{ij} = \begin{pmatrix} 1+h & g_1 & g_2 & g_3 \\ g_1 & -1 & 0 & 0 \\ g_2 & 0 & -1 & 0 \\ g_3 & 0 & 0 & -1 \end{pmatrix}$$

Basic equation

$$\frac{\partial g_\alpha}{c\partial t} - \frac{1}{2} \cdot \frac{\partial h}{\partial x^\alpha} = f_\alpha / mc^2$$

$$\mathbf{g}(t),\ h(\mathbf{r})$$

$$f_\alpha = -\partial\varphi/\partial x^\alpha\ ,\ h = 2\varphi/mc^2$$

(gravitational potential)

For the first time Einstein "suspected the time" (1905)

Curved Spaces: Gauss ($\sim$ 1830)

Riemann: *Uber die Hypothesen welche der Geometrie zugrunde liegen*, 1854

Grassman, Christoffel, Ricci, Levi-Civita

Klein: *Programm zum Eintritt in die philosophische Fakultat in Erlangen*, 1872

Einstein, Poincare, Minkowski, Sommerfeld, Kottler, Weyl, Hilbert

Motion depends on subjectivity, though a "universal subjectivity" ("inter-subjectivity"?)

Pauli 1921

Covariance; Dirac

Non-Uniform Translation

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}(t') \ , \ t = t'$$

$$m\frac{d\mathbf{v}'}{dt'} = \mathbf{f}' - md\mathbf{V}/dt'$$

Inertial force

$$\mathbf{g} = -\mathbf{V}/c$$

Equivalent with a curved space (**Principle of equivalence**)

Coordinate Transformations

From a flat space to a curved space

$$dx^i = a^i_j dx'^j,\ dx'^i = b^i_j dx^j,\ dx_i = b^j_i dx'_j\ a^i_k b^k_j = b^i_k a^k_j = \delta^i_j$$

$$ds^2 = \eta_{ij} dx^i dx^j = \eta_{ij} a^i_k a^j_l dx'^k dx'^l$$

$$g_{ij} = \eta_{lm} a^l_i a^m_j,\ g^{ij} = \eta^{lm} b^i_l b^j_m, \text{ where}$$
$$\eta^{lm} = \eta_{lm}(+,---)$$

$$dt = \frac{(1+h)dt' + (g+\beta\Delta)dx'/c}{\sqrt{(1+h)((1-\beta^2)}}\ ,\ dx = \frac{c\beta(1+h)dt' + (\beta g+\Delta)dx'}{\sqrt{(1+h)((1-\beta^2)}}$$

$\beta = dx/cdt$, $\Delta = \sqrt{1 + h + g^2}$

Two frames, relative velociy β, one flat, the other curved

-For h, $\mathbf{g} = 0$ - Lorentz transformations

-We put $\beta = V/c = -\mathbf{g}$ as before (to give a sense to our curved space)

-We put $h(\mathbf{r})$ and $\mathbf{g}(t)$

-We use h, $\mathbf{g} \ll 1$, to get corrections to the relativistic motion

-We get

$$dt = (1 + h/2)dt' \ , \ dx = dx' - cgdt' = dx' + V dt'$$

Life in a curved space

Proper time $d\tau = \sqrt{1+h}dt$

Distance $ds^2 = c^2(1+h)[dt + \mathbf{g}d\mathbf{r}/c(1+h)]^2 - [d\mathbf{r}^2 + (\mathbf{g}d\mathbf{r})^2/(1+h)]$

or

$$ds^2 = c^2dt'^2 - d\mathbf{l}^2$$

Light propagates along curved geodesics ($ds = 0$) with velocity c

Time is indefinite, $d\tau$ and dt', depends on path

PART III : MOTION as a COORDINATE TRANSFORM; EINSTEIN's VIEW

A GENERALIZED HAMILTON-JACOBI EQUATION

General Theory of Motion

-Free motion $t \to x$; Motion under forces $t \to x'$

-Had we know $x \to x'$, *i.e.* a global coordinate transf, we solve the motion

-Einstein's line of thought

-We have not that global coordinate transformation

(cannot get the 10 g_{ij} with four functions; local flat spaces, but axes are different from point to point)

