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Lectures on Morse Theory

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ABSTRACT. The main elements of Morse theory are presented in a selfconsistent form. The concepts of differentiable triads, critical points of the differentiable functions, Morse functions are presented. The types of homotopy of the differentiable manifolds by means of the critical values, as well as the Morse inequality are described. Finally, perfect Morse functions on complex Grassman manifolds are studied. As an application to quantum mechanics, a perfect Morse function is constructed on the Slater determinant manifold.

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LIST OF NOTATIONS

Z (Z_+, R, C respectively) denotes the set of integers (positive integers, real and complex numbers respectively). Z_n denotes the factor ring Z/nZ.

$R^n = \{(x^1,\dots,x^n) \mid x^i \in R,\ i = 1, \dots, n\}$,

$R^n_+ = \{(x^1,\dots,x^n) \mid x^n \geq 0,\ x^i \in R,\ i = 1,\dots,n\}$,

$C^n = \{(z^1,\dots,z^n) \mid z^i \in C,\ i = 1,\dots,n\}$.

The scalar product of two vectors $x = (x^1,\dots,x^n)$, $y=(y^1,\dots,y^n)$ is denoted by $\langle x, y\rangle$ and has the value

$\langle x, y\rangle = \sum_{i=1}^{n} \bar{x}^i y^i$, if the vectors $x, y \in C^n$, where $\bar{z} = a - ib$ if $z = a+ib$, $a, b \in R$, $z \in C$, or

$\langle x, y\rangle = \sum_{i=1}^{n} x^i y^i$, if the vectors $x, y \in R^n$.

The norm of the vector x in the Euclidean space R^n or C^n is denoted by $\|x\| = \langle x, x\rangle^{1/2}$.

If A, B are subsets of a topological space, we shall denote by

$\mathring{A} = \operatorname{int} A$, the interior of A, $\bar{A}$ the closure of A, CA the complement of A, and $A \setminus B = A \cap CB$ the difference of the sets A, B.

If X is a topological space, E a vector space and $f: X \to E$ a function, the support of the function f is denoted by

$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}$.

If $u : M \to N$ is a morphism of P-modules, we denote by

$\operatorname{Ker} u = \{x \in M \mid u(x) = 0\}$,

$\operatorname{Im} u = \{y \in N \mid$ there exists $x \in M$, such that $u(x) = y\}$.

$\operatorname{Hom}_P(M,N)$ the set of all morphisms from M to N.

Matrices over the field C will be considered. We will denote by

$\mathcal{M}(p,q)$ the set of matrices with p rows and q columns,

$\mathcal{M}(p)$ the subset of the matrices in $\mathcal{M}(p,p)$ where the matrix determinant is nonzero and the set of matrices $\mathcal{M}(p)$ is identified over C with the general linear group $GL(p, C)$;

$\mathcal{T}(p)$ the subset of the upper triangular matrices in $\mathcal{M}(p)$ over the field C with positive diagonal elements:

$\mathcal{T}(p) = \{T \in \mathcal{M}(p) \mid t_{\alpha\alpha} > 0,\ \alpha = 1,\dots,p;\ t_{\alpha\beta} = 0,\ 1 \leq \beta < \alpha \leq p\}$.

I_n denotes the unit matrix of the group $GL(n, C)$.

If $A = (a_{ij})_{1\leq i,j\leq n}$, the trace of matrix A is denoted by

$\operatorname{Tr} A = \sum_{i=1}^{n} a_{ii}$.

A^+ stands for the hermitian conjugate matrix of matrix A,

$A^+ = (a^+_{ij}) = (\bar{a}_{ji})$, $1 \leq i,\ j \leq n$.

If A is a linear continuous operator that acts on then A^+ denotes the adjoint operator.

I. INTRODUCTION

In the last few years, a particular attention was paid to the application of geometrical ideas and methods in quantum physics. Specifically, the right application of the variational principles is conditioned on establishing some global topological and geometrical properties of certain quantum states manifolds determined by the physical nature of the problems under consideration. Global variational methods are studied by the Morse theory.

Morse theory studies the relationship between the critical points of real valued functions on a differentiable manifold and the topology of the manifold they are defined on. Morse himself regarded his papers as part of an effort for developing analysis in the large, a field which he considered had been founded by H.Poincaré. In the foreward to his book /17/, printed in 1934, Morse emphasized that: " Any problem which is non-linear in character , which involves more than one coordinate system or more than one variable, or whose structure is initially defined in the large, is likely to require considerations of topology and group theory in order to arrive at its meaning and its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the group theory or topology ".

The current relavance of these affirmations is illustrated by the dynamical systems theory (particularly systems with symmetry and Morse-Smale systems) /1/, Hamiltonian analysis (e.g., periodicity problems) /1/, /19/, Lagrangian analysis (recent formalization based on the Morse index and Maslov index of the semiclas-

sical approximation) /14/, the connection between the geometric quantization method and the functional integral /10/, the Dirac variational principle /13/ on coherent states manifolds (generalization of the time-dependent Hartree-Fock, variational principle, only applied so far in the small), the gauge field theory (e.g., the topological and geometrical description of extreme manifolds for the Yang-Mills functional) /2/, /6/, the non-linear field theory (e.g., quantification for Gross-Neveu models based on symplectic manifolds) /5/, supersymmetry theories (parricularly the spontaneous supersymmetry breaking in Morse theory) /27/.

This paper covers the main elements of Morse theory which is based upon two theorems. Theorem A essentially asserts that if a real-valued differentiable function on the differentiable manifold M, $f : M \to R$, has no critical points in the submanifold $f^{-1}[a,b]$, $a<b$, $a,b \in R$, then submanifolds $M^a=f^{-1}(-\infty,a]$ and $M^b = f^{-1}(-\infty,b]$ are diffeomorphic (moreover, M^a is the deformation retract of M^b). We owe this result to Morse (see /17/). Theorem B of Morse theory specifies what happens when between a and b, $a, b \in R$, $a<b$, there is only one critical value of the function f which corresponds to only one nondegenerate critical point. In this case, M^a has the same homotopy type as M^b to which a cell is attached whose dimension is equal to the function index in the critical point. The way to prove theorem B presented here was found by S.Smale (the proof was published after 1950; the other proof of the theorem, earlier devised by Bott, Thom and Pitcher is dealt with at length in /16/ and /21/). We shall further note that, as far as the above outlined theorems A and B of Morse theory are concerned, it will be enough if proper functions f and compact manifolds M are considered.

To give a unitary exposition of theorems A and B, we have closely followed Milnor's presentation in his book /16/ printed in 1965, Sections 1-3, wherein cobordism theory elements have been used.

The basic elements of this approach are summarized in Chapter 1 including definitions of such concepts as the differentiable manifold with and without boundary, the differentiable triads and some cobordism theory elements.

Chapter 2 defines other notions such as the critical point of a function defined on differentiable manifolds, the nondegenerate critical point, the Hessian index. The main result in this chapter is to be found in Proposition 2.4 which is owed to Morse (Morse's lemma, 1925).

Chapter 3 covers at length some properties of the Morse functions defined on manifolds with and without boundaries. The existence of the Morse functions defined on differentiable triads is proved and the topological space of the Morse functions is studied (Theorems 3.7 and 3.8). Proposition 3.12 provides a significant example of differentiable triads, while Proposition 3.13 shows that any cobordism may be expressed as a product of cobordisms of Morse numbers 1. Such cobordisms, known as elementary cobordisms, are studied in Chapter 4.

The notion of gradient-like vector field is introduced in Chapter 4 (Definition 4.1) and the existence of such fields is then proved (4.2). Theorem 4.4 represents Morse's theorem A in terms of cobordims. Corollaries 4.5 and 4.7 establish the connection with Morse's initial study (see /17/) (and the form of theorem A in Milnor's processing in his 1963 book /15/).

Definitions 4.13 and 4.14 introduce the Morse surgery operation. Examples have been taken from /26/. Theorems 4.15 and

4.16 show that passing from a noncritical level line to another noncritical level line over a nondegenerate critical point is equivalent to Morse surgery. Theorem 4.17 is Morse's theorem B and it lays the ground for a large number of applications (see /15/), one of which is described in the following chapter.

Chapter 5 is devoted to Morse inequalities (Morse , 1925) (specific cases of these inequalities, namely the last relation (5.23), have been obtained in Poincaré (1885) for bidimensional manifolds, and by Lefschetz (1925) and Hopf (1926) for n-manifolds).

Morse inequalities impose restrictions to the number of critical points of index λ as a function of the Betti numbers R_λ that contains the topological information on the manifolds (Theorem 5.8). For the sake of selfconsistency, the chapter starts with some notions from the theory of homology, mainly selected according to /25/. Several results that are required to prove Morse inequalities are included in lemmas 5.5-5.7. The chapter ends on corollary 5.9 establishing that if $C_{\lambda+1}=C_{\lambda-1}=0$, then $R_\lambda=C_\lambda$. This corrolary can be employed in particular cases to find out the Betti numbers of some differentiable manifolds on which Morse functions of such properties as in the corollary can be effectively constructed.

A perfect Morse function is constructed in Chapter 6 for the complex Grassman manifold G_p (C^n) of the complex p-dimensional vector subspaces of space C^n (Theorem 6.4), i.e. , a Morse function for which Morse inequalities (5.19) are satisfied as equalities. The Morse function is chosen in the conditions of Corollary 5.9, so that the number of critical points of odd index is zero. This construction makes it easy to determine the homology groups of the complex Grassman manifolds (Corollary 6.5). The manifold

$G_p(C^n)$ has been endowed with a differentiable structure induced by the structure of the homogeneous space $U(n)/U(n-p) \times U(p)$ with which $G_p(C^n)$ is diffeomorphic (6.2). At the end of the chapter, a direct application of Morse theory to quantum mechanics is illustrated. The first specifications concern the structure of the complex Grassman manifold $G(C^n)$ of the Fock space made up of A fermionic particles within a space of n fermionic states in the maps obtained while proving Proposition 6.2. Morse theory imposes restrictions to the minimum number of Hartree-Fock states for self-adjoint polynomial Hamiltonians (according to /22/). The paper is closed by Proposition 6.6 establishing the physical significance of such quantum Hamiltonians for which the associated energy function is the perfect Morse function on the complex Grassman manifold constructed in Theorem 6.4.

Morse inequalities have been applied to classical mechanics /19/, the Hartree-Fock problem /12/, /14/ , supersymmetry theories /27/. As an important result, it is worth mentioning the one recently obtained by Atiyah and Bott /2/ who applied Morse inequalities to determine the manifold of Yang-Mills functional minima using the topological characteristics of the unstable critical manifolds. Perfect Morse functions are also useful in studying the geometry of coherent states manifolds /3/.

We would further note that the "theorem A" and "theorem B" of Morse theory as well as the "perfect Morse function" have been called that according to Bott /7/.

To increase the fluency of our presentation, we have confined ourselves to a minimum number of eferences, while trying to preserve selfconsistency.

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1. DIFFERENTIABLE TRIADS

Definition 1.1. An n-dimensional differentiable (smooth) manifold W is a separate topological space of countable base, locally homeomorphic with the space R^n or R^n_+, endowed with a differentiable structure $\mathcal{S}$. $\mathcal{S}$ is a family of pairs $\{U_\alpha, h_\alpha\}_{\alpha\in I}$ which satisfy the following four conditions :

1. For each $\alpha \in I$, $U_\alpha \subset W$, U_α is an open set and h_α maps U_α homeomorphically onto its image $h_\alpha(U_\alpha)$ which is open in R^n or R^n_+.

2. $W = \bigcup_{\alpha\in I} U_\alpha$.

3. If (U_α, h_α), (U_β, h_β) belong to the family $\mathcal{S}$ then the mapping $h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.

4. The $\mathcal{S}$ family is maximal with respect to property (3), that is for any pair $(U, h) \notin \mathcal{S}$, property (3) does not hold any longer for $\mathcal{S} \cup \{(U,h)\}$.

The pairs (U_α, h_α) are called local charts or local coordinate systems, the U_α sets are called coordinate neighborhoods, and the h_α mappings, coordinate mappings. The components of an h_α mapping : $U_\alpha \to R^n$ are called coordinate functions.

An example of differentiable manifold is the Grassman manifold which will be described at length in Chapter 6 (see 6.2).

Remark. As a topological space, being a locally compact Hausdorff space of countable base, the differentiable manifold is a paracompact space (/23/, p.64, /25/, p.85), after Bourbaki). A paracompact space is normal (/25/, p.92, after Dieudonne). A normal space of countable base is metrizable. (/25/, p.99,after Urfson).

Definition 1.2. *The boundary* of a differentiable manifold W is the set $\partial W = \{p\in W |$ there exists a chart (h, U) with the property that $p \in U$ and $h(U) \in R^n_+$ and $h(p) \in \partial R^n_+ = R^{n-1} \times 0\}$.

It is proved that ∂W has an n-1 dimensional differentiable manifold structure. The differentiable structure of the ∂W boundary consists of the pairs $(U_\alpha \cap \partial W, h_\alpha \mid U_\alpha \cap \partial W)$ with $(U_\alpha, h_\alpha) \in \mathcal{S}$ /18/.

Definition 1.3. The triplet $(W; V_0, V_1)$ is called an n-dimensional *differentiable triad*, if W is a compact and smooth n-manifold and $\partial W = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$, where V_0, V_1, are n-1 dimensional differentiable manifolds.

The Proposition 3.12 will provide an important class of triads.

Let $(W; V_0, V_1)$ and $(W'; V_1', V_2')$ be two triads and $h : V_1 \to V_1'$ a diffeomorphism. Then a third triad $(W \cup_h W' ; V_0, V_2')$ can be formed,where $W \cup_h W'$ is the space obtained from the disjoint union of the manifolds W and W' identifying V_1 and V_1' under h. The manifold $W \cup_h W'$ will be endowed with a differentiable structure (see Theorem 4.10).

Definition 1.4. Given two closed differentiable n-manifolds M_0 and M_1 (which hence M_0, M_1 compact and $\partial M_0 = \partial M_1 = \emptyset$),a cobordism from M_0 to M_1 is a system $(W; V_0, V_1; h_0, h_1)$ where $(W; V_0, V_1)$ is a triad and $h_i : V_i \to M_i$, $i = 0, 1$, are diffeomorphisms.

A straightforward example of triad is provided by the triplet $(V \times I; V \times 0, V \times 1)$, where V is a closed differentiable manifold (see Definition 4.3.). For the triad $(W; V_0, V_1)$, the system $(W; V_0, V_1; id_{V_0}, id_{V_1})$ is a cobordism from V_0 to V_1. Further examples of cobordisms will be given in Chap. 4 (Fig. 5).

2. CRITICAL POINTS OF THE DIFFERENTIABLE FUNCTIONS

Let M and N be m-respectively n-dimensional differentiable manifolds and $\varphi : M \to N$ a differentiable mapping. In charts terms, let the local charts (U, h) and (V, g) be so that $h : U \to h(U) \subset R^m$, $g : V \to g(V) \subset R^n$ and the coordinate functions $\{x^i\}, i = 1, \dots m$ and $\{y^j\}$, $j = 1, \dots, n$ for the coordinate mappings h and g respectively. The differentiability of the mapping φ means that the mapping $g\circ\varphi\circ h^{-1} : R^m \to R^n$, given as $y^i = \varphi_i(x^1, \dots x^m)$, $i = 1, \dots n$ is also differentiable.

Let TM_p (TN_q), where $q = \varphi(p)$, $p \in M$, $q \in N$, denote the tangent space to the manifold M (respectively N) at point p (respectively q). Let φ_* be the (tangent) mapping induced by the mapping φ between the tangent spaces

$$\varphi_* : TM_p \to TN_q.$$

According to the notations above, in the chart (U,h), let $X = (X^1, \dots, X^m)$ be the vector field given by the whole of the differentiable functions $X^i : U \to R$. The vector field X has the form $X = \sum_{i=1}^{n} X^i(\partial/\partial x^i)$ and at the point $p, \partial/\partial x^i = (\partial/\partial x^i)_p$ and $X^i = X^i(x(p))$.

For the function $f = M \to R$, let $y = f\circ h^{-1}$ or otherwise, symbolically, in local coordinates, $y = f(x^1, \dots, x^n)$. The action of f_* on the field X is

$$f_*(X) = \left(\sum_{i=1}^{n} X^i(p) \frac{\partial y}{\partial x^i}\right)\left(\frac{\partial}{\partial y}\right)$$

where $\frac{\partial y}{\partial x^i}$ represents $\frac{\partial(f\circ h^{-1})}{\partial x^i}\Big|_{h(p)}$.