- Our transformations are Local! ($x = (ct, \mathbf{r})$, $x' = (ct', \mathbf{r}')$, $dx \longleftrightarrow dx'$)

One Exception: Special Theory of Relativity

From rest to motion (principle of inertia, $ds^2 = const$)

$$x = c\beta\tau/\sqrt{1-\beta^2}\ ,\ t = \tau/\sqrt{1-\beta^2}$$

A vector: momentum $\mathbf{p} = \partial S/\partial \mathbf{r}$, energy $p_0 = E/c = -\partial S/c\partial t$

Apply these transf to this vector

$$\mathbf{p} = \mathbf{v}E/c^2\ ,\ E = E_0/\sqrt{1-\beta^2}\ ,\ E_0 = mc^2$$

Eqs of motion

$$d\mathbf{p}/dt = \mathbf{f}$$

Additional "relativistic" forces ($\sim v^2$)

Hamilton-Jacobi equation $E^2 - c^2p^2 = m^2c^4$

$$(\partial S/\partial t)^2 - c^2(\partial S/\partial \mathbf{r})^2 = m^2c^4$$

This is the entire theory of special relativity

Hamilton-Jacobi Equation in Curved Space

Motion in curved space

Let ($P_0 = E_0/c, -\mathbf{P}$) be the (cov) momentum of a free motion in the flat space, constant, $P_0^2 - \mathbf{P}^2 = m^2c^2$

Apply the coord transf for our curved space

$$p_0 = (1+h)p^0 + gp^1 = \sqrt{1+h} \cdot \frac{P_0 - \beta P_1}{\sqrt{1-\beta^2}}$$

$$p_1 = gp^0 - p^1 = \frac{(g+\beta\Delta)P_0 - (g\beta+\Delta)P_1}{\sqrt{(1+h)(1-\beta^2)}}$$

An integral of motion, already (by using $p_i = mcdu_i/ds$)? NO! Different x and x'!

Use $P_0^2 - \mathbf{P}^2 = m^2c^2$ for $g = -\beta$

Hamilton-Jacobi Equation in curved space

$$(E - c\mathbf{g}\mathbf{p})^2 - c^2(1 + h + g^2)(p^2 + m^2c^2) = 0$$

or

$$(\partial S/\partial t + c\mathbf{g}\partial S/\partial \mathbf{r})^2 - c^2(1+h+g^2)[(\partial S/\partial \mathbf{r})^2 + m^2c^2] = 0$$

Euler-Lagrange Motion

The action

$$S = -mc\int ds = -mc^2\int dt\cdot(1+h+2\mathbf{g}\mathbf{v}/c - v^2/c^2)^{1/2} = \int dt\cdot L$$

$$m d\mathbf{p}/dt = \mathbf{F}$$

$$\mathbf{F} = \partial L/\partial \mathbf{r} = -(mc^2/2) \cdot \frac{\partial h/\partial \mathbf{r}}{(1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2}}$$

$$E = \mathbf{pv} - L = \frac{mc^2(1 + h) + mc\mathbf{vg}}{(1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2}}$$

We get again the Hamilton-Jacobi equation given before

$$(E - c\mathbf{gp})^2 - c^2(1 + h + g^2)(p^2 + m^2c^2) = 0$$

Lagrange Motion

$$S = -mc \int ds$$

$$\delta ds^2 = 2ds\delta ds = \delta(g_{ij}dx^i dx^j) = 2g_{ij}dx^i d\delta x^j + dx^i dx^j (\partial g_{ij}/\partial x^k)\delta x^k$$

$$p_i = -\partial S/\partial x^i = mcu_i \ , \ p_i = (p_0, -\mathbf{p})$$

Hamilton: $E = -\partial S/\partial t$; it follows $p_0 = -E/c$

Since $p_i p^i = m^2 c^2$, *i.e.* $g^{ij} p_i p_j = m^2 c^2$ we get again the Hamilton-Jacobi equation in curved space

$$(\partial S/\partial t + c\mathbf{g}\partial S/\partial \mathbf{r})^2 - c^2(1+h+g^2)[(\partial S/\partial \mathbf{r})^2 + m^2c^2] = 0$$

Contravariant metric

$$g^{ij} = \frac{1}{\Delta^2} \begin{pmatrix} 1 & g_1 & g_2 & g_3 \\ g_1 & -\Delta^2 + g_1^2 & g_1 g_2 & g_1 g_3 \\ g_2 & g_1 g_2 & -\Delta^2 + g_2^2 & g_2 g_3 \\ g_3 & g_3 g_1 & g_3 g_2 & -\Delta^2 + g_3^2 \end{pmatrix}$$

Eikonal Equation

-Waves go by $k_i dx^i = -d\Phi$, the eikonal (phase); flat space $k_i = (k_0 = \omega/c, -\mathbf{k})$, frequency and wavevector, $k_i k^i = (\omega/c)^2 - \mathbf{k}^2 = 0$; straight line, $\Phi = -\omega t + \mathbf{kr}$; geometric optics

-Since $k_i k^i = g^{ij} k_i k_j = 0$ we get the eikonal equation in curved space

$$(\partial\Phi/\partial t + c\mathbf{g}\partial\Phi/\partial\mathbf{r})^2 - c^2(1 + h + g^2)(\partial\Phi/\partial\mathbf{r})^2 = 0$$

-Solve it!

-neglect g^2

-first term does not depend on the time t (the second doesnt!)

$$\partial\Phi/c\partial t + \mathbf{g}\partial\Phi/\partial\mathbf{r} = -\omega_0/c$$

where ω_0 the frequency in the flat space; in addition

$$(\partial\Phi/\partial\mathbf{r})^2 = k^2 = \frac{1}{1+h} \cdot (\omega_0/c)^2 = \frac{1}{1+h} \cdot k_0^2$$

-It follows

$$\partial\Phi/c\partial t = -\omega_0/c - \mathbf{g}\mathbf{k}_0$$

-What we measure? We measure the local, proper-time frequency

$$\omega/c = -\partial\Phi/c\partial\tau = -\frac{1}{\sqrt{1+h}}\cdot\partial\Phi/c\partial t = \frac{1}{\sqrt{1+h}}\cdot\omega_0/c+\mathbf{g}\mathbf{k}_0$$

-Therefore a shift in frequency

$$\Delta\omega/\omega_0 = -h/2 + c\mathbf{g}\mathbf{k}_0/\omega_0$$

First term - the **red shift**; second term - **Doppler effect** (long)

-Time-dependent part of the eikonal: $\Phi_t(t) = -\omega_0 t + \mathbf{k_0 R}(t)$: a translation, as expected

-The path? $(\partial\Phi/\partial\mathbf{r})^2 = (1-h)k_0^2$

-Write it in spherical coordinates; separate variables by $\Phi = \Phi_r(r) + M\varphi$, M a constant; $\partial\Phi/\partial M = const$ gives the equation of the trajectory (M is a generalized coordinate, its momentum is constant)

The **deflection angle** (distance M/k_0)

$$\Delta\varphi = -(k_0^2/2)\int_\infty^r dr \cdot \frac{h \cdot M/r^2}{(k_0^2 - M^2/r^2)^{3/2}}$$

(4 times smaller than in grav field of a point mass; our metric is not that metric!)

PART IV : QUANTIZATION

BASIC CHANGES

Quantization

What are we doing?