Definition 2.1. Let $f : M \to R$ be a differentiable function. The point $p \in M$ is *critical* if the induced application $f_* : TM_p \to TR_{f(p)}$ is zero at that point. In view of the observations above, the necessary and sufficient condition that p be a critical point is

$$\frac{\partial(f\circ h^{-1})}{\partial x^i} = 0, \qquad i = 1, \dots, n,$$

or merely :

$$\frac{\partial f}{\partial x^1} = \dots = \frac{\partial f}{\partial x^n} = 0. \qquad (2.1)$$

Let now ξ, η be the vectors in the tangent space to the M manifold at p, and $\tilde{\xi}$, $\tilde{\eta}$ the vector fields with the property $\tilde{\xi}|p = \xi, \tilde{\eta}|_p = \eta$. Let us have in a local coordinate system

$$\xi = \sum_{i=1}^{n} \xi^i\left(\frac{\partial}{\partial x^i}\right)_p \quad , \qquad \eta = \sum_{i=1}^{n} \eta^i\left(\frac{\partial}{\partial x^i}\right)_p.$$

From here it is found :

$$\tilde{\xi}_p(\tilde{\eta}(f)) = \sum_{i=1}^{n} \xi^i\left(\frac{\partial}{\partial x^i}\right)_p \sum_{j=1}^{n} \eta^j\left(\frac{\partial f}{\partial x^j}\right) = \sum_{i,j=1}^{n} \xi^i \frac{\partial \eta^j}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)_p +$$

$$+ \sum_{i,j=1}^{n} \xi^i\eta^j\left(\frac{\partial^2 f}{\partial x^2 \partial x^j}\right)_p.$$

If p is a critical point, then the condition (2.1) leads to the relation

$$\tilde{\xi}_p(\tilde{\eta}(f)) = \sum_{i,j=1}^{n} \xi^i\eta^j\left(\frac{{}^2 f}{\partial x^i \partial x^j}\right)_p \qquad (2.2)$$

A mapping $f_{**} : TM_p \times TM_p \to R$ can thus be defined

$$f_{**}(\xi,\eta) = \tilde{\xi}_p(\tilde{\eta}(f)) = \tilde{\eta}_p(\tilde{\xi}(f)). \qquad (2.3)$$

In view of earlier observations (2.2) and (2.3), this mapping is a symmetric bilinear functional which does not depend on vector fields $\tilde{\xi}$ and $\tilde{\eta}$.

Definition 2.2. The functional f_{**}(2.3) is called the *Hessian* of the function f at point p.

The matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p$ is the matrix of the bilinear functional f_{**} with respect to the basis $(\frac{\partial}{\partial x^1})_p, \ldots, (\frac{\partial}{\partial x^n})_p$

Let λ_i, $i = 1, \ldots, n$, be the eigenvalues of the matrix $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right)_p$. Since the matrix is real and symmetric, the eigenvalues are also real (see /8/, p.258). The number of negative eigenvalues (considered each time with their multiplicity) is the *index of the hessian* f_{**}, and is usually called just the index of the function of (at the critical point p). It is equal to the maximal dimension of the linear subspace on which the functional f_{**} is negatively defined. If, by means of an orthogonal transformation, the matrix is reduced to diagonal form, then the index of the f function is equal to the number of negative coefficients in the quadratic form (/8/, p.157). The number of zero eigenvalues is called *degeneracy degree*. It is equal to the maximal dimension of the vector subspace $\xi \in TM_p$, for which $f_{**}(\xi, \eta) = 0$, for any $\eta \in TM_p$ (null space). Linear algebra properties assures that neither the index nor the degeneracy degree depend on the coordinate system (theorem of inertia of quadratic forms /8/, p.269). Particularly a non-degenerate matrix whose degeneracy degree is zero is non-degenerate in any system of coordinates.

Lemma 2.3. Let f be a function of class C^∞ in a convex neighbourhood V of the point $O \in R^n$ and $f(O) = 0$. Then the functions $g_i \in C^\infty$, $i = 1, \ldots, n$, exist in V such that

$$f(x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i g_i(x^1, \ldots, x^n), \qquad (2.4)$$

and moreover

$$g_i(O) = \frac{\partial f}{\partial x^i}(O) \qquad (2.5)$$

Proof. Since V is conex, if the points O and $\vec{x} \in V$, then $t\vec{x} \in V$, $t \in [0,1]$. As $f(O) = 0$, it follows that :

$$f(\vec{x}) = \int_0^1 \frac{df(t\vec{x})}{dt}\, dt = \int_0^1 \sum_{i=1}^{n} x^i \frac{\partial f(t\vec{x})}{\partial x^i} dt = \sum_{i=1}^{n} x^i g_i(\vec{x}),$$

where $g_i(\vec{x}) = \int_0^1 \partial f(t\vec{x}) / \partial x^i dt$. Note that $f \in C^\infty$ implies $g_i \in C^\infty$, $i = 1, \ldots, n$. Additionally,

$$g_i(O) = \int_0^1 \frac{\partial f}{\partial x^i}(O)\, dt = \frac{\partial f}{\partial x^i}(O).$$

Proposition 2.4. (Morse's lemma/17/, p.172). Let p be a non-degenerate critical point of the function f of index λ. Then in a neighbourhood U of the point p there exists a local coordinate system $(y^1, \ldots y^n)$ such that $y^i(p) = 0$, $i = 1, \ldots, n$, and for any $x \in U$:

$$f(x) = f(p) - (y^1)^2 - \ldots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \ldots + (y^n)^2. \qquad (2.6)$$

Proof. One may consider the situation in which the function f and the coordinate mappings are selected so that $f(\varphi^{-1}(O)) = 0$, that is in brief $f(O) = 0$ and O is the critical point. From the preceding lemma, eq.(2.4) , it follows that

$$f(\vec{x}) = \sum_{i=1}^{n} x^j g_j(\vec{x})$$

and moreover from (2.5) it follows that $g_i(O) = \frac{\partial f}{\partial x^i}(O) = 0$. Lemma 2.3. is again applied for each g_j, $j = 1, \ldots, n$:

$$g_j(\vec{x}) = \sum_{i=1}^{n} x^i g_{ij}(\vec{x}),$$

where $g_{ij}(O) = \partial g_j(O)/\partial x^i = \partial^2 f(O)/\partial x^i \partial x^j$.

Hence

$$f(x^1, \ldots, x^n) = \sum_{i,j=1}^{n} x^i x^j g_{ij}(\vec{x}) \qquad (2.7)$$

Of course, one can chose a symmetric matrix for g_{ij}, or otherwise one passes to matrix $1/2\ (g_{ij} + g_{ji})$.

It will further be shown that there exists a non-degenerate change of coordinates that brings f into its form in the lemma. Let the notation $(u^1)^2=(x^1)^2|g_{11}|$ and $u^1=\text{sgn}\ g_{11}\ \sqrt{(x^1)^2|g_{11}|}$ and one make the change of variables that keeps the other coordinates unaltered. Here one assumes $g_{11}(0) \neq 0$. If this condition is not satisfied, a linear transformation can be carried out and the relation will be true in the new system. From the relation defining u^1, one can get x^1 as a function of u^1 and the other arguments that keep uncharged, as the matrix is non-degenerate in origin, hence by continuity, it is novanishing in a neighborhood of the latter. Let us now assume that by induction, in a neighborhood U_1 of the point O, a coordinate system $u^1, \ldots, u^n$ has been found in which the relation (2.7) takes the form:

$$f = \pm(u^1)^2 \pm \ldots \pm (u^{r-1})^2 + \sum_{i,j\geq r} u^i u^j H_{ij}(u), \qquad (2.8)$$

where H_{ij} is a symmetric matrix. By means of a linear transformation of coordinates, one can take $H_{rr}(0)\neq 0$. Let another coordinate change hold in a neighborhood $U_2 \subset U_1$ of zero in which $H_{rr}(u)\neq 0$:

$$\begin{cases} v^r(u^1,\ldots,u^n) = \sqrt{|H_{rr}|}\,(u^r + u^{r+1}\frac{H_{r+1,r}}{H_{rr}} + \ldots + u^n \frac{H_{nr}}{H_{rr}}), \\ v^i = u^i, \quad i \neq r. \end{cases} \qquad (2.9)$$

From here it follows that

$$u^r = \frac{v^r}{\sqrt{H_{rr}}} - u^{r+1}\frac{H_{r+1,r}}{H_{rr}} + \ldots ,$$

and hence $(u^r)^2 H_{rr} = (v^r)^2 + \ldots$.

But

$$\frac{\partial v^r}{\partial u^r} = \sqrt{|H_{rr}|}\ (1 + \ldots) + u^r\left(\frac{\partial\sqrt{|H_{rr}|}}{\partial u^r} + \ldots\right)$$

and in the neighborhood of zero, $\det\left|\frac{\partial v}{\partial u}\right| \neq 0$, as $H_{rr}(0) \neq 0$, hence the change of variables is possible in a neighborhood $U \subset U_2 \subset U_1$.

Introducing (2.9) in (2.8) it follows that

$$f = \sum_{i\leq r} \pm(v^i)^2 + \sum_{i,j>r} v^i v^j H'_{ij}(v) .$$

The possibility of diagonalizing the quadratic form in a neighborhood of the origin has thus been effectively proved by induction. Further the definition of the index is taken into account.

Corollary 2.5. Non-degenerate critical points are isolated.

Indeed, in a neighborhood of the non-degenerate critical point, f has the form (2.6) from which we infer $\partial f/\partial y^i = \pm 2y^i$, $i = 1, \ldots n$, and the unique solution in the given neighborhood of the non-degenerate critical point is $y = 0$.

Remark. On the basis of Morse's lemma it may be noted that for a critical point p with the property that $f(p) = c$, the set $f^{-1}(c) \cap h(U)$ has the equation

$$-(y^1)^2 - \ldots -(y^\lambda)^2 + (y^{\lambda+1})^2 + \ldots + (y^n)^2 = 0 \qquad (2.10)$$

where U is the neighborhood in the lemma.

In R^2 this equation defines two lines if $\lambda = 1$, and a point if $\lambda = 0$ or $\lambda = 2$.

In R^3, (2.10) represents a cone with its vertex at the origin if $\lambda = 1$ or $\lambda = 2$, and the origin if $\lambda = 0$ or $\lambda = 3$. If $\lambda = 0$ or $\lambda = n$, the point p is an isolated point of the set $f^{-1}(c)$. If

$\lambda=0$, p is a relative minimum point of the function f, whereas if $\lambda=n$, p is a relative maximum point. The equation (2.10) generally describes a cone of index λ with an $n-\lambda$ dimensional centre.

Remark. There exist unisolated degenerate critical points, e.g., the origin for $f(x) = e^{-1/x^2}\sin^2 1/x$.

Example (the height function on the torus).

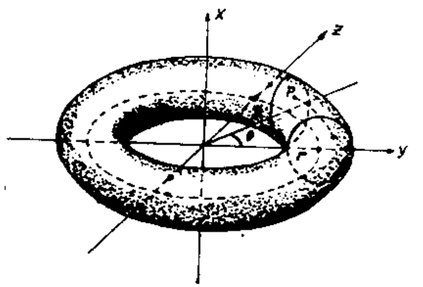


Fig. 1

The equation of a point on the torus surface can be found by intersecting the sphere with centre (x_o, y_o, z_o) in the plane (z,y), of radius r, with the plane whose normal is the perpendicular from the plane (z,y) to the direction determined by the origin and the point (x_o, y_o, z_o):

$$(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2 = R^2$$

$$-y\sin\theta + z\cos\theta = 0,$$

where

$$x_o = 0,$$

$$y_o = R\cos\theta,$$

$$z_o = R\sin\theta.$$

Solving the system with respect to $\sin\theta$, $\cos\theta$, it results

$$\sin\theta = \frac{z}{2R}\,\frac{x^2+y^2+z^2+R^2-r^2}{z^2+y^2},$$

$$\cos\theta = \frac{y}{2R}\,\frac{x^2+y^2+z^2+R^2-r^2}{z^2+y^2}.$$

If the parameter θ is eliminated and a change of variables is made so that to obtain z = 0 at the "lowest" point of the torus, it follows that :

$$z = R+r+\varepsilon_1((R+\varepsilon_2(r^2-x^2)^{1/2})^2 - y^2)^{1/2}$$

where $\varepsilon_i^2 = 1$, i = 1,2.

Expanding in a series in the neighborhood of the four critical points, it results

$$z = R(1+\varepsilon_1)+r(1+\varepsilon_1\varepsilon_2) - \frac{\varepsilon_1\varepsilon_2 x^2}{2r} - \frac{\varepsilon_1 y^2}{2(R+\varepsilon_2 r)} + \dots$$

The values of ε_i together with the value of z in the neighborhood of the critical points are shown in the adjoining figure :

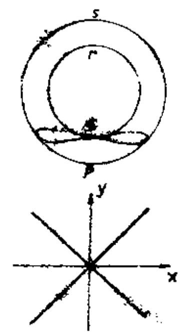


Fig.2

ε_1	ε_2	z
+	+	$2R+2r - \frac{x^2}{2r} - \frac{y^2}{2(R+r)}$
+	-	$2R + \frac{x^2}{2r} - \frac{y^2}{2(R-r)}$
-	-	$2r - \frac{x^2}{2r} + \frac{y^2}{2(R-r)}$
-	+	$\frac{x^2}{2r} + \frac{y^2}{2(R+r)}$

With the change of coordinates as above in the neighborhood of the critical point p, by a dilatation of the variables x, y, it is found that, in the plane xy, the set formed by the two bisectors corresponds to the image of the critical level of q (eq.(2.10)).

3. MORSE FUNCTIONS

Definition 3.1. Let $(W; V_o, V_1)$, be a differentiable triad. The differentiable function $f : W \to [a, b]$, $a,b \in R$, is called a

Morse function if it satisfies the following conditions:

1. $f^{-1}(a) = V_o$, $f^{-1}(b) = V_1$.

2. All critical points of the function f are interior points, i.e. they belong to $W \setminus \partial W$.

3. All critical points are nondegenerate.

Remark. It follows from corollary 2.5. that the nondegenerate critical points of a Morse function are isolated. Since W is compact, there exists a finite number of critical points.

Definition 3.2. The *Morse number* μ of the triad $(W; V_o, V_1)$ is defined as the minimum number of critical points of the function f, considered for all functions f of Morse type.

In order that the definition make sense, the existence of Morse functions will be proved. Effective examples of Morse functions will be provided. The set of Morse functions will be described . To prove the existence of Morse functions, several steps are needed. First, a function will be constructed that verifies conditions 1 and 2 in definition 3.1. The f function is then approximated by functions that, besides 1. and 2, satisfy in addition condition 3.

Lemma 3.3. There exists a differentiable function $f = W \to [0,1]$ with $f^{-1}(0) = V_o$, $f^{-1}(1) = V_1$ and f has no critical points in a neighborhood of the boundary of W.

Proof. Let the $W = \bigcup_{\alpha=1}^{k} U_\alpha$ be a covering of W with coordinate neighborhoods U_α . The covering is finite as W is assumed to be compact. It can be assumed that the covering is such that for every $\alpha = 1, \dots, k$ or $U_\alpha \cap (V_o \cup V_1) = \emptyset$, or, if $U_\alpha \cap \partial \neq \emptyset$, then the set $h_\alpha(U_\alpha)$ belongs to the intersection between the set R^n_+ and the unit open ball. For every set U_α, $\alpha = 1, \dots, k$, a map $f_\alpha : U_\alpha \to [0,1]$

is defined with the following properties:

$$f_\alpha |_{U_\alpha} = x^n, \quad (1-x^n) \quad \text{if} \quad U_\alpha \cap V_o \neq \emptyset \, (U_\alpha \cap V_1 \neq \emptyset) \; ;$$

$$f_\alpha |_{U_\alpha} = 1/2, \qquad U_\alpha \cap \partial W \neq \emptyset;$$

$$f_\alpha |_{U_\beta} = 0 \;, \qquad \beta \neq \alpha$$

If an open covering of W, $\{U_\alpha\}_{\alpha = 1,\dots,k}$ is considered, the theorem of partition of unity ensures the existence of functions $\varphi_\alpha : W \to R$, $\alpha = 1, \dots, k$, with the property that the family $\{\text{supp}\, \varphi_\alpha\}$ is locally finite and inscribed in the family $\{U_\alpha\}_{\alpha=1,\dots,k}$. In addition

$$\sum_{i=1}^{k} \varphi_i(p) = 1$$

for any $p \in W$ (see /23/, p.64). Then the differentiable function

$$f(p) = \sum_{i=1}^{k} \varphi_i(p)\, f_i(p)$$

has the properties $f|V_o = 0$, $f|V_1 = 1$, $f(p) \in [0,1]$, $p \in W$. It remains to be verified that on the boundary ∂W, the differential $df \neq 0$. Using continuity, it will result that there exists a neighborhood of the boundary on which f has no critical points.

Let the point $q \in V_o (q \in V_1)$. According to the theorem of partition of unity, when $q \in U_i$, there exists at least one i such that $\varphi_i(q) > 0$. Choosing the coordinate mappings $h_i(p) = (x^1(p), \dots, x^n(p))$ we obtain:

$$\frac{\partial f}{\partial x^n} = \sum_{j=1}^{k} f_j \frac{\partial \varphi_j}{\partial x^n} + \varphi_i \frac{\partial f_i}{\partial x^n} + \sum_{\ell \neq i} \varphi_\ell \frac{\partial f_\ell}{\partial x^n}$$

In the first term of the sum, $f_j = 0$ for all $j=1,\dots,k$ at point $q \in U_i$ and

$$\sum_{j=1}^{k} \partial \varphi_j / \partial x^n = \partial/\partial x^n \Big(\sum_{j=1}^{k} \varphi_j \Big) = \partial/\partial x^n \, 1 = 0$$

hence only the term $\partial f_\ell / \partial x^n$ contributes to the sum. But $\partial f_i/\partial x^n = 1(-1)$ and besides, in the sum $\sum_{\ell \neq i} \varphi_\ell \partial f_\ell/\partial x^n$ all terms that contribute are of the same sign as $\partial f_i/\partial x^n$, because only those ℓ indices contribute to the sum for which $p \in U_\ell$, so $f_\ell \sim x^n (f_\ell \sim 1 - x^n)$ and it is obtained that $\partial f_\ell/\partial x^n \sim 1\ (-1)$.