Nothing Good (though NEW), even WORSE than before

Because it is bad to solve for a non-inertial motion; we just solve for an inertial frame and do the translation (for instance, the Ham-Jac eq is solved with h for Mercury's perihelia precession; then apply the translation, etc)

This is perfectly true for classical motion with trajectory

Things Change Fundamentally for the Quantal Motion

Quantization

$S = -i\hbar \ln \psi$; $E \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar\partial/\partial \mathbf{r}$

No trajectory, wavefunction ψ

No determined physical quantities (E, $\mathbf{p}$) (operators)

Means and deviations: statistical meaning

$|\psi|^2$density of probability (conservation)

Apply this procedure to the Ham-Jac eq $E^2 - c^2p^2 = m^2c^2$

Get the **Klein-Gordon equation**

$$\partial^2\psi/\partial t^2 - c^2\partial^2\psi/\partial\mathbf{r}^2 + (m^2c^4/\hbar^2)\psi = 0$$

Troubles: the conserved quantity is $\psi^*(\partial\psi/\partial t) - (\partial\psi^*/\partial t)\psi$, both positive and negative (due to negative energies $E = -\sqrt{p^2c^2 + m^2c^4}$)- nonsense

Dirac: $i\hbar\partial\psi/\partial t = (\alpha c\mathbf{p} + \beta mc^2)\psi$, matrices α and β; get a probability, but ψ is a spinor; so, the question remains for the Klein-Gordon eq

Approximate Klein-Gordon equation

Apply the quantization to the Ham-Jac eq in curved space

$$(E - c\mathbf{g}\mathbf{p})^2 - c^2(1+h)(\mathbf{p}^2 + m^2c^2) = 0$$

(neglecting g^2)

Troubles: $1+h$ does not commute with $\mathbf{p}^2 + m^2c^2$, ambiguities

We may transfer it to the *lhs* as $1/(1+h)$, and neglect the gh-commutator

Get then an approx Klein-Gordon eq in curved space

$$(i\hbar\partial/\partial t - c\mathbf{gp})^2\psi - c^2(1+h)(p^2+m^2c^2)\psi = 0$$

Still troubles, since we do not know where to put $1+h$ with respect to $p^2+m^2c^2$

However, in the non-relativistic limit this ambiguity does not matter, and we get the Schrodinger equation (recall $h = 2\varphi/mc^2$)

$$i\hbar\partial\psi/\partial t = H\psi = (mc^2 + p^2/2m + \varphi)\psi + c\mathbf{gp}\psi$$

The Fundamental Fact

The eigenstates are no more conserved due to the non-uniform translation

We get quantal transition

An observer in a non-uniform translation sees quantal transitions

Esssential thing: do not conserve the momentum; the presence of the external potential φ(*i.e.* h) is essential

The approximate Klein-Gordon eq can be solved by pert theory

$$(i\hbar\partial/\partial t - c\mathbf{gp})^2\psi - c^2(1+h)(p^2+m^2c^2)\psi = 0$$

Define $H^2 = c^2(1+h)(p^2+m^2c^2)$, solve in the first order, get $E^2 = (1+\bar{h})(p^2c^2+m^2c^4)$, the wavefunctions $\varphi(\mathbf{p})$-plane waves plus a weak admixture of plane waves (due to h); then we have $(i\hbar\partial/\partial t - c\mathbf{gp})\psi = E\psi$

Get the transition amplitude

$$-(i/\hbar)\int dt \cdot e^{-i[E(p)-E(p')]t/\hbar} c\mathbf{gp}_{\mathbf{p}'\mathbf{p}}$$

Conclusion: we do have quantal transitions!

Restricting to the first-order of the perturbation theory we get also a Dirac equation

$$(i\hbar\partial/\partial t - c\mathbf{g}\mathbf{p})\psi = (\alpha c\mathbf{p} + \beta mc^2)\psi$$

with leads to the same conclusion

A "profound" argument

Let our eq be

$$(\partial/\partial t + c\mathbf{g}\partial/\partial\mathbf{r})^2\psi - c^2(1+h)[\partial^2\psi/\partial\mathbf{r}^2 - (m^2c^2/\hbar^2)\psi] = 0$$

like above

Fourier transform; a homogeneous matricial equations in labels $(\omega, \mathbf{k})$; solve it by zeroing the determinant; get the eigenvalues; they are labelled by points $(\omega, \mathbf{k})$ conveniently ordered; consequently, the eigenvalues are useless, they do not provide an algebraic relationship between ω and $\mathbf{k}$

That means that for an ω we have many $\mathbf{k}$ and for a $\mathbf{k}$ we have many ω

That means that the plane waves scatter both in ω and in $\mathbf{k}$

That means that the quantization with plane waves is the only way to understand such solutions of the 2nd order diff eqs, and more, we have for them a statistical meaning; this is **The Quantal Fields Theory**!

PART V : FIELDS; HOW THEY ARE and What THEY DO in a CURVED SPACE

Fields

Cannot forget that the above Klein-Gordon or Dirac equations in curved space are only approximate

Way out: The Fields!

Real Scalar Field (general note: covariant derivative)

$$S = \int dx^0 d\mathbf{r}\sqrt{-g} \cdot \left[(\partial_i\psi)(\partial^i\psi) + (m^2c^2/\hbar^2)\psi^2\right]$$

Eqs of motion

$$(i\hbar\partial/\partial t - c\mathbf{g}\mathbf{p})^2\psi - c^2(1+h)(p^2+m^2c^2)\psi + (i\hbar c^2/2)(\partial h/\partial\mathbf{r})\mathbf{p}\psi = 0$$

This is the real Klein-Gordon equation in curved space

Note the additional interacting term $(\partial h/\partial\mathbf{r})\mathbf{p}$

Supports a similar treatment with the perturbation theory; same conclusion: quantal transitions

(Note: compare it with the KG eq in an electromagnetic field

$$(i\hbar\partial/\partial t - e\varphi)^2\psi - c^2[(i\hbar\partial/\partial \mathbf{r} + e\mathbf{A}/c)^2 + m^2c^2] = 0$$

Quite different! (Gauge fields!))

The Hamiltonian of the Real Scalar Field

Quantization by $\Pi = \partial L/\partial(\partial\psi/\partial t)$, Hamiltonian by $\Pi(\partial\psi/\partial t) - L$, the Lagrangian in $S = \int dt \cdot L$

$$H = H_0 + H_{1h} + H_{1g}$$

$$H_0 = \int d\mathbf{r} \cdot \left[c^2\Pi^2/4 + (\partial\psi/\partial\mathbf{r})^2 + (m^2c^2/\hbar^2)\psi^2\right] =$$

$$= \sum_{\mathbf{p}} (\varepsilon/2)(a_{\mathbf{p}} a_{\mathbf{p}}^{+} + a_{\mathbf{p}}^{+} a_{\mathbf{p}})$$

$$H_{1h} = \int d\mathbf{r} \cdot (\sqrt{1+h} - 1)\left[c^2\Pi^2/4 + (\partial\psi/\partial\mathbf{r})^2 + (m^2c^2/\hbar^2)\psi^2\right]$$

$$H_{1g} = -(c/2) \int d\mathbf{r} \cdot [\Pi(\mathbf{g}\partial\psi/\partial\mathbf{r}) + (\mathbf{g}\partial\psi/\partial\mathbf{r})\Pi] =$$

$$= -(c/2) \sum_{\mathbf{p}} (\mathbf{g}\mathbf{p})(a_{\mathbf{p}} a_{\mathbf{p}}^{+} + a_{\mathbf{p}}^{+} a_{\mathbf{p}})$$

Systematic perturbation theory; scattering in the hg-order ($\varepsilon = \sqrt{p^2c^2 + m^2c^4}$)

Electromagnetic Field. Photons

$$S = -(1/16\pi c)\int dx^0 d\mathbf{r}\cdot\sqrt{-g}F_{ij}F^{ij}$$

$F_{ij} = \partial_i A_j - \partial_j A_i$; $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ (free Maxwell eqs)

Interacting Maxwell eqs $\partial_j(\sqrt{-g}F^{ij}) = 0$

$$div[(\mathbf{E} + g \times \mathbf{B})/\Delta] = div\mathbf{D} = 0$$

$$\frac{\partial}{c\partial t}[(\mathbf{E} + \mathbf{g} \times \mathbf{B})/\Delta] = curl[\Delta\mathbf{B} + \mathbf{g} \times \mathbf{E}/\Delta]$$