Three lemmas concerning Euclidean spaces will be presented below, with their aid the function constructed in Lemma 3.3. will be modified, removing the degenerate critical points.

Lemma 3.4. Let $f : U \subset R^n \to R$, U an open set and the function $f \in C^2$, and a function $L \in Hom_R(R^n, R)$. Then, except for a set of zero measure in $Hom_R(R^n, R) \cong R^n$, the function $f + L$ has only non-degenerate critical points.

Proof. Sard's theorem will be employed : if V and W are two differentiable manifolds of the same dimension and the map $\Pi : V \to W$, $\Pi \in C^1$, then the image through Π of the set of critical points of V is a set of zero measure in W. Here critical point of the map Π means a point at which the Jacobian of the Π map vanishes.

Let M be the submanifold of the manifold $U \times Hom_R(R^n, R)$, $M = \{(x,L) / d(f(x) + L(x)) = 0\}$. If $L(x^1,\dots,x^n) = \sum_{i=1}^{n} a_i x^i$, then the condition defining the manifold M leads to $L(x^1,\dots,x^n) = -\sum_{i=1}^{n} x^i (\partial f/\partial x^i) = -(df)$, and the correspondence $x \to (x, -df(x))$ is a diffeomorphism of the open set U onto the submanifold M. The critical point of the function $f + L$, which is degenerate when $\det(\partial^2 f/\partial x^i \partial x^j) = 0$, corresponds to any point $(x,L) \in M$. Now let Π be the projection $\Pi : M \to Hom_R(R^n, R)$, $\Pi(x,L) = L = -df(x)$. The mapping Π is critical at point $(x,L) \in M$, when $d\Pi = -(\partial^2 f/\partial x^i \partial x^j)$ is a singular matrix, that is when the critical point of the function $f + L$ is degenerate. In fact, $f + L$ has a degenerate critical point at a given point x if and only if L is the image of a critical point of the map Π. Applying Sard's theorem in which V and W are M^n and R^n respectively, it follows that the image of the critical points of the map Π has measure zero in R^n.

Lemma 3.5. Let K and U be two sets, $K \subset U \subset R^n$, K compact, U open and the function $f : U \to R$, $f \in C^2$ and assume the function f has only non-degenerate critical points in K. Then there exists a $\delta > 0$ so that from the inequality satisfied in K

$$\left| \frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i} \right| < \delta, \qquad (3.1)$$

$$\left| \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 g}{\partial x^i \partial x^j} \right| < \delta, \qquad (3.2)$$

where the function $g : U \to R$, $g \in C^2$, it follows that g too has only nondegenerate critical points in K.

Proof. Let the notation $|df| = [(\partial f/\partial x^1)^2 + \dots + (\partial f/\partial x^n)^2]^{1/2}$. Then if $|df| = 0$ the relation $\det(\partial^2 f/\partial x^i \partial x^j) = 0$ cannot occur in K, since f has no degenerate critical points in K according to the hypothesis. Hence, on the set K, the relation $|df| + \det(\partial^2 f/\partial x^i \partial x^j) > 0$ is true. Since the set K is compact and the function f continuous, there exists an $\mu > 0$ so that in K

$$|df| + \left|\det \frac{\partial^2 f}{\partial x^i \partial x^j}\right| \geq \mu > 0.$$

Let now the number δ be sufficiently small for the inequalities (3.1.) and (3.2) to imply

$$\big|\,|df| - |dg|\,\big| < \mu/2$$

$$\left|\det\left(\frac{\partial^2 f}{\partial x^i \partial x^j}\right) - \det\left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right)\right| < \mu/2,$$

respectively.

Then the latest two bounds and the choice of μ imply the inequalities

$$|dg| + |\det(\frac{\partial^2 g}{\partial x^i \partial x^j})| > |df| + |\det(\frac{\partial^2 f}{\partial x^i \partial x^j})| - \frac{\mu}{2} - \frac{\mu}{2} \geq 0.$$

Consequently, $|dg| + |\det(\frac{\partial^2 g}{\partial x^i \partial x^j})| > 0$ on the compact K, that is neither the function g has non-degenerate critical points in the lemma conditions.

Lemma 3.6. Let h be the function $h : U \to U'$, h a diffeomorphism, U, U' ,open sets in R^n, and $K \subset U$, $K' \subset U'$, K, K' compact sets with the property that $h(K) = K'$. For any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that the strict bound δ of the functions $|f|$, $|\partial f/\partial x^i|, |\partial^2 f/\partial x^i \partial x^j|$ in K' , implies the strict bound ε of the functions $|foh|$, $|\partial foh/\partial x^i|, |\partial^2 foh/\partial x^i \partial x^j|$ on K $(i,j= 1,\ldots,n)$.

Proof. The functions foh, $\partial foh/\partial x^i$, $\partial^2 foh/\partial x^i \partial x^j$ are polynomial functions in the partial derivatives of the functions f and h of maximum second order, and they vanish when the derivatives of f vanish. But the derivatives of h are bounded on the compact set K since h is a diffeomorphism.

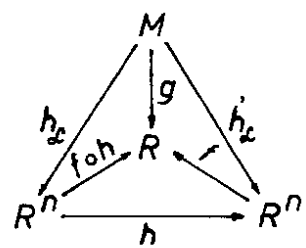
Let F(M,R) denote $\{f| \ f:M \to R, f \in C^2\}$, where M is a compact manifold (with boundary). A basis of neighborhoods of zero is constructed, where zero is the zero function in the additive group F(M,R). Let $\mathscr{S} = \{U_\alpha, h_\alpha\}$ be the differentiable structure of a manifold realized with a finite covering $\{U_\alpha\}$ and $\{C_\alpha\}$ a compact refinement of the family $\{U_\alpha\}$. For every $\delta > 0$, consider $N(\delta) \subset F(M,R)$ as a base of neighborhoods of the zero function, the set of functions

$$N(\delta) = \{g: M \to R| \ |g_\alpha| < \delta, \ |\frac{\partial g_\alpha}{\partial x^i}| < \delta, \ |\frac{\partial^2 g_\alpha}{\partial x^i \partial x^j}| < \delta\}$$

at all points on the set $h_\alpha(C_\alpha)$, where $g_\alpha = goh_\alpha^{-1}$. The sets of functions of the form $N(f,\delta) = \{F \in F(M,R)| F = f + N(\delta)\}$ make up a base of neighborhoods of the functions $f \in F(M,R)$. Explicitly, for $f \in F(M,R)$, $g \in N(F,\delta)$:

$$|f_\alpha - g_\alpha|, \ |\frac{\partial f_\alpha}{\partial x^i} - \frac{\partial g_\alpha}{\partial x^i}|, \ |\frac{\partial^2 f_\alpha}{\partial x^i \partial x^j} - \frac{\partial^2 g_\alpha}{\partial x^i \partial x^j}| < \delta$$

at all points of the set $h_\alpha(C_\alpha)$. The base of neighborhoods of functions of F(M,R) defines a C^2 topology on F(M,R). The definition is proved to be consistent by noting that for another atlas (h'_α, U'_α) and another compact subcovering C'_α, for any neighborhood $N(\delta)$ of zero in C^2, a neighborhood of zero in C'^2, $N'(\delta')$, is found, having the property that $N'(\delta') \subset N(\delta)$. Indeed, let us take the diagram :



where $h_\alpha : U_\alpha \subset M \to h_\alpha(U_\alpha) \subset R^n$, $h'_\alpha : U'_\alpha \subset M \to h'_\alpha(U'_\alpha) \subset R^n$, and consider the compact sets $C_\alpha \subset U_\alpha$, with properties from the construction of the C^2 topology and $f = goh_\alpha^{-1}$, $h = h_\alpha o h'^{-1}_\alpha$, where $h : h'_\alpha(C_\alpha \cap C'_\alpha) \to h_\alpha(C_\alpha \cap C'_\alpha)$. It is clear, from the definition of the differentiable manifold M, that the application h is a diffeomorphism on the two sets earlier described.

Under the conditions and notations of the preceding lemma, it follows that for any number $\varepsilon > 0$, there exists a number $\delta > 0$, so that if $|f|, |\partial f/\partial x^i|$, $|\det(\partial^2 f/\partial x^i \partial x^j)|$ are bounded by δ this implies a bound ε of the functions $|foh|, |\partial foh/\partial x^i|$, $|\partial^2 foh/\partial x^i \partial x^j|$. Hence, we get the inclusion $N'(\delta) \subset N(\delta)$. Consequently, $C'^2 \subset C^2$. Analogously, $C^2 \subset C'^2$, hence $C'^2 = C^2$.

The case $(W;\ V_o,\ V_1) = (M;\ \emptyset,\ \emptyset)$ will be firstly considered.

Theorem 3.7. The set of Morse functions of a compact manifold without boundary is an open set, dense in $F(M,R)$ (relative to the C^2 topology).

Proof. Let $\mathcal{S} = (U_\alpha, h_\alpha)_{\alpha=1,\dots,k}$ be the maximal atlas of M. One can find the compact subrefinement $\{C_i\}_{i=1,\dots,k}$ of the family of open sets $\{U_i\}_{i=1,\dots,k}$. The function $f \in F(M,R)$ is <u>good</u> on the set $S \subset M$ provided it has not any degenerate critical points on S.

It is first pointed out that the Morse functions set is open. Let $f : M \to R$ be a Morse function. Lemma 3.5 shows that in a neighborhood N_i of f in $F(M,R)$, any function is good in C_i. To apply lemma 3.5 formulated for functions defined on open sets $U \subset R^n$ and compact sets $K \subset U$, account is taken of the existence of coordinate mappings, so that the function foh_α is defined on the set $h_\alpha^{-1}(C_\alpha) \subset h_\alpha^{-1}(U_\alpha)$. Thus, in the finite intersection (for the compact sets covering is finite as M is a compact manifold) of the neighborhoods of f, $N = N_1 \cap \dots \cap N_k$, any function will be good in $\bigcup_{i=1}^{k} C_i = M$.

It is proved further on that the Morse functions set is dense in $F(M,R)$. Let N be a neighborhood of the function $f \in F(M,R)$ in the C^2 topology . Since the set M is compact, let λ be a function $\lambda : M \to [0,1]$, $\lambda \in F(M,R)$, such that $\lambda = 1$ in a neighborhood of C_1 and $\lambda = 0$ in a neighborhood of $M \setminus U_1$. From lemma 3.5, it follows that for almost any map $L : R^n \to R$, the function

$$f_1(p) = f(p) + \lambda(p)\ L(h_1(p))$$

will be good on $C_1 \subset U_1$. One can note that if the coefficients of

the linear map L are sufficiently small then the function $f_1 \in N$.

Indeed, f_1 differs from f only on the set supp $\lambda \subset U_1$. If $L(x) = \sum_{i=1}^{n} a^i x^i$ then

$$f_1 o h_1^{-1}(x) - f o h_1^{-1}(x) = (\lambda o h_1^{-1}(x)) \sum_{i=1}^{n} a^i x^i \qquad (3.3)$$

for any point $x \in h_1$ (supp λ). If a^i are sufficiently small on the compact set h_1 (supp λ), the difference (3.3) can be made arbitrary small, and so can the first two derivatives. As the functions h_1 and L are diffeomorphisms, lemma 3.6 ensures that $f_1 \in N$. Hence it is found $f_1 \in N$, good on C_1. By means of lemma 3.5, one can obtain a neighborhood N_1 of the function f_1, $N_1 \subset N$, such that any function in N_1 still is good on the compact set C_1. Using the same procedure, a function f_2 in N_1 is obtained from f_1 which will be good on C_2 and yet another neighborhood $N_2 \subset N_1$, so that any function in N_2, still good on C_2 and the function f_2 is implicitly good on C_1 as it is in N_1. Finally, it is obtained $f_k \in N_k \subset \dots \subset N_1 \subset N$, good on $C_1 \cup \dots \cup C_k = M$.

Theorem 3.8. There exists a Morse function on any triad $(W;V_o,V_1)$.

Proof. Lemma 3.3. proves the existence of a function $f: W \to [0,1]$ that satisfies the conditions

(i) $f^{-1}(0) = V_o$, $f^{-1}(1) = V_1$,

(ii) f has no critical points in a neighborhood of the boundary ∂W.

Employing lemmas 3.4 - 3.6, a function is constructed that has no degenerate points in $W \setminus \partial W$ while keeping the properties (i), (ii) of the function f. Let the open set U be a neighborhood of the boundary $\partial W = V_o \cup V_1$, with the property that the function f has no critical points in U. Since the space W is normal, there exists a neighborhood V of ∂W with the closure $\bar{V} \subset U$. Let the finite covering of W by coordinate neighborhoods $\{U_i\}$ be such that either $U_i \subset U$ or $U_i \subset W \setminus \bar{V}$, $i = 1, \ldots, k$. Let $\{C_i\}$ be a compact refinement of $\{U_i\}$ and $C_o = \{\bigcup_{i=1}^{k} C_i | C_i \subset U, i = 1, \ldots, k\}$.

Lemma 3.5 implies the existence of a neighborhood N of the function f such that for $g \in N$, the function g has no degenerate critical points in C_o. Moreover, the condition (i) implies that $f(p) \in (0,1)$ for $p \in W \setminus V$. One can find a neighborhood $N' \in F(W,R)$ such that for $g \in N'$, $g(p) \in (0,1)$ for $p \in W \setminus V$. Let the set $N_o = N \cap N'$.
Let $U_1, \ldots, U_k$ be the coordinate neighborhoods of $W \setminus V$.
Lemma 3.4 ensures the existence of the function $f_1 \in N_o$, good on C_1 and that of the neighborhood N_1 of f_1, $N_1 \subset N_o$, in which any function is good in C_1. Proceeding similarly for all coordinate neighborhood U_i and compact sets C_i $(i = 1, \ldots, k)$, one finds the function $f_k \in N_k \subset N_{k-1} \subset \ldots \subset N_o$, which is good on $\bigcup_{i=0}^{k} C_i = W$. As for $f_k \in N_o \subset N'$, f_k verifies (ii) and as $f_k|V = f|V$, it also satisfies (i). In fact, f_k is a Morse function on $(W; V_o, V_1)$.

Remark . As in theorem 3.7, it has been proved that the Morse functions form a dense open set in the set of differentiable mappings $f : (W; V_o, V_1) \to ([0,1]; 0,1)$.

If a Morse function f is given, it is possible that several of its critical points be at the same level, i.e., have through f the same image on the real axis.

Proposition 3.9. Let $f : W \to [0,1]$ be a Morse function of the triad $(W; V_o, V_1)$ with the critical points $p_1, \ldots, p_k$. Then f may be approximated by a Morse function g with the same critical points so that $g(p_i) \neq g(p_j)$, $i \neq j$.

Proof. Let the critical points p_1, p_2 be such that $f(p_1) = f(p_2)$. Let $\lambda : W \to [0,1]$ be a smooth function with the property that $\lambda|U = 1$, $p_1 \in U$, U open and $\lambda|_{CN} = 0$ (C - complementary) where the set $N \supset U$, N is the neighborhood of U and $N \subset W \setminus \partial W$, and moreover, from among the critical points of f only p_1 is contained in $\bar{N}$. Let the number $\epsilon_1 > 0$ be sufficiently small for $f_o = f + \epsilon_1 \lambda$, $f_o \in [0,1]$, $f_o(p_1) \neq f_o(p_i)$, $i > 1$.

The notion of gradient (to be used later in the proof) is now reminded.

Definition 3.10. Let $(W; V, V')$ be a triad. A Riemannian metric is chosen on W(/23/, p.97). Let $\langle X, Y \rangle$ denote the scalar product of two vectors X, Y in the tangent space. If $f \in F(W,R)$, then the <u>gradient</u> of the f function is the vector field defined on W, denoted by grad f, defined by the relation

$$\langle X, \text{grad } f \rangle = X(f),$$

where $X(f)$ is the derivative of f along the direction X, while X is an arbitrary vector field. In a given local coordinate system, the Riemannian metric is provided by the symmetric matrix $g_{ij}(x)$ and if $X = \sum_{i=1}^{n} x^i \partial/\partial x^i$, $Y = \sum_{i=1}^{n} Y^i \partial/\partial x^i$, then $\langle X, Y \rangle = \sum_{i,j=1}^{n} g_{ij} X^i Y^j$. It thereby follows that in a local coordinate system grad f has the components $\sum_{i=1}^{n} g^{ij} \partial f/\partial u^j$, where $\sum_{j=1}^{n} g_{ij} g^{jk} = \delta_i^k$.

Returning to the proof of Proposition 3.9, let K be the closure of the set $\{p \in W | \lambda(p) \in (0,1)\}$. Since K is a compact set, there exist positive numbers c and c' such that $0 < c \leq |\text{grad } f|$ and $|\text{grad } \lambda| \leq c'$. Let the function $f_1 = f + \epsilon\lambda$. Outside of the set K, $|\text{grad } \lambda| = 0$, hence $|\text{grad } f_1| = |\text{grad } f|$, hence outside of K, f_1 and f have the same singularities $p_1, \ldots, p_k$. The number ϵ is chosen such that f_1 has no critical points in K. It will thus be seen that f_1 is still a Morse function and $f_1(p_1) \neq f(p_i)$, $i > 1$. Going ahead with this procedure, a Morse function is constructed that has the property required in the Proposition. The following relations occur on the set K :

$$|\text{grad}(f + \varepsilon\lambda)| \geq |\text{grad } f| - \varepsilon\, |\text{grad } \lambda| \geq c - \varepsilon c'$$

If the condition $\varepsilon < \min(\varepsilon_1, c/c')$ is imposed then f_1 has the required property.

Definition 3.11. Let there be a differentiable function $f: W \to R$. The image of a critical point f is called *critical value* of f.

Proposition 3.12. Let $f: (W; V_o, V_1) \to ([0,1]; 0,1)$ be a Morse function and $0 < c < 1$, where c is different from the critical value of the function f. Then $f^{-1}[0,c]$ and $f^{-1}[c,1]$ are differentiable manifolds (with boundary).

Proof. If $w \in f^{-1}(c)$ and c is not a critical point, then in a local coordinate system $x^1, \ldots, x^n$ in the neighborhood of w, according to the implicit function theorem f has the expression of the projection $R^n \to R$, $(x^1,\ldots,x^n) \to x^n$. A chart of a neighborhood of w is thus defined. A chart for $f^{-1}(o)$ already exists as $f^{-1}(0) = V_o$. Analogously, $f^{-1}(0,c)$ is an open set in W for which a system of charts is available.

Corollary 3.13. Any cobordism may be expressed as a composition of cobordisms with Morse number 1.

Proof. The Morse function of distinct critical values is considered. This is possible according to Proposition 3.9. Noncritical values of the function are then chosen that range between the subsequent critical values. To the noncritical values there correspond

noncritical levels. By means of lemma 3.12, the product of cobordisms on the noncritical levels is made.

Thus, using Corollary 3.13, a given cobordism can be factored as cobordisms of Morse number 1. Elementary cobordisms will be studied in the following section.

4. DESCRIPTION OF THE TYPES OF HOMOTOPY OF DIFFERENTIABLE MANIFOLDS BY MEANS OF THE CRITICAL VALUES

Definition 4.1. Let f be a Morse function for the triad $(W; V, V')$. A vector field ξ on W^n is called a *gradient-like vector field* for the function f if

1. $\xi(f) > 0$ on the set complementary to the set of critical points of f;

2. for any critical point p of the function f, in a neighborhood U of point p, there exist the coordinates $(\vec{x},\vec{y}) = (x^1,\ldots,x^\lambda, x^{\lambda+1},\ldots,x^n)$ such that $f = f(p) - \vec{x}^2 + \vec{y}^2$ and ξ has the coordinates $(-x^1,\ldots,-x^\lambda, x^{\lambda+1},\ldots,x^n)$ on U. The existence of gradient-like fields will be proved below.

Proposition 4.2. There is a gradient-like field ξ for any Morse function on the triad $(W; V, V')$.

Proof. The proof is based on Morse's lemma, the implicit function theorem and the partition of unity theorem. For simplicity, let us assume the function f with a single critical point. According to Morse's lemma, there exists a neighborhood U_o of point p, in which a coordinate system $(\vec{x},\vec{y})$, $\vec{x} = (x^1,\ldots x^\lambda)$, $\vec{y} = (x^{\lambda+1},\ldots,x^n)$ can be chosen such that $f = f(p) - \vec{x}^2 + \vec{y}^2$.

Let U be a neighborhood of point p with the closure $\bar{U} \subset U_o$. The set $W \setminus U_o$ contains no critical points. From the implicit function theorem, it follows that in the neighborhood U' of point p', there exist the coordinates $(x^1)',\dots,(x^n)'$ such that $f = (x^1)' +$ constant in U', where $p' \in W \setminus U_o$.

Since $W \setminus U_o$ is compact, there exist the neighborhoods $U_1,\dots,U_k$ with the properties:

1. $W \setminus U_o \subset U_1 \cup \dots \cup U_k$,
2. $U \cap U_i = \emptyset$, $i = 1,\dots,k$,
3. U_i has the coordinates $x_i^1,\dots,x_i^n$ and $f = ct + x_i^1$ on U_i, $i = 1,\dots,k$.

On the set U_o, there exists a vector field ξ^o of coordinates $(-x^1,\dots,-x^\lambda, x^{\lambda+1},\dots,x^n)$ and on each set $U_i, i=1,\dots,k$, there exists a vector field $\xi^i = \partial/\partial x_i^1$ of coordinates $(1,0,\dots,0)$. By a partition of unity $\varphi_i, i = 0,\dots,k$, subordinate to the covering $\bigcup_{i=0}^{k} U_i$, where

$$\sum_{i=0}^{k} \varphi_i(p) = 1, \quad \operatorname{supp} \varphi_i \subset U_i, \quad i = 0,1,\dots,k,$$

the field

$$\xi = \sum_{i=0}^{k} \xi^i \varphi_i$$

has the required property. An analogous procedure is employed when several critical points are concerned.

Remark. The gradient-like vector field is a vector field orthogonal to the manifold $f^{-1}(f(q))$, $q \in W$. The positivity condition $\xi(f) > 0$ is imposed in order to obtain the one-parameter group of diffeomorphisms φ_t associated to the field ξ (/123/, p.101) so that the functions f be increasing along the trajectories φ_t, as follows from theorem 4.4.

Remark. The triad $(W; V_o, V_1)$ may be identified with the cobordism $(W; V_o, V_1; i_o, i_1)$, where $i_o : V_o \to V_o$ $i_1 : V_1 \to V_1$ are the identity maps.

Definition 4.3. The triad $(W; V_o. V_1)$ is called a ***product cobordism*** if it is diffeomorphic with the triad $(V_o \times [0,1]; V_o \times 0, V_o \times 1)$.

Theorem 4.4. If the Morse number μ of the triad $(W; V_o, V_1)$ is zero, then the triad is a product cobordism.

Proof. Let $f: W \to [0,1]$ be a Morse function with no critical points. Proposition 4.2. assures the existence of a gradient like vector field for the function f. Since the Morse function has no critical points, it follows from condition 1 of Definition 4.1. that $\xi(f) : W \to R$ is strictly positive. The field $\xi' = \xi/\xi(f)$ has the property $\xi'(f) = 1$ at all points of the manifold W. If the point belongs to the boundary ∂W, then f, expressed in a coordinate system $x^1,\dots,x^n$, $x^n \geq 0$ can be extended to a smooth function g defined on an open set $U \subset R^n$. In this coordinate system, the corresponding field ξ' extends to the open-set U. Let φ_t be one-parameter group having as generator the field ξ', which thereby verifies the differential equation

$$\frac{d\varphi_t}{dt} = \xi' \tag{4.1}$$

or

$$\frac{d}{dt}(f \circ \varphi_t) = \xi'(f),$$

where the vector field ξ' has been identified to the isomorphic vector in R^n. Since the manifold is compact, the local solution of the differential equation (4.1) are the integral curves $\varphi_t : [a, b] \to W$(/23/ , Theorem 8.1., p.102),

Definition 3.10 and the above relations lead to

$$\frac{d}{dt}(f\circ\varphi_t) = \langle \frac{d\varphi_t}{dt}, \text{grad } f\rangle = \langle \xi', \text{grad } f\rangle .$$

Hence

$$\frac{d}{dt}(f\circ\varphi_t) = \frac{\langle \xi, \text{grad } f\rangle}{\xi(f)} = 1. \qquad (4.2)$$

Integrating the differential equation (4.2) between t_o and t, at a fixed point q, one finds

$$f(\varphi_t(q)) = t - t_o + f(\varphi_{t_o}(q)) \qquad (4.3)$$

$$f(\varphi_t) = ct + t. \qquad (4.4)$$

After a change of variable $ct + t = s$ and a change of function $\varphi_{s-ct} = \psi(s)$ in relation (4.3), it is found

$$f(\psi(s)) = s, \qquad (4.5)$$

that is exactly what was asserted in Proposition 4.2. But each integral curve can be extended in a unique manner to a maximum interval which in the given case is $[0,1]$, as W is compact. In fact, for each point $y \in W$, there is a unique maximal integral curve

$$\psi_y : [0,1] \to W$$

that passes through the point y and satisfies condition (4.5) $f(\psi_y(s)) = s$. Moreover, $\psi_y(s)$ is a differentiable function as shown by the dependence of the differential equation's solution on the critical conditions.

Consequently, the diffeomorphism

$$h : V_o \times [0,1] \to W$$

has been established by the relations

$$h(y_o, s) = \psi_{y_o}(s), \qquad (4.6)$$

$$h^{-1}(y) = (\psi_y(o), f(y)). \qquad (4.7)$$

Now let f be a function $f : M \to R$ and denote $M^a = \{x \in M \mid f(x) \leq a\}$.

Corollary 4.5. (*Morse* /17/, p.150). Let $f: M \to R$ be a proper differentiable function defined on the differentiable manifold M and let us assume that the set $f^{-1}[a, b]$ ($a < b$, $a, b \in R$) does not contain any critical point of the function f. Then M^a is diffeomorphic with M^b.

Proof. Let $\varepsilon > 0$ be such that $f^{-1}[a - \varepsilon, a]$ do not contain any critical points. Let $T_e = (f^{-1}[a - \varepsilon, e] ; f^{-1}(a - \varepsilon), f^{-1}(e))$, $e = a, b$, be two triads. Proposition 3.12 implies that the triad T_e is differentiable, as the interval $[0,1]$ in the statement may be represented diffeomorphically on $[a - \varepsilon, e]$ by a linear transformation ($e = a, b$). With the same remark, Theorem 4.4 ensures the diffeomorphisms :

$$h_e : f^{-1}[a - \varepsilon] \times [a-\varepsilon, e] \sim f^{-1}[a-\varepsilon, e] , \quad e = a, b.$$

Let the isomorphism

$$h : f^{-1}(a - \varepsilon) \times [a - \varepsilon, a] \to f^{-1}(a-\varepsilon) \times [a-\varepsilon, b] ,$$

$$h(x,y) = (x, (y-a)(b-a+\varepsilon)/c + b)$$

Then $k = h_a \circ h^{-1} \circ h_b^{-1}$ is a diffeomorphism $f^{-1}[a-\varepsilon, b] \to f^{-1}[a - \varepsilon, a]$ between the given manifolds.

Let us define

$$g(x) = \begin{cases} k(x), & x \in f^{-1}[a - \varepsilon, b] , \\ x & x \in M^a - \varepsilon \end{cases}$$

g is the diffeomorphism $M^b \to M^a$ we have looked for.

Remark. This is the first part of theorem 3.1. in Milnor's book (/15/, p.12). One can notice that in Morse's paper the diffeomorphism along the trajectories orthogonal to the level curves is established in the sense of Corollary 4.5. Moreover, Theorem 3.1. in the above mentioned paper sets that M^a is a deformation re -

tract of M^b. We will remind the definitions (see /21/, p.81, p.92).

Definition 4.6. The map $r : X \to A$ of the space X onto the subspace A is called *retraction* if it is the extension of the identity map $1_A : A \to A$, that is $r \circ i = 1_A$ where $i : A \to X$ is the inclusion map. The subspace A of space X, for which the retraction map r exists, is called the *retract* of X.

If X is space, the homotopy $\xi_t : X \to X$, $t \in [0,1]$ with the property that $\xi_0 = 1_X$ is called a *deformation* of space X.

The continuous map $h : X \to X$ is called a *homotopy identity* if it is homotopic with the identity map 1_X, that is if there exists $\xi_t : X \to X$ as a deformation of space X with $\xi_1 = h$.

The map $f : X \to Y$ is a *homotopy equivalence* if there exist a map $g : Y \to X$ and the maps $g \circ f : X \to X$, $f \circ g : Y \to Y$ are homotopy identities. The spaces X, Y are called *homotopically equivalent* if there exists at least one homotopy equivalence $f : X \to Y$.

Let $r : X \to A$ be a retraction. The subspace A is said to be a *deformation retract* of space X if the retraction r is a homotopy equivalence. It follows that the map of retraction r is a homotopy equivalence if and only if $i \circ r : X \to X$ is a homotopy identity.

Corollary 4.7. Under conditions in Corollary 4.5., the set M^a is a deformation retract of the set M^b.

Proof. Using formula (4.3), a map $r_t : M^b \to M^b$ is defined as

$$r_t(q) = \begin{cases} q, & f(q) \le a, \\ \varphi_{t(a-f(q))}(q), & a \le f(q) \le b. \end{cases}$$

We have

$$r_0(q) = \begin{cases} q, & f(q) \le a, \\ \varphi_0(q) = q, & f(q) \in [a, b], \end{cases}$$

as

$$f(r_t(q)) = \begin{cases} f(q), & f(q) \le a, \\ f(\varphi_{t(a-f(q))}(q)) = f(q) + t(a - f(q)), & a \le f(q) \le b. \end{cases}$$

The last equality leads for the point $q' = r_t(q)$ to the relations:

$$a = a(1-t) + at \le f(q') \le b(1-t) + at < b, \quad f(q) \in (a,b).$$

Hence

$$r_t = \begin{cases} 1_{M^a} : M^a \to M^a, \\ f^{-1}(a,b) \to f^{-1}(a,b), \end{cases} \qquad t \in [0,1)$$

and

$$r_1 = \begin{cases} 1_{M^a} : M^a \to M^a, \\ f^{-1}[a,b] \to f^{-1}[a]. \end{cases}$$

Corollary 4.8. (Collar neighborhood theorem). Let W be a compact differentiable manifold with boundary. Then there exists a neighborhood of the boundary ∂W which is diffeomorphic with $\partial W \times [0,1)$.

Proof. By means of lemma 3.3., a function $f : W \to R_+$ is found which has the property that $f^{-1}(0) = \partial W$, and a neighborhood U of the boundary ∂W in which $df \ne 0$ is also found. Then f is a Morse function on the manifold $f^{-1}[0, \varepsilon/2]$ where $\varepsilon = \inf f(x)|_{x \in W \setminus U}$. Owing to Theorem 4.4, there exists a diffeomorphism $f^{-1}[0, \varepsilon/2) \sim \partial W \times [0,1)$.

A closed submanifold $M^{n-1} \subset W^n \setminus \partial W^n$ is called a *two-sided* manifold if there exists a neighborhood V of M^{n-1} in W^n such that $V\setminus M^{n-1}$ has two connected components.

Corollary 4.9. (Bicollar neighborhood theorem). Let M^{n-1} be a connected submanifold of the two-sided compact differentiable manifold W^n. Then there exists a neighborhood of M^{n-1} in W^n, diffeomorphic with $M^{n-1} \times (-1,1)$, such that M^{n-1} corresponds to $M^{n-1} \times 0$.

Proof. To simplify writing we remove the indices n and $n-1$ from the manifolds W^n and M^{n-1}. Let U be a set $U \subset W\setminus \partial W$, U an open neighborhood of M, with the property that $\bar{U}$ is compact and placed in a neighborhood of M which is cut into two components when M is eliminated. We then have $U = U_1 \cup U_2$, $U_1 \cap U_2 = M$ where U_1, U_2 are submanifolds with $\partial U_1 = \partial U_2 = M$. If the construction from Lemma 3.3 is repeated, a covering by open sets and a partition of unity may be chosen that would lead to a differentiable mapping $\varphi: U \to R$ with the property that $d\varphi \neq 0$ on M, $\varphi < 0$ on $\bar{U}\setminus U_1$, $\varphi = 0$ on M, $\varphi > 0$ on $\bar{U}\setminus U_2$. Moreover, a neighborhood V of the submanifold M may be chosen such that $\bar{V} \subset U$, and on which φ has no critical points. Let $2\varepsilon_1$ be the least upper bound on the set $\bar{U}_1\setminus V$ and $2\varepsilon_2$ = the greatest lower bound on the set $\bar{U}_2\setminus V$. Then the set $\varphi^{-1}[\varepsilon_1,\varepsilon_2]$ is a compact n-dimensional submanifold of V with the boundary $\varphi^{-1}(\varepsilon_1) \cup \varphi^{-1}(\varepsilon_2)$ and φ is a Morse function on $\varphi^{-1}[\varepsilon_1,\varepsilon_2]$. Owing to the Theorem 4.4. it follows that $\varphi^{-1}(\varepsilon_1,\varepsilon_2)$ is a bicollar neighborhood of M in V, hence in W too.

Remark. The theorem is still true if M manifolds consisting of several components ~~are considered~~ as the components of M can be covered by open sets from W.

With theorems of collar and bicollar neighborhood it is defined the differentiable structure on the manifold that results from piecing two triads.

Theorem 4.10. Let $(W; V_o, V_1)$ and $(W'; V_1', V_2')$ be differentiable triads and the mapping $h : V_1 \to V_1'$ a diffeomorphism. Then there exists a differentiable structure $\mathcal{S}$ for $W \cup_h W'$ compatible with the given structures on W and W'. The structure $\mathcal{S}$ is unique up to a diffeomorphism that leaves the manifolds V_o, $h(V_1) = V_1'$ and V_2' fixed.

Proof. 1) Existence. The problem is to provide the topological space $W \cup_h W'$ with a differentiable structure that would be consistent with that of W and W'. $W \cup_h W'$ is the topological space that is obtained by the disjoint union of sets W' and W on which an equivalence relation is introduced by identifying the points $x \in V_1$ and $h(x) \in V_1'$. The topology on the set $W \cup_h W'$ is given by the quotient topology. It will be noted that $W \cup_h W' = (W\setminus V_1) \cup W'$, $W\setminus V_1 = (W \cup_h W')\setminus W'$ which justifies the term of piecing spaces by the isomorphism h. To define the differentiable structure it will be enough to define the differentiable structure on open sets that cover the manifold (Theorem 1.1 in /23/, p.44). Consider the inclusions $j : W \to W \cup_h W'$ and $j' : W' \to W \cup_h W'$. Then $W \cup_h W'$ is covered by $j(W\setminus V_1)$ and $j'(W'\setminus V_1)$ and this leaves to define the covering for V_1 identified to V_1' by the mapping h. Indeed, Corollary 4.8. shows there exist the neighborhoods $U_1(U_1')$ of boundaries $V_1(V_1')$ in W(and W' respectively) and diffeomorphisms $g_1: V_1 \times (0,1] \to U_1$, $g_1(x,1) = x$, $g_2 : V_1' \times [1,2) \to U_1'$, $g_2(y,1) = y$. The mapping $g: V_1 \times (0,2) \to W \cup_h W$ is defined by

$$g(x,t) = \begin{cases} j(g_1(x,t)) & 0 < t \leq 1, \\ j'(g_2(h(x),t)), & 1 \leq t \leq 2. \end{cases}$$

Then $g(V_1 \times (0,2))$ defines the image of the neighborhood $V_1 \subset W \cup_h W'$ in $W \cup_h W'$.

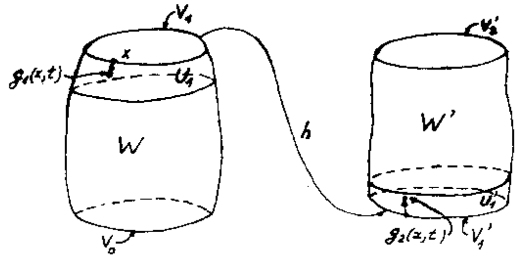


Fig.3

2) Uniqueness. It will now be proved that any smooth structure $\mathscr{S}'$ on the manifold $W \cup_h W'$ compatible with the given structures on the manifolds W and W' is isomorphic with the structure constructed above by piecing the collar neighborhoods of the boundaries V_1 and V_1'. From corollary 4.9, it follows that there exists bicollar neighborhood U of the set $j(U_1) = j'(U_1')$ in $W \cup_h W'$ and a diffeomorphism $g: V\times(-1,1) \to U$ with respect to the differentiable structure $\mathscr{S}$ with the property that $g(x,o) = j(x)$ for $x \in V$. Hence $j^{-1}(U \cap j(W))$ and $j'^{-1}(U \cap j'(W))$ are collar neighborhoods of V_1 and V_1' in W and W' respectively. These are diffeomorphic with the collar neighborhoods constructed in the first part (that on existence) of the theorem. It is seen that the uniqueness occurs up to a diffeomorphism which leaves the submanifolds V_o, $h(V_1) = V_1'$ and V_2' fixed.

Let $(W; V_o, V_1)$ and $(W; V_1', V_2')$ be two triads with f and f' the Morse functions on $[0,1]$ and $[1,2]$ respectively. According to Proposition 4.2 there is a gradient-like field ξ, ξ' on either triad, that can be normalized to 1 when acting on the functions f and f' respectively, except for small neighborhoods of the critical

points. Taking again the proof of Theorem 4.10 and adding the condition for piecing the integral curves of the gradient-like fields ξ and ξ' in the collar neighborhoods V_1 and V_1', which leads to a gradient-like field uniquely defined on the bicollar neighborhood $V_1' = h(V_1)$, it is found:

Corollary 4.11. Under the conditions of theorem 4.10, h being the diffeomorphism $h: V_1 \to V_1'$, there exists a unique differentiable structure on $W \cup_h W'$, which is consistent with the structures on W and W' so that the Morse functions f and f' piece together to give a unique Morse function on $W \cup_h W'$, and so do the gradient-like fields ξ and ξ'.

We will describe further on a method for passing from a noncritical level $f^{-1}(c-\varepsilon)$ to another such level $f^{-1}(c+\varepsilon)$ when a single critical value f(c) is overpassed and the critical level $f^{-1}(c)$ contains a single nondegenerate critical point.

Definition 4.12. An elementary cobordism is a triad $(W; V, V')$ which has a single critical point p.

It will be shown that an elementary cobordism is not a product cobordism and by virtue of Theorem 4.4 it follows that $\mu(W;V,V')=1$. At the same time, Proposition 5.5 will prove that the index of an elementary cobordism is properly defined, i.e. it does not depend on the choice of the Morse function f.

Now the meaning of these assertions will be specified. Let $(W; V, V')$ be a differentiable triad with the Morse function f and ξ the gradient-like field for f.

Let us assume that point $p \in W$ is a critical point and let the levels $V_o = f^{-1}(c_o)$ and $V_1 = f^{-1}(c_1)$ be such that $c_o < f(p) < c_1$ and $c = f(p)$ be the only critical value in the interval $[c_o, c_1]$.

Let the notation

$$D^n_r = \{x \in R^n \mid \|x\| < r\} \quad (4.8)$$

$$\overset{\circ}{D}{}^n_r = \text{int } D^n_r, \quad (4.9)$$

$$D^n = D^n_1, \quad (4.10)$$

$$S^{n-1} = \{x \in R^n \mid \|x\| = 1\}. \quad (4.11)$$

Since ξ is a gradient-like vector field, there exists neighborhood U of the point p in W and a coordinate mapping $g : D^n_{2\varepsilon} \to U$ such that $f o g(\vec{x},\vec{y}) = c-\vec{x}^2+\vec{y}^2$, while ξ has the coordinates $(-x^1,\dots,-x^\lambda, x^{\lambda+1},\dots,x^n)$ in the neighborhood U, where λ is an integer from the interval $[0,n]$. Here $\vec{x} = (x^1,\dots,x^\lambda) \in R^\lambda$, $\vec{y} = (x^{\lambda+1},\dots,x^n) \in R^{n-\lambda}$. Let the levels $V_{\pm\varepsilon} = f^{-1}(c\pm\varepsilon^2)$. Imposing the condition that $4\varepsilon^2 < \min(c-c_o, c_1-c)$, a situation as shown in Fig. 4 occurs

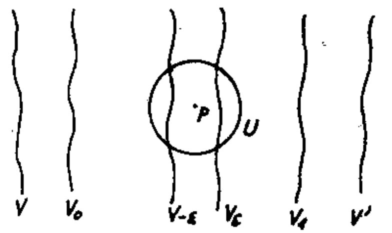


Fig. 4

To simplify writing, the following notations are introduced

$$A^{n-1} = S^{\lambda-1} \times \overset{\circ}{D}{}^{n-\lambda}, \quad (4.12)$$

$$B^{n-1} = \overset{\circ}{D}{}^\lambda \times S^{n-\lambda-1}, \quad (4.13)$$

$$Y^{n-1} = S^{\lambda-1} \times S^{n-\lambda-1} \times (0,1) \quad (4.14)$$

and the mappings are defined

$$F : Y \to A,\ F(u,v,t) = (u,vt),\ u \in S^{\lambda-1},\ v \in S^{n-\lambda-1}, \quad (4.15)$$

$$G : Y \to B,\ G(u,v,t) = (tu,v),\ t \in [0,1). \quad (4.16)$$

It will be noticed that :

$$F(Y) = S^{\lambda-1} \times (\overset{\circ}{D}{}^{n-\lambda} \setminus 0) \subset A^{n-1}, \quad (4.17)$$

$$G(Y) = (\overset{\circ}{D}{}^\lambda \setminus 0) \times S^{n-\lambda-1} \subset B^{n-1},$$

$$A^{n-1} = F(Y) \cup (S^{\lambda-1} \times o),\ F(Y) \cap (S^{\lambda-1} \times o) = \emptyset,$$

$$B^{n-1} = G(Y) \cup (o \times S^{n-\lambda-1}),\ G(Y) \cap (o \times S^{n-\lambda-1}) = \emptyset.$$

A mapping $\varphi_L : A^{n-1} \to V_o$ will be defined. Let $\varphi : A^{n-1} \to V_{-\varepsilon}$ defined by $\varphi(u,\theta v) = g(\varepsilon u \text{ ch } \theta, \varepsilon v \text{ sh } \theta)$. Using the isomorphism $(u,v,\theta) \to (u,v\theta)$, $u \in S^{\lambda-1}$, $v \in S^{n-\lambda-1}$, $\theta \in [0,1)$ the identification $S^{\lambda-1} \times S^{n-\lambda-1} \times [0,1) \to A^{n-1} = S^{\lambda-1} \times \overset{\circ}{D}{}^{n-\lambda}$ has been made. The assertion is verified as $f \circ \varphi(u,\theta v) = c-\varepsilon^2 u^2 \text{ch}^2\theta + \varepsilon^2 v^2 \text{sh}^2\theta = c-\varepsilon^2$. Moreover, the mapping φ is a homeomorphism, an immersion, hence φ is a embedding. The integral curve of the gradient-like field ξ which passes through $\varphi(u,\theta v) \in V_{-\varepsilon}$, intersects the level curve V_o in a unique point $\varphi_L(u,\theta v)$.

Definition 4.13. The mapping $\varphi_L : A^{n-1} \to V_o$ as defined above is called a *characteristic embedding*. Let $S_L = \varphi_L(S^{\lambda-1} \times o)$ denotes the *left-hand sphere* of p in V_o. From the relations (4.17), it is observed that $S_L = \varphi_L(A^{n-1} \setminus F(Y))$. S_L is just the intersection of the set V_o with all of the integral curves of the gradient like-field ξ that lead to p.

Indeed, in the earlier defined neighborhood of the point, U, the field ξ has the following components with respect to chart g :

$$\xi^i = -x^i / \|x\|, \qquad i = 1,\dots,\lambda,$$

$$\xi^\alpha = x^\alpha / \|x\|, \qquad \alpha = \lambda+1,\dots,n.$$

In the neighborhood U, the parametric equations in local coordinates of the integral curves of ξ have the form:

$$x^i = x^i_o e^{-u(t)}, \qquad i = 1,\dots,\lambda,$$

$$x^{\alpha} = x_o^{\alpha} e^{u(t)}, \qquad \alpha = \lambda+1,\dots,n,$$

where u(t) is the solution of the differential equation

$$\frac{du}{dt} = (e^{-2u} \sum_{i=1}^{\lambda} (x_o^i)^2 + e^{2u} \sum_{\alpha=\lambda+1}^{n} (x_o^{\alpha})^2)^{-1/2}.$$

In order that a trajectory tends to zero it is necessary that $x^i(t) \to 0$ for $t\to\infty$, $i = 1,\dots,n$. Since $du/dt > 0$, it is necessary that $\lim_{u\to\infty} x^i(u) = 0$, which imposes the condition that $\vec{x}_o = 0$ or $\vec{y}_o = 0$. It follows that $\varphi_L(S^{\lambda-1} x\ o)$ are the only points in $V_{-\varepsilon}$ from which such trajectories are leaving that pass through the critical point p.

We also define the left-hand disk D_L as the union of all integral segments starting from S_L and ending in p.

Analogously, a mapping $\varphi': B^{n-1} \to V_{\varepsilon}$ is defined by $\varphi'(\theta u,V) = g(\varepsilon u\ sh\theta,\ \varepsilon\ vch\theta)$, where again the identification $S^{\lambda-1} \times [0,1) \times S^{n-\lambda-1} \to B^{n-1} = \overset{\circ}{D}^{\lambda} \times S^{n-\lambda-1}; (u,\theta,v) \to (\theta u,v), u\in S^{\lambda-1}$, $v\in S^{n-\lambda-1}$, $\theta\in[0,1)$ was used. Then $\varphi'(u\theta,v)$ is translated till the manifold V_1 along the integral curves of the gradient-like field ξ till to the point $\varphi_R(\theta u,v)$.

The right-hand sphere of point p in V_1 is defined as $S_R = \varphi_R(OxS^{n-\lambda-1}) = \varphi_R(B^{n-1}\setminus G(Y))$ and this is the boundary of the *right-hand disk* D_R, defined as the union of the segments of the integral curves of ξ that pass through p and end in S_R.

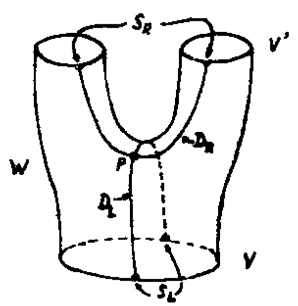


Fig. 5

Fig. 5 shows an elementary cobordism of dimension n=2, index λ=1 and function $f : W \to R$, that has a single nondegenerate critical point p. The surface is the rotation torus from Ch.2.

Definition 4.14. Considering the (n-1 dimensional) manifold V and the embedding $\varphi : A^{n-1} \to V$, let $\chi(V,\varphi) = [V\setminus \varphi(A^{n-1}\setminus F(Y))] \cup_Y B^{n-1}$ be the manifold thus obtained: into the disjoint union of the set $\tilde{V} \equiv V\setminus \varphi(A^{n-1}\setminus F(Y)) = V\setminus \varphi(S^{\lambda-1} \times o)$, with $B^{n-1} = \overset{\circ}{D}^{\lambda} \times S^{n-\lambda-1}$ an equivalence relation is introduced

$$R : \varphi(F(y)) = G'y), \qquad y\in Y.$$

The set $\chi(V,\varphi) \equiv \tilde{V} \cup B^{n-1}/R$ is constructed . As coordinate mapping of the manifold $\chi(V,\varphi)$ are taken $\alpha o h, \alpha o h'$ where $\alpha: \tilde{V} \cup B^{n-1} \to \chi(V,\varphi)$ is the canonical projection and h, h' the coordinate mappings for $\tilde{V}$ and B^{n-1}. If V' is a manifold diffeomorphic with $\chi(V,\varphi)$, it is said that V' is obtained from V by (Morse) <u>surgery of type (λ,n-λ).</u>

From (4.17), one can see that by the equivalence relation R and the construction of $\chi(V,\varphi)$, part of the points of A are not sent to the points of $\chi(V,\varphi)$, namely the points removed are $\varphi(A^{n-1}\setminus F(Y)) = \varphi(S^{\lambda-1} x o)$, that is a set homeomorphic with a sphere $S^{\lambda-1}$, embedded into an n-dimensional space.

Relation (4.17) $B^{n-1}\setminus G(Y) = o \times S^{n-\lambda-1}$ shows that the sphere $o \times S^{n-\lambda-1}$ represents the B^{n-1} points which are not equivalent to points in $\tilde{V}$. Consequently , the (λ,n-λ) type surgery is replacing a sphere of dimension λ-1 by another sphere of dimension n-λ-1.

Using /26/ we will illustrate Morse surgery in the case n = 3, λ = 2. V is a bidimensional manifold, $S^{\lambda-1}=S^1$ is a circle, $D^{n-\lambda}=D^1$ a segment, $\overset{\circ}{D}^2 = D^2\setminus S^1$ an open disk, $A^2 = S^1 \times [0,1)$ a cylinder , $B^2 = \overset{\circ}{D}{}^2 \times S^0$ two open disks, $Y^1 = S^1 \times S^0 \times (0,1)$ the disjoint union of two cylinders (without the bases).

By Morse surgery (2,1), the sphere $S^1, \varphi(S^1 \times \{a\})$, is removed and the points $o \times \{a\}$, $o \times \{b\}$ are added. Now if a (1,2) surgery is applied to the surface $\chi(V,\varphi)$ obtained, then $S'^{\lambda-1} = \{\alpha, \beta\}$ are two points, $S'^{n-1} = S^1$ is a circle, $D'^{\lambda} = [0,1]$ a segment, $D^{n-\lambda} = D^2$ a disk, $\overset{\circ}{A}{}^{n-1} = \{\alpha, \beta\} \times \overset{\circ}{D}{}^2$ two disks, $B'^{n-1} = (0,1) \times S'$ a cylinder, $G(Y)$ two cylinders and $F(Y)$ two disks. It should be remarked that a set $S^{\lambda-1} \times o$ is removed, that is two points, while another set $o \times S^{n-\lambda-1}$, which is a circle, is added.

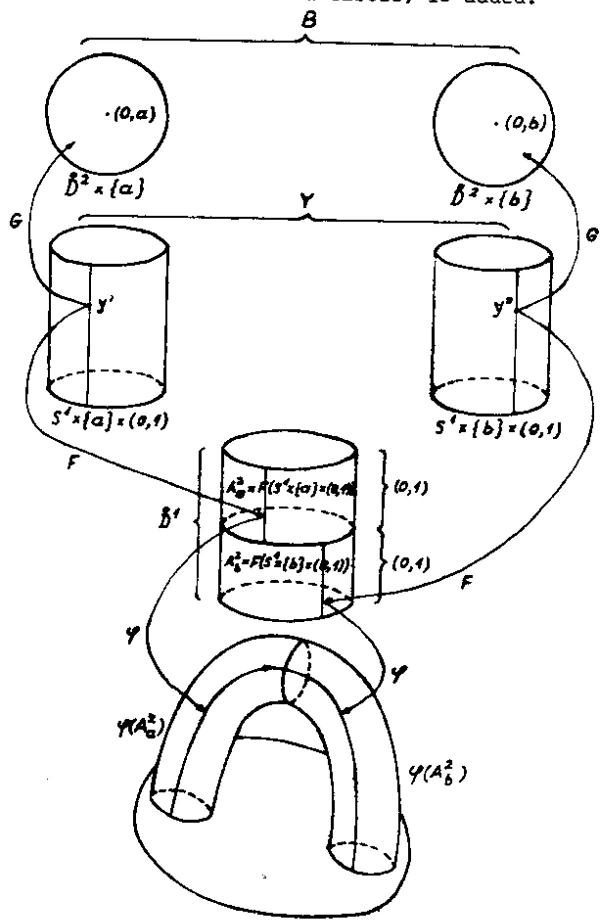


Fig. 6.

For instance applying the (1,2) surgery to the sphere S^2, a one handle orientable compact surface is obtained, provided the mappings φ' are taken such that the orientations induced on the circles $\varphi'(S^1 \times \{a\})$ and $\varphi'(S^1 \times \{b\})$ both correspond to the same orientation of the sphere S^2.

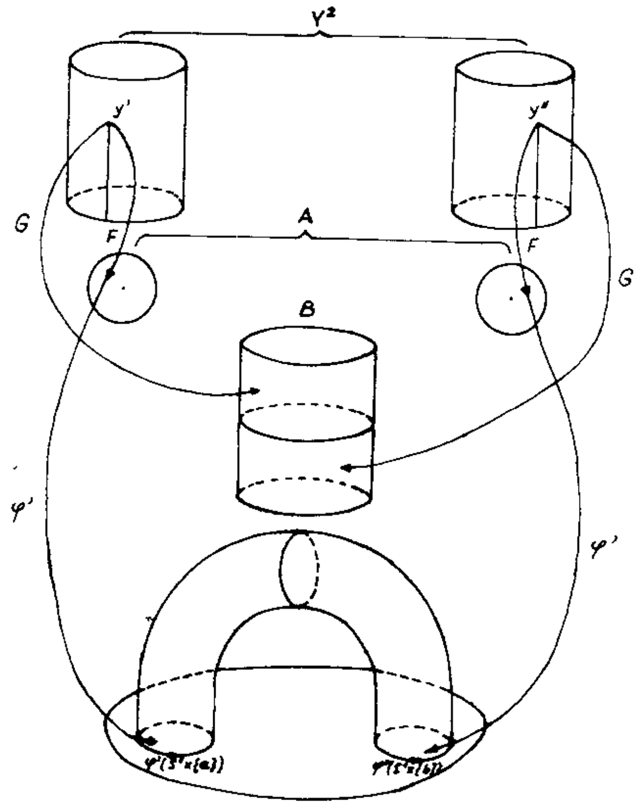


Fig. 7

The situation is outlined in Fig. 7. By Morse surgery, two points, the centers of the two disks in A^2, are removed and a circle S^1 is added (the circle on the cylinder which is the boundary of the image through G of the two cylinders in Y). The identification $\varphi(F(y)) = G(y)$, $y \in Y$ is performed. By Morse surgerry of the type (1,2) the torus has been changed into a sphere.

The two theorems below show that passing from one level curve to another while crossing a critical level is equivalent to a Morse surgery.

Theorem 4.15. If $V' = \chi(V,\varphi)$ can be obtained from V by a surgery of the type $(\lambda, n-\lambda)$, then there exists an elementary cobordism $(W; V, V')$ and a Morse function $f : W \to R$ with a critical point of index λ.

Proof. Let

$$L_\lambda = \{ \vec{x} \in R^\lambda, \vec{y} \in R^{n-\lambda} \mid |\vec{x}|\,|\vec{y}| < \text{ch } 1 \cdot \text{sh } 1,\ -1 \leq -\vec{x}^2 + \vec{y}^2 \leq 1 \}, \tag{4.18}$$

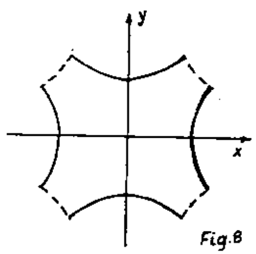


Fig.8

L_λ is a two-boundary differentiable manifold. The "left hand" boundary $-|\vec{x}|^2 + |\vec{y}|^2 = -1$ is diffeomorphic with the product $S^{\lambda-1} \times \overset{\circ}{D}{}^{n-\lambda}$ by the correspondence $(u, \theta v) \leftrightarrow (u \text{ch}\theta, v \text{sh}\theta)$, $0 \leq \theta < 1$. Indeed, let $\vec{x}$, $\vec{y}$ be such that $-\vec{x}^2 + \vec{y}^2 = 1$. It follows that $\text{ch}\theta = |\vec{x}|$, $\text{sh } \theta = |\vec{y}|$, $u = \vec{x}/\text{ch}\theta$, $v = \vec{y}/\text{sh}\theta$ if $\vec{y} \neq 0$, and $v = 0$ if $\vec{y} = 0$. The right hand boundary $-|\vec{x}|^2 + |\vec{y}|^2 = 1$ is diffeomorphic with the product $\overset{\circ}{D}{}^\lambda \times S^{n-\lambda-1}$ by the correspondence $(\theta u, v) \leftrightarrow (u \text{ sh } \theta, v \text{ ch } \theta)$.

In Fig. 8 above, we have symbolically $\vec{x} \in R^\lambda$ (actually $\lambda = 1$), $\vec{y} \in R^{n-\lambda}$ $(n-\lambda=1)$. The dotted piece corresponds to the equilateral hyperbola arcs that appear in eq. (4.18).

Let us now consider the trajectories orthogonal to the surface $-|\vec{x}|^2 + |\vec{y}|^2 = \text{ct}$. The trajectories passing through the point $(\vec{x}, \vec{y})$ can be parametrized in the form $t \to (t\vec{x}, t^{-1}\vec{y})$. If $\vec{y} = 0$ $(\vec{x} = 0)$, the trajectory is a segment belonging to the "plane $\vec{y} = 0$ $(\vec{x} = 0)$, which tends towards the origin. If $|\vec{x}|.|\vec{y}| \neq 0$, the trajectory is a hyperbola sending a well-defined point $(u\text{ch } \theta, v\text{sh}\theta)$ from the left-hand boundary of L_λ into the corresponding point $(u\text{sh } \theta, v \text{ ch } \theta)$ on the right boundary.

An n-manifold $W = \omega(V,\varphi)$ is constructed as follows. Starting from the disjoint union $(V \setminus \varphi(S^{\lambda-1} \times O)) \times D^1 + L_\lambda$, an equivalence relation R' is introduced: on the disjoint (topological) sum above, by means of mapping φ carrying $(V \setminus \varphi(S^{-1} \times O))$ in L_λ, the point $(u, v, \theta, c) \in Y \times [-1,1] = S^{\lambda-1} \times S^{n-\lambda-1} \times (0,1) \times [-1,1]$ is identified to the unique point $(\vec{x}, \vec{y}) \in L_\lambda$ to be found by the procedure:

1. $-\vec{x}^2 + \vec{y}^2 = c$

2. $(\vec{x}, \vec{y})$ lies on the orthogonal trajectory passing through the point $(u \text{ ch } \theta, v \text{ sh } \theta)$.

Condition 2 requires that an orthogonal trajectory pass through the point $(u \text{ ch } \theta, v \text{ sh } \theta) \in L_\lambda$, $(u, v, \theta) \subset Y$. The parametric equations of this trajectory are $(tu \text{ ch } \theta, \frac{1}{t} v \text{ sh } \theta)$. Parameter t is derived from the condition that the trajectory intersect the hyperboloid which verifies condition 1. An equation in t

$$-t^2 \text{ch}^2\theta + \text{sh}^2\theta / t^2 = c,$$

is obtained, and hence

$$t = [-c + (c^2 + 4\text{sh}^2\theta \text{ch}^2\theta)^{1/2}]/2 \text{ ch}^2\theta.$$

By the equivalence R' a diffeomorphism has been established between

$$\varphi(S^{\lambda-1} \times (\overset{\circ}{D}{}^{n-\lambda} \setminus O)) \times D^1 \leftrightarrow L_\lambda \setminus [(R^\lambda \times o) \cup (R^{n-\lambda} \times o)].$$

The right hand set is obviously equal to

$L_\lambda \cap [(R^\lambda \setminus 0) \times (R^{n-\lambda} \setminus 0)]$.

Putting

$$W = \omega(V,\varphi) = [(V \setminus \varphi(S^{\lambda-1} \times o)) \times [-1,1]] \cup L_\lambda / R',$$

W is a differential manifold having two boundaries that correspond to the values $c = -|\vec{x}|^2 + |\vec{y}|^2 = -1$ and $c = 1$. The left-hand boundary may be identified to V while establishing the correspondence for $z \in V$ with

$$\begin{cases} (z,-1) \in (V \setminus \varphi(S^{\lambda-1} \times o)) \times D^1, & z \notin \varphi(S^{\lambda-1} \times o), \\ (u \text{ ch } \theta,\ v \text{ sh } \theta) \in L_\lambda & z = \varphi(u,\theta v) \end{cases} \tag{4.19}$$

The right hand boundary may be identified to $\chi(V,\varphi)$:

$$\begin{cases} (z,1) \in (V \setminus \varphi(S^{\lambda-1} \times o)) \times D^1, & z \in (S^{\lambda-1} \times o), \\ (u \text{ sh } \theta,\ v \text{ ch } \theta) \in L_\lambda\ , & (\theta u, v) \in \overset{o}{D}{}^{r-\lambda} \times S^{n-\lambda-1}. \end{cases} \tag{4.20}$$

A Morse function f is then defined which has a single critical point of index λ :

$$\begin{cases} f(z,c) = c, & (z,c) \in (V \setminus (S^{\lambda-1} \times o)) \times D^1, \\ f(\vec{x},\vec{y}) = -|\vec{x}|^2 + |\vec{y}|^2\ , & (\vec{x},\vec{y}) \in L_\lambda . \end{cases} \tag{4.21}$$

The following theorem is in a way the reciprocal of Theorem 4.15.

Theorem 4.16. Let $(W;V,V')$ be an elementary cobordism with the characteristic embedding $\varphi_L : S^{\lambda-1} \times \overset{o}{D}{}^{n-\lambda} \to V$. Then the triad $(W;\ V,\ V')$ is diffeomorphic with the triad $(\omega(V,\omega_L);\ V, \chi(V,\omega_L))$.

Proof. Definition 4.13 will be employed using V instead of V_o and V' instead of V_1. One can note the following diffeomorphisms :

$$(W;\ V,V') \approx (W\ ;\ V_{-\varepsilon},\ V_\varepsilon)\ , \tag{4.22}$$

and

$$(\omega(V,\varphi_L); V, \chi(V,\varphi_L)) \approx (\omega(V_{-\varepsilon},\varphi); V_{-\varepsilon}, \chi(V_{-\varepsilon},\varphi)). \tag{4.23}$$

Here $W_\varepsilon = f^{-1}[c-\varepsilon^2,\ c+\varepsilon^2]$. Theorem 4.4 and Corollary 4.5 prove that the triads $(f^{-1}[c_o, c-\varepsilon^2];\ V,\ V_{-\varepsilon})$ and $(f^{-1}[c+\varepsilon^2, c_1];\ V_\varepsilon, V')$ are product cobordisms. The second diffeomorphism (4.23) arises from the properties of the integral curves. To prove the theorem, one has to prove the diffeomorphism of the triads

$$(W_\varepsilon;\ V_{-\varepsilon}, V_\varepsilon),\quad (\omega(V_{-\varepsilon},\varphi);\ V_{-\varepsilon},\ \chi(V_{-\varepsilon},\ \varphi)). \tag{4.24}$$

Let $k : \omega(V_{-\varepsilon},\varphi) \to W_\varepsilon$ be the diffeomorphism defined as follows. If $(z,t) \in (V_{-\varepsilon} \setminus (S^\lambda \times o)) \times D^1$, then $k(z,t)$ is the (unique) point on the integral curve passing through z and lying on the level curve $\varepsilon^2 t + c(\in [c - \varepsilon^2,\ c + \varepsilon^2])$. If the point $(\vec{x},\vec{y}) \in L_\lambda$, then $k(\vec{x},\vec{y}) = g(\varepsilon\vec{x},\ \varepsilon\vec{y})$, hence $f \circ k(\vec{x},\vec{y}) = -\varepsilon^2 \vec{x}^2 + \varepsilon^2 \vec{y}^2 + c \in [c - \varepsilon^2,\ c + \varepsilon^2]$ as $-\vec{x}^2 + \vec{y}^2 \in [-1,1]$. Diffeomorphism (4.24) results from the definitions of the mapping φ and of the manifold $\omega(V_{-\varepsilon},\varphi)$, as well as from g sends trajectories which are orthogonal to the boundary L_λ in integral curves in W_ε.

The following theorem describes the geometrical meaning of passing over a critical point. It is important both as a consequence of Theorem 4.16 and in establishing the homotopy type of differentiable manifolds.

Theorem 4.17. Let $(W;\ V,V')$ be an elementary cobordism with a Morse function with a single critical point of index λ . Let D_L be the left-hand disk associated to a gradient-like vector field. Then $V \cup D_L$ is the deformation retract of W.

Proof. By virtue of the preceding theorem, one can take $(W;V,V') = (\omega(V,\varphi_L);\ V, \chi(V,\varphi))$, where φ_L is the characteristic embedding $\varphi_L : S^{\lambda-1} \times \overset{o}{D}{}^{n-\lambda} \to V$. Then D_L is: $D_L = \{(\vec{x},\vec{y}) \in R^\lambda \times R^{n-\lambda} \mid |x| \le 1,\ \vec{y} = 0\}$ and it is clear that $D_L \subset L_\lambda$.

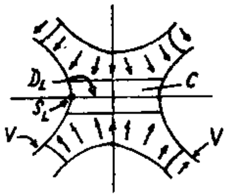


Fig. 9

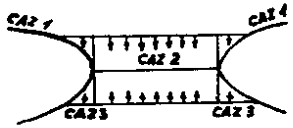


Fig. 10

Let now C be neighborhood of D_L :

$$C = \{ (\vec{x},\vec{y}) \in L_\lambda \mid \quad |\vec{y}| \le 1/10\}$$

The retraction from W to $V \cup D_L$ is defined by composition of two retractions

$$r_t : W \to V \cup C \qquad \text{and} \qquad r'_t : V' \cup C \to V \cup D_L .$$

1. Construction of the retraction r_t. For each point $(\vec{x},\vec{y})$ in W, consider the hyperboloid passing through the given point, hence having the value $-\vec{x}^2 + \vec{y}^2$, and then follow the trajectories orthogonal to this hyperboloid, inside L_λ until neighborhood C or V is reached, and outside L_λ up to V.

For each $(v,c) \in [V \setminus {}_L(S^{\lambda-1} \times o)] \times D^1$, let $r_t(v,c) = (v, c - t(c+1))$. One can note that $c'(t,c) = c - t(c+1)$, as a function of t, is decreasing, as $-(c+1) \le 0$, hence for $t = 0$, $c'(c,t)|_{t=0} = c$ and $c'(c,t)|_{t=1} = -1$, that is $c'(c,t) \in [-1,1]$. Consequently, r_t carries $[V \setminus \varphi_L(S^{\lambda-1} \times O) \times D^1]$ in itself, while being of course a continuous function of t.

For $t = 0$, $r_o(v,c) = (v,c)$, hence r_o is the identity map and $r_1(v,c) = (v,-1)$, thereby belonging to the boundary as $(v,-1) \in (V \setminus \varphi(S^{\lambda-1} \times o)) \times D^1$, $v \notin \varphi(S^{\lambda-1} \times o)$ is identified to $v \in V$ (according to (4.19)). Thus r_1 is a retraction from $W \setminus L_\lambda$ to $V \subset V \cup C$.

For $(\vec{x},\vec{y}) \in L_\lambda$, the retraction r_t is defined

$$r_t(\vec{x},\vec{y}) = \begin{cases} (\vec{x},\vec{y}), & |\vec{y}| \le 1/10, \\ (\rho^{-1}\vec{x}, \rho\vec{y}), & |\vec{y}| \ge 1/10, \end{cases}$$

where $\rho = \rho(\vec{x},\vec{y},t)$ is the maximum of $1/10|\vec{y}|$ and the real positive solution ρ' of the equation

$$-\frac{\vec{x}^2}{\rho'^2} + \rho'^2 \vec{y}^2 = (-\vec{x}^2+\vec{y}^2)(1-t)-t$$

Taking into account that $-\vec{x}^2 + \vec{y}^2 \in [-1,1]$ (as $(\vec{x},\vec{y}) \in L_\lambda$), such considerations as those made above for the function $c'(c,t)$ lead to the remark that $r_t : W \to W$. The solution ρ' has the expression:

$$\rho'^2 = \frac{(1-t)(-\vec{x}^2+\vec{y}^2)-t + \sqrt{4\vec{x}^2\vec{y}^2 + [(1-t)(-\vec{x}^2+\vec{y}^2)-t]^2}}{2\vec{y}^2}$$

hence it is continuously dependent on t.

One can readily see that $r_o(\vec{x},\vec{y}) = (\vec{x},\vec{y})$ for $(\vec{x},\vec{y}) \in L_\lambda$. It remains to be proved that r_1 is a retraction from L_λ to $V \cup C$. In case $t = 1$, ρ' has the values

$$\rho'^2 = \frac{-1 + \sqrt{1+4\vec{x}^2\vec{y}^2}}{2\vec{y}^2} .$$

If $\rho = 1/(10|\vec{y}|)$, then the point $(\vec{x},\vec{y})$ is send to the point $(10|\vec{y}|\vec{x}, \vec{y}/(10|\vec{y}|))$ whose $\vec{y}$-type variable module is equal to 1/10, i.e. it belongs to the boundary C. If $\rho' < \rho$, then the point $(\vec{x},\vec{y})$ is send to a point on the boundary V (with $|\vec{y}| \ge 1/10$), as then $-\vec{x}^2/\rho'^2 + \vec{y}^2\rho'^2 = -1$. Of course, for $|\vec{y}| \le 1/10$, $r_1(\vec{x},\vec{y}) = (\vec{x},\vec{y})$, that is the identity on C. In fact, r_1 carries points from L_λ into $V \cup C$, hence r_t is a retraction.

2. A mapping $r'_t : V \cup C \to V \cup D_L$ is constructed as in Fig. 10, taking into account three distinct cases. For <u>case 1</u> when $(\vec{x},\vec{y}) \notin C$, the point $(\vec{x},\vec{y}) \in V$ and $r'_t = 1$. In <u>case 2</u>, $|x|^2 \le 1$

and it is considered $r'_t(\vec{x},\vec{y}) = (\vec{x},(1-t)\vec{y})$, continuous in t and moreover, r'_t carries the domain $|x| \leq 1$, $|y| \leq 1/10$ into itself. It can be seen that $r'_1(\vec{x},\vec{y}) = (\vec{x},o) \in D_L$. In <u>case 3</u> $1 \leq |x| \leq \sqrt{101}/10$, where the value on the right-hand side of the last inequality corresponds to the point $|\vec{y}| = 1/10$, $-\vec{x}^2+\vec{y}^2 = -1$ in $L_\chi \cap V$.

Let r'_t be

$$r'_t(\vec{x},\vec{y}) = (\vec{x},\alpha\vec{y})$$

where

$$\alpha = (1-t) + t\left[(|\vec{x}|^2 - 1)/|\vec{y}|^2\right]^{1/2}$$

Then

$$r'_t(\vec{x},\vec{y}) = (1-t)\vec{y} + t\left[|x|^2 - 1|\right]^{1/2} \vec{y}/|\vec{y}|,$$

and since $0 \leq t \leq 1$, then $r'_t(\vec{x},\vec{y}) = (1-t)|\vec{y}| + (|\vec{x}|^2 - 1)^{1/2} \leq$

$\leq (1-t)/10 + t/10 = 1/10$, as $|\vec{x}| \geq 1$ and $|\vec{y}| \leq 1/10$. Hence $|r'_t(\vec{x},\vec{y})| \leq 1/10$.

It can be noted that if $|\vec{x}|^2 = 1$, case 3 coincides with case 2. If we have $y \to 0$ and obviously $|\vec{x}|^2 - 1 \to 0$ to keeps us in case 3, then $r'_t(\vec{x},\vec{y}) \to (1-t)\vec{y} \to 0$, as $\vec{y}/|\vec{y}|$ is finite, but $\vec{x}^2 - 1 \to 0$. Hence this type of points are sent to points in $S_L = \partial D_L$.

It is seen that $r'_0 = 1|_{V \cup C}$ and

$$r'_1(\vec{x},\vec{y}) = (\vec{x}, |\vec{y}|^{-1}\vec{y}(|\vec{x}|^2 - 1)^{1/2}),$$

that is $r'_1(\vec{x},\vec{y}) \in V$. Finally it has been proved that r'_t is a deformation retraction $V \cup C \to V \cup D_L$. But $V \cap D_L = S_L$, hence $V \cup D_L$ means precisely attaching the cell D_L to the manifold V by the mapping of identifying S_L with the corresponding manifold in V.

The following corollary is a direct application of this theorem (see J.Milnor, Theorem 3.2., Ch.1, /15/).

Corollary 4.18. Let $f : M \to R$ be a differentiable function and p a nondegenerate critical point of index λ. Suppose that the set $f^{-1}[c-\varepsilon, c+\varepsilon]$, where $c = f(p)$, is a compact set which does not contain any critical points of the function f, different from p for a given number ε. Then for any $\varepsilon > 0$ sufficiently small, the set $M^{c+\varepsilon}$ has the homotopy type $M^{c-\varepsilon}$ with attached cell of dimension λ.

Proof. Under the corollary conditions, we have the triad $(W;V,V')$ where $W = f^{-1}[c-\varepsilon, c+\varepsilon]$, $V = f^{-1}(c-\varepsilon)$, $V' = f^{-1}(c+\varepsilon)$. It has been established that the set $V \cup D_L$ is the deformation retract of the manifold W, that is there exists a continuous function $r_t : W \to W$, with $r_0 = 1_W$, $r_1 : W \to D_L \cup V$, where r_1 is retraction. Let $M^{c+\varepsilon} = W \cup M^{c-\varepsilon}$. We are defining the deformation retraction $r'_t : M^{c+\varepsilon} \to M^{c+\varepsilon}$ by putting $r'_t|_W = r_t$, $r'_t|_{M^{c-\varepsilon}} = 1_{M^{c-\varepsilon}}$. It is clear that r'_t is the deformation retraction of $M^{c+\varepsilon}$ into $M^{c-\varepsilon} \cup D_L$, for r'_t is a continuous function in t, $r'_0 = 1_{M^{c+\varepsilon}}$, $r'_1 : M^{c+\varepsilon} \to M^{c-\varepsilon} \cup D_L$ is a retraction.

Example (according to /15/).

Let us return to the example in Ch.2 with the torus and the height function. We have 5 situations as follows:

1. $a < 0$
2. $a \in (f(p), f(q))$,
3. $a \in (f(q), f(r))$,
4. $a \in (f(r), f(s))$,
5. $a > f(s)$.

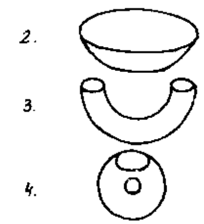


Fig.11

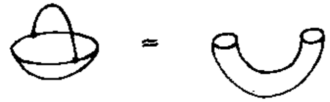
In case 1. $M^a = \emptyset$;

2. $M^a \sim$ two dimensional cell,
3. $M^a \sim$ cylinder,
4. $M^a \sim$ compact manifold of genus one whose boundary is a circle,
5. $M^a \sim$ the whole torus.

The sign ∿ as employed above designates the homeo - morphism.

It is found that passing from one part of the critical level to the other is indeed equivalent to attaching to the manifold under the critical level a cell whose dimension equals the index of the critical point, i.e. the number of negative signs in the expression of f, the height function on the torus in the neighborhood of the critical point.

The passage 1 → 2 is thus equivalent to attaching a 0 - cell,



the passage 3 → 4 is equivalent to attaching a 1 - cell,

while the passage 4 → 5 is equivalent to attaching a 2 - cell.

At the same time, the torus correspondent of the curves in L_λ for $\lambda = 1$ (behaviour around point q) are presented below

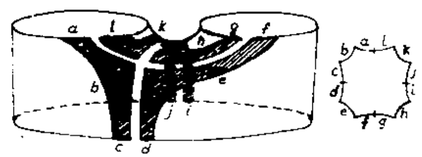


Fig. 12

Note 4.19. Theorem 4.17 may be generalized for the case in which the Morse function f has k critical points at the same level, $p_1,\dots,p_k$, each of index $\lambda_1,\dots,\lambda_k$ respectively. Taking a gradient like field for the function f, the characteristic dis-

joint embeddings $\varphi_i = S^{\lambda_i-1} \times D^{n-\lambda_i} \to V$, $i = 1,\dots,k$, are obtained. A manifold $\omega(V;\varphi_1,\dots,\varphi_k)$ is constructed as follows: Starting with the disjoint union $(V \setminus \bigcup_{i=1}^{k} \varphi_i(S^{\lambda_i-1} \times 0)) \times D^1 + L_{\lambda_1} + \dots + L_{\lambda_k}$, an equivalence relation as in Theorem 4.17, is introduced identifying for each $u \in S^{\lambda_i-1}$, $v \in S^{n-\lambda_i-1}$, $0 < \theta < 1$, $c \in D^1$, the point $(\varphi_i(u,\theta v),c)$ in the first summand with the point $(\bar{x},\bar{y}) \in L_\lambda$ that satisfied conditions 1 and 2 of Theorem 4.15. Then W is shown to be diffeomorphic with $\omega(V;\varphi_1,\dots,\varphi_k)$. It follows as in Theorem 4.17 that the set $V \cup D_1 \cup \dots \cup D_k$ is the deformation retract of W where D_i is the left-hand disk of the point p_i, $i = 1,\dots,k$.

5. *MORSE INEQUALITIES*

For the sake of consistency a few definitions will be firstly recalled.

Let M be a differentiable manifold . Let $L_q(M)$ be the free Abelian group generated by the set of a q-dimensional singular simplexes of the manifold M, $q \in Z_+$. The continuous mapping of the standard simplex $\Delta^q \subset R^{q+1}$ into M is called a q-dimensional singular simplex of the manifold M (cf. /25/ p.106).

We recall that

$$\Delta^q = \{y^i \in R^{q+1},\ i \in \{0,1,\dots,q\}\ |0 \le y^i \le 1,\ \sum_{i=0}^{q} y^i = 1\}$$

is the standard simplex (cf. /20/, p.25).

The differential operator ∂ is defined as the homomorphism

$$\partial_q : L_q(M) \to L_{q-1}(M),$$

$$\partial_q = \sum_{i=0}^{q} (-1)^q \partial_{q-1,i},$$

where

$$\partial_{n,i} : \Delta^n \to \Delta^{n+1},$$

$$\partial_{n,i}(y^o,\dots,y^{i-1},y^i,y^{i+1},\dots,y^n) =$$

$$= (y^o,\dots,y^{i-1}, 0, y^i,\dots,y^n)$$

and the property $\partial_q \partial_{q+1} = 0$ is verified.

The set $L(M) = \{(L_q(M), \partial_q)\}_{q \in Z_+}$ forms a chain-complex. The elements of the group of cycles $Z_q(M) = \text{Ker}\ \partial_q$ and boundaries $B_q(M) = \text{Im}\ \partial_{q+1}$ are called singular cycles and singular boundaries respectively. The graded group $\mathscr{H}(M) = \{H_q(M)\}_{q \in Z_+}$, $H_q(M) = Z_q(M)/B_q(M)$, is called the singular homology group of the space M (cf. /25/, p.107).

Introducing the relative homology groups (cf. /25/, p.113) is quite important for the next steps. The following property (cf. /25/, p. 43 and 57) will be used.

Theorem of the exact homology sequence . To each short exact sequence of chain complexes

$$0 \to \mathscr{L}' \xrightarrow{i} \mathscr{L} \xrightarrow{j} \mathscr{L}'' \longrightarrow 0 \tag{5.1}$$

the exact sequence of Abelian groups is associated

$$\dots \longrightarrow H_q(\mathscr{L}') \xrightarrow{i_q^*} H_q(\mathscr{L}) \xrightarrow{j_q^*} H_q(\mathscr{L}'') \xrightarrow{\Delta_q} H_{q-1}(\mathscr{L}') \longrightarrow \dots \tag{5.2}$$

Here

$$\Delta_q \langle z''\rangle_q = \langle i_{q-1}^{-1} \circ \partial_q \circ j_q^{-1}(z'')\rangle_{q-1}$$

where $\langle z \rangle_q$ designates the homology class of the q-cycle $z \in Z_q(\mathscr{L})$ (modulo $B_q(z)$).

The standard definition of exact sequences has been employed: the sequence is exact provided in relation (5.1) Ker j = Im i (cf. /25/, p.42).

Now let $i : N \to M$ be the inclusion map of a topological subspace N into the space M. Let us apply this property while

taking $\mathscr{L}' = \{L_q(N)\}$, $\mathscr{L} = \{L_q(M)\}$, $\mathscr{L}'' = \{L_q(M)/L_q(N)\}$, the inclusion $i = \{i_q\}$, $i_q : L_q(N) \to L_q(M)$ and the cannonic projection $j_q : L_q(M) \to L_q(M)/L_q(N)$, $q \in Z_+$. One obtains the exact homology sequence

$$\dots \xrightarrow{\Delta_{q+1}} H_q(N) \xrightarrow{i_q^*} H_q(M) \xrightarrow{j_q^*} H_q(M,N) \xrightarrow{\Delta_q} H_{q-1}(N) \xrightarrow{i_{q-1}^*} \dots,$$

where $H_q(M,N)$ denotes the homology group of the factor complex $L_q(M)/L_q(N)$. $H_q(M,N)$ are called be relative homology groups associated to the pair (of topological spaces) M, N, $M \supset N$ (cf./25/ p.113).

If a chain complex $\mathscr{L} = \{L_q, \partial_q\}$ is given, let us assume it has the following properties : a) L_q is a finitely generated Abelian group for any $q \geq 0$; b) the sequence $\{L_q\}$ is null except for a finite number of indices $q \in Z_+$. For such a complex the *Euler-Poincaré characteristic* is defined as

$$\chi(\mathscr{L}) = \sum_{q \geq 0} (-1)^q \text{rank}\,(L_q) \tag{5.3}$$

The rank of the q-dimensional homology group $H_q(\mathscr{L})$, denoted $R_q(\mathscr{L})$, is called the q-dimensional Betti number of the chain complex $\mathscr{L}$.

We will recall here the notion of rank of a finitely generated Abelian group . As is known (/127,p.112), an Abelian group can be organized as a Z-module. Then the invariant factor theorem (/12/, p.209) provides the structure of the modules finitely generated over a principal ring. Let M be a module finitely generated over the principal ring P. Then,

i) There exist $m, n \in Z_+$, $m \leq n$ and $x_1,\dots,x_n \in M$ such that

$$M = Px_1 \oplus Px_2 \oplus \dots \oplus Px_m \oplus Px_{m+1} \oplus \dots \oplus Px_n,$$

where $\mu_{x_i} \neq 0$, $\mu_{x_i} \notin$ to the ring of unities $1 \leq i \leq m$, and

$\mu_{x_1} | \ldots | \mu_{x_m}$, and $\mu_{x_i} = 0$, $m < i \le n$, where μ_{x_i} is the order of element x_i and $a|b$ means that a divides b, a, b$\in$P.

ii) Numbers m and n are uniquely determined and so are the orders of x_i elements, $i = 1, \ldots, m$, except for a divisibility association.

Applying the theorem to finitely generated Abelian groups, the structure theorem of the latter is obtained. If a finitely generated Abelian group G is given, there exist the (uniquely determined) numbers $m, n, d_1, \ldots, d_m$, $m \le n$, such that

$$G \approx Z_{d_1} \oplus Z_{d_2} \oplus \ldots \oplus Z_{d_m} \oplus Z^{n-m},$$

$d_i > 1$, $1 \le i \le m$ and $d_1 | d_2 | \ldots | d_m$. The number n-m is called the rank of group G.

Now returning to the Euler-Poincaré characteristic (5.3), this has a property as expressed in

Lemma 5.1.

$$\chi(L) = \sum_{q \ge 0} (-1)^q \operatorname{rank} L_q = \sum_{q \ge 0} (-1)^q R_q. \quad (5.4)$$

Proof. From the exactness of the short sequence

$$0 \longrightarrow Z_q(L) \xrightarrow{i_q} L_q(L) \xrightarrow{\partial_q} B_{q-1}(L) \longrightarrow 0 \quad (5.5)$$

follows that

$$\operatorname{rank} L_q = \operatorname{rank} Z_q + \operatorname{rank} B_{q-1},$$

wherein, to simplify writing the dependence on L was omited .

But

$$H_q = Z_q / B_q$$

hence

$$\operatorname{rank} Z_q = \operatorname{rank} H_q + \operatorname{rank} B_q,$$

which implies that

$$\operatorname{rank} L_q = \operatorname{rank} B_{q-1} + \operatorname{rank} B_q + \operatorname{rank} H_q.$$

Multiplying by $(-1)^q$ and summing, the relation (5.4) is found.

We now move on to some specific properties of the Betti numbers associated to the pairs X, Y, $X \supset Y$.

The Euler-Poincaré characteristic for the pair X, Y is defined to be

$$\chi(X,Y) = \sum_{q \ge 0} (-1)^q R_q(X,Y) \quad (5.6)$$

where $R_q(X,Y)$ is the rank of the previously defined relative homology group.

Definition 5.2. Let S be a function that associates integers to each pair of spaces X, Y, $X \supset Y$. If $X \supset Y \supset Z$, the function is called <u>subadditive</u> if

$$S(X,Z) \le S(X,Y) + S(Y,Z) \quad (5.7)$$

Lemma 5.3. Let S be a subadditive function and $X_o \subset X_1 \ldots \subset X_n$.

Then

$$S(X_n, X_o) \le \sum_{i=1}^{n} S(X_i, X_{i-1}). \quad (5.8)$$

If S is additive, the inequalit above turns into equality.

Proof. The proof is made by induction. For n = 2 and $X_o \subset X_1 \subset X_2$

$$S(X_2, X_o) \le S(X_1, X_o) + S(X_2, X_1)$$

according to relation (5.7).

Let us assume that the result is true for n - 1 :

$$S(X_{n-1}, X_o) \le \sum_{i=1}^{n-1} S(X_i, X_{i-1}).$$

According to the definition of subadditivity for $X_o \subset X_{n-1} \subset X_n$

$$S(X_n, X_o) \le S(X_{n-1}, X_o) + S(X_n, X_{n-1}) \le \sum_{i=1}^{n} S(X_i, X_{i-1}),$$

i.e. relation (5.8).

If $X_o = \emptyset$, then $S(X,\emptyset) = S(X)$ and

$$S(X_n) \leq \sum_{i=1}^{n} S(X_i, X_{i-1}). \tag{5.9}$$

Let now the notation

$$S_q(X,Y) = \sum_{k=0}^{q} (-1)^k R_{q-k}(X,Y). \tag{5.10}$$

Proposition 5.4. The function χ (5.6) is additive, and the functions R_q, S_q (5.10) are subadditive.

Proof. The property of the exact complex sequence (5.1), (5.2) will be used. Let (X,Y,Z) be a triplet where $X \supset Y \supset Z$. Let $i : (Y,Z) \longrightarrow (X,Z)$, $j : (X,Z) \longrightarrow (X,Y)$ be the cannonic inclusions. Then, according to (5.2). the sequence

$$\ldots \xrightarrow{\Delta_{q+1}} H_q(Y,Z) \xrightarrow{i_q^*} H_q(X,Z) \xrightarrow{j_q^*} H_q(X,Y) \xrightarrow{\Delta_q} H_{q-1}(Y,Z) \xrightarrow{i_{q-1}^*} \ldots \tag{5.11}$$

is an exact sequence. For each $q \geq 0$, the exact short sequence is constructed

$$0 \longrightarrow \mathrm{Im}\, i_q^* \xrightarrow{\text{injection}} H_q(X,Z) \xrightarrow{j_q^*} \mathrm{Im}\, j_q^* \longrightarrow 0.$$

The relation

$$R_q(X,Z) = \mathrm{rank}\,(\mathrm{Im}\, i_q^*) + \mathrm{rank}\,(\mathrm{Im}\, j_q^*) \tag{5.12}$$

is true. But

$$\mathrm{rank}\,(\mathrm{Im}\, i_q^*) = R_q(Y,Z) - \mathrm{rank}\,(\mathrm{Ker}\, i_q^*) = R_q(Y,Z) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_{q+1}),$$

and replacing in (5.12) it follows that

$$R_q(X,Z) = R_q(Y,Z) + \mathrm{rank}\,(\mathrm{Im}\, j_q^*) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_{q+1}). \tag{5.13}$$

But relation (5.11) implies the equality

$$\mathrm{rank}\,(\mathrm{Im}\, j_q^*) = \mathrm{rank}\,(\mathrm{Ker}\, \Delta_q) = R_q(X,Y) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_q).$$

Replacing in (5.13), it is found that

$$R_q(X,Z) = R_q(Y,Z) + R_q(X,Y) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_{q+1}) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_q), \tag{5.14}$$

i.e. the function R_q is subadditive.

If in the relation (5.14) a multiplication by $(-1)^q$ and a summation up to a given q are performed and account is taken that $\mathrm{Im}\, \Delta_o = 0$, it is obtained that

$$S_q(X,Z) = S_q(X,Y) + S_q(Y,Z) - \mathrm{rank}\,(\mathrm{Im}\, \Delta_{q+1}).$$

If q is taken to be the largest number for which the chain $\{L_q\}\ q \in Z_+$ is nonzero, then the additivity of the Euler-Poincaré characteristic (5.6) according to Lemma 5.1. is obtained.

A statement will follow, which is in fact a corollary of Theorem 4.17. Let $(W;\ V,\ V')$, be an elementary cobordism (cf. Definition 4.12), i.e. the triad admitting a Morse function with a single critical point of index λ. Then the following property occurs.

Proposition 5.5. The relative homology group $H_q(W,V)$ (with integer coefficients) is isomorphic with the integers group in the $q = \lambda$ dimension and zero in any other cases ($q \neq \lambda$).

Remark. As in Remark 4.19, when several critical points at the same level are concerned, if $\lambda_1 = \ldots = \lambda_k = \lambda$, the corollary above shows that $H(W,V) \approx \bigoplus_{i=1}^{k} Z$ in the dimension λ and zero in the other cases.

Proof. (of Proposition 5.5). Theorem 4.17, according to which $V \cup D_L$ is the deformation retract of W, hence $H(W,V) \approx H(V \cup D_L, V)$. will be employed. Use is made of the excision property (cf. /25/, p.157), asserting that if $i : (X_1, Y_1) \to (X_2, Y_2)$ is an inclusion map and $X_1 \setminus Y_1 = X_2 \setminus Y_2$, then the induced map in the relative homology group

$$i^* : H(X_1,Y_1) \longrightarrow H(X_2,Y_2)$$

is an isomorphism. Here $Y_i \subset X_i$ and Y_i is the subspace of X_i, $i = 1,2$. In this case X_2, Y_2, X_1, Y_1 are identified with $V \cup D_L$, V, D_L and S_L respectively.

It is thus verified that $X_1 \setminus Y_1 = D_L \setminus S_L = \mathring{D}_L$ and $X_2 \setminus Y_2 = (V \cup D_L) \setminus V = (V \cup D_L) \cap CV = D_L \cap CV = \mathring{D}_L$. Hence

$$H(W,V) \approx H(D_L,S_L). \quad (5.15)$$

Then the proof of Proposition 5.5. is a consequence of the following two statements.

Proposition 5.6. The homology groups of the sphere $S^n (n \geq 1)$ are given by the formulae

$$H_q(S^n) \approx 0, \quad q \neq 0,n; \quad H_o(S^n) \approx H_n(S^n) \approx Z. \quad (5.16)$$

Proof. We will start with a remark (cf. /9/ ,p.184). Let $a \in S^n$ be a fixed point. Then $S^{n-1} = \{x \in S^n \langle a,x \rangle = 0\}$ is the Euclidean sphere in a Euclidean space orthogonal to a. Let $U \subset S^n$ be the open set given by the relation $U = \{x \in S^n | -\varepsilon < \langle a,x \rangle < \varepsilon\}$ for a fixed number $\varepsilon \in (0,1)$. It can be seen that S^{n-1} is the deformation retract of the set U. Indeed, let $i : S^{n-1}$ be the inclusion map and let $r : U \to S^{n-1}$, be the mapping , $r(x) = (x - a\langle x,a \rangle) / \| x - \langle x,a \rangle a \|$, $x \in U$. It follows that $\langle r(x),a \rangle = 0$ (since $\langle a,a \rangle = 1$ and $x - \langle x,a \rangle a \neq 0$ for $x \in U$). Moreover, $r \circ i = 1_{S^{n-1}}$. Let the mapping $r_t : [0,1] \times U \to U, r_t(x) = (x - t\langle x,a \rangle a) / \| x - ta\langle x,a \rangle \|$. r_t is a homotopy between $i \circ r$ and the identity map on U.

Let now S^n be the sphere embedded in the Euclidean space R^{n+1}. Since the sphere S^n is connected, $H_o(S^n) \approx Z$ (cf./20/,p.55).

Let now A and B be the sets $A = \{x \in S^n | \langle x,a \rangle > -\varepsilon\}$ and $B = \{x \in S^n | \langle x,a \rangle < \varepsilon\}$. Hence $S^n = A \cup B$, and $A \cap B$ contains S^{n-1} as a deformation retract according with the note above. Let the exact sequence of homology groups and chains (according to properties (5.1) and (5.2)) :

$$\ldots \to H_{p+1}(S^n) \to H_p(A \cap B) \to H_p(A) \oplus H_p(B) \longrightarrow H_p(S^n) \longrightarrow \ldots$$

As A and B contractible, the sequence of relations may be written:

$$\ldots \longrightarrow H_{p+1}(S^n) \to H_p(S^{n-1}) \to H_p(\text{point}) \oplus H_p(\text{point}) \to H_p(S^n) \to \ldots$$

It is obtained that

$$0 \to H_1(S^n) \to H_o(S^{n-1}) \to H_o(\text{point}) \oplus H_o(\text{point}) \to H_o(S^n) \to 0 \quad (5.17)$$

$$0 \to H_{p+1}(S^n) \approx H_p(S^{n-1}) \to 0 ; \quad p \geq 1, \quad (5.18)$$

and consequently,

$$0 = \dim H_1(S^n) - \dim H_o(S^{n-1}) + 2 \dim H_o(\text{point}) - \dim H_o(S^n).$$

Since S^{n-1} is connected if $n \geq 2$, and S^o consists of two points, we get

$$H_1(S^n) \approx \begin{cases} Z, & n = 1, \\ 0, & n > 1. \end{cases}$$

The second exact sequence (5.18) shows that $H_p(S^n) \approx H_1(S^{n-p+1})$ for $1 \leq p \leq n$ and relations (5.16) are obtained.

Proposition 5.7. The relative homology groups of the pair (D^n, S^{n-1}) are trivial except for the group $H_n(D^n,S^{n-1}) \approx Z$.

Proof. The contractibility of the D^n ball leads to $H_o(D^n) \approx Z$. Using proposition 5.6, the exact sequence is obtained

$$\xrightarrow{\Delta_{q+1}} H_q(S^{n-1}) \xrightarrow{i_q^*} H_q(D^n) \xrightarrow{j_q^*} H_q(D^n, S^{n-1}) \xrightarrow{\Delta_q} H_{q-1}(S^{n-1}) \xrightarrow{i_{q-1}^*}$$

$$\longrightarrow H_{q-1}(D^n) \ldots \xrightarrow{j_1^*} H_1(D^n, S^{n-1}) \xrightarrow{\Delta_1} H_o(S^{n-1}) \xrightarrow{i_o^*} H_o(D^n)$$

which for $q = n$ takes the form

$$0 \xrightarrow{j^*} H_n(D^n, S^{n-1}) \xrightarrow{\Delta} Z \xrightarrow{i^*} 0$$

hence

$$H_n(D^n, S^{n-1}) \approx Z$$

For $q \neq 0, n$, there exists the exact sequence

$$0 \xrightarrow{i^*} H_q(D^n, S^{n-1}) \xrightarrow{\Delta} 0,$$

and moreover $H_o(D^n, S^{n-1}) \approx 0$, as any chain of order zero from D^n is a boundary (to which a chain from S^{n-1} can be added).

Theorem 5.8. (Morse inequalities. /17/,p.143). Let C_λ be the number of critical points of index λ of a compact differentiable n-manifold M. Then, the following relations occur:

$$R_\lambda(M) \leq C_\lambda, \qquad (5.19)$$

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^\lambda R_\lambda(M) = \sum_{\lambda=0}^{n} (-1)^\lambda C_\lambda, \qquad (5.20)$$

$$S_\lambda(M) \leq \sum_{p=0}^{\lambda} (-1)^p C_{\lambda-p}. \qquad (5.21)$$

Proof. Let f be a differentiable function on the manifold M whose critical points are all isolated and nondegenerate. Let $a_1 < \ldots < a_k$ such that M^{a_i} contain exactly i critical points, $i = 1, \ldots, k$ and $M^{a_k} = M$. Then $M^{a_i} \approx M^{a_{i-1}} \cup_S D^{\lambda_i}$ (homotopy equivalent), according to Corollary 4.18. According to Proposition 5.5 the relation

$$H_{\lambda_i}(M^{a_i}, M^{a_{i-1}}) \approx \text{group of } Z \text{ coefficients}$$

is true. Then

$$C_\lambda = \sum_{i=1}^{k} R_\lambda(M^{a_i}, M^{a_{i-1}}). \qquad (5.22)$$

Applying Lemma 5.3 to the subadditive function R_λ (Proposition 5.4) and the sequence $\emptyset = M^{a_o} \subset M^{a_1} \ldots \subset M^{a_k} = M$, it is obtained

$$R_\lambda(M) \leq \sum_{i=1}^{k} R_\lambda(M^{a_i}, M^{a_{i-1}})$$

and according to relation (5.22) , it follows that

$$R_\lambda(M) \leq C_\lambda,$$

i.e. the Morse (weak) inequalities (5.19).

Taking into account the additivity of the Euler-Poincaré characteristic (Proposition 5.4), Lemma 5.1 and relation (5.22), it is found that

$$\chi(M) = \sum_{i=1}^{k} \chi(M^{a_i}, M^{a_{i-1}}) = \sum_{i=1}^{k} \sum_{q\geq 0} (-1)^q R_q(M^{a_i}, M^{a_{i-1}}) = \sum_{q=0}^{q} (-1)^q C_q.$$

The last relation of Theorem 5.8 can be proved by using the subadditivity of the function S_λ (Cf. Proposition 5.4). We have

$$S_\lambda(M) \leq \sum_{i=1}^{k} S_\lambda(M^{a_i}, M^{a_{i-1}}) = \sum_{i=1}^{k} \sum_{p=0}^{\lambda} (-1)^p R_{\lambda-p}(M^{a_i}, M^{a_{i-1}}) =$$

$$= \sum_{p=0}^{\lambda} (-1)^p \sum_{i=1}^{k} R_{\lambda-p}(M^{a_i}, M^{a_{i-1}}) = \sum_{p=0}^{\lambda} (-1)^p C_{\lambda-p}.$$

Remark If written explicitly for different values of λ, the inequality (5.21) leads to the following sequence of inequalities

$$C_o \geq R_o,$$

$$C_o - C_1 \leq R_o - R_1 ,$$

$$C_o - C_1 + C_2 \geq R_o - R_1 + R_2, \qquad (5.23)$$

$$\dots$$

$$C_o - C_1 + \dots + (-1)^n C_n = R_o - R_1 + \dots + (-1)^n R_n.$$

The last relation in sequence (5.23) is precisely equality (5.20). The above inequalities are called strong Morse inequalities.

Writting two consecutive Morse inequalities type (5.21)

$$R_\lambda - R_{\lambda-1} + \dots \pm R_o \leq C_\lambda - C_{\lambda-1} \dots \pm C_o,$$

$$R_{\lambda-1} - R_{\lambda-2} + \dots \mp R_o \leq C_{\lambda-1} - C_{\lambda-2} + \dots \mp C_o,$$

and summing the (weak) inequality (5.19) for $\lambda \leq n$ is obtained.

It is found that there exists at least $R_o + R_1 + \dots + R_n$ critical points on the manifold M. This is in a way the number of critical points which are topologically necessary. Morse (/17/, p.144) calls the numbers $Q_i = C_i - R_i$, $i = 1,\dots,n$, the number of critical points in excess of those topologically necessary . They verify the relations

$$Q_{i-1} + Q_{i+1} \geq Q_i, \qquad i = 1,\dots,n-1.$$

For $\lambda \geq n$, by comparing two consecutive terms of inequality (5.21), the last equality of (5.23) is obtained.

Example. Let $C_{\lambda+1} = 0$, then $R_{\lambda+1} \leq C_{\lambda+1} = 0$. Comparing the expressions of inequalities (5.21) for $S_{\lambda+1}$ to that for S_λ, it is found that

$$R_\lambda - R_{\lambda-1} + \dots \pm R_o = C_\lambda - C_{\lambda-1} + \dots \pm C_o. \qquad (5.24)$$

If also $C_{\lambda-1} = 0$, then $R_{\lambda-1} = 0$ and analogously

$$C_{\lambda-2} - C_{\lambda-3} + \dots \pm C_o = R_{\lambda-2} - R_{\lambda-3} + \dots \pm R_o.$$

Comparing to equality (5.24), it follows that $R_\lambda = C_\lambda$.

Corollary 5.9. If $C_{\lambda+1} = C_{\lambda=1} = 0$, then $R_\lambda = C_\lambda$ and $R_{\lambda+1} = R_{\lambda-1} = 0$.

Milnor emphasizes in his book (/15/, p.31) that the result of this corollary can be used in order to determine the homology groups of the complex projective space. In the following Chapter, Corollary 5.9 will be employed to determine the homology groups of the complex Grassman manifold $G_p(C^n)$. The case of the complex projective space corresponds