Perturbation theory; scattering, both in $\mathbf{k}$ and ω

The Hamiltonian of the Photons

$$S = (1/8\pi)\int dt d\mathbf{r}\cdot\Delta(\mathbf{D}^2 - \mathbf{B}^2) =$$

$$= (1/8\pi)\int dt d\mathbf{r}\cdot(1/\Delta)(\mathbf{E}^2 + 2\mathbf{E}(\mathbf{g}\times\mathbf{B}) - \Delta^2\mathbf{B}^2)$$

$$H = H_0 + H_{1h} + H_{1g}$$

$$H_0 = \int d\mathbf{r}\cdot(c^2\Pi^2/4 + B^2) = \sum_{\alpha\mathbf{p}}(\varepsilon/2)(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}})$$

$$H_{1h} = \int d\mathbf{r}\cdot(\sqrt{1+h} - 1)(c^2\Pi^2/4 + \mathbf{B}^2)$$

$$H_{1g} = -\sum_{\alpha\mathbf{p}} (\mathbf{gp}/2)(a^{+}_{\alpha\mathbf{p}} a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}} a^{+}_{\alpha\mathbf{p}})$$

Systematic theory of perturbations ($\varepsilon = cp = c\hbar k$)

Photons are scatterred in frequency, as a consequence of a non-uniform translation, when in an external field (like a static grav field)

Other Fields. Quantum Gravity

Similar for other fields (spin-1/2 Dirac field) (technically more cumbersome; vierbeins)

Gravitons; quantized (with troubles); moving in a curved space $S = \int dx^0 d\mathbf{r} \cdot \sqrt{-g}R$; $g = g_0 + \delta g$, background and gravitons; scattering of gravitons, *i.e.* of the space-time, on space-time, *i.e.* on matter or on the non-inertial motion

PART VI : OTHER non-INERTIAL MOTIONS and MISCELLANEA; CONCLUSIONS

Rotations

$$d\mathbf{r}' = d\mathbf{r} + (\Omega \times \mathbf{r})\, dt$$

$$d\mathbf{v} = d\mathbf{v} + \left(\dot{\Omega} \times \mathbf{r}\right) dt + 2\,(\Omega \times \mathbf{v})\, dt + [\Omega \times (\Omega \times \mathbf{r})]\, dt$$

Non-uniform rotation, Coriolis, centrifugal

$$H = mv^2/2 - m(\Omega \times \mathbf{r})^2/2 + \varphi = p^2/2m - \Omega(\mathbf{r} \times \mathbf{p}) + \varphi =$$

$$= p^2/2m - \Omega\mathbf{L} + \varphi$$

No Coriolis, no centrifugal; just $\mathbf{L}$ we may neglect Ω^2

The above coordinate transformation gives the metric

$$g_{ij} = \begin{pmatrix} 1+h & g_1 & g_2 & g_3 \\ g_1 & -1 & 0 & 0 \\ g_2 & 0 & -1 & 0 \\ g_3 & 0 & 0 & -1 \end{pmatrix}$$

with

$$\mathbf{g} = -\Omega \times \mathbf{r}/c$$

as before

Two distinctions: $g(t, \mathbf{r})$

$\Omega r/c \ll 1$

Coupling through the angular momentum $\mathbf{L}$

Conclusions

Non-inertial motion (for instance of the observer) produces quantal transitions in the presence of an external field

The coupling is through momentum $\mathbf{p}$ for translations or through the angular momentum $\mathbf{L}$ for rotations; so, the external field must not conserve these quantities

For instance, photons in a static gravitational field are scattered toward the blue (the blue shift) while seen from a non-uniform translation (or rotation)

Relation to the Unruh effect - quite distinct (the observer in the U effect sees its own motion as a bath of photons)

Another more practical **Conclusion:**

The quantization in curved spaces has no meaning or it has the meaning given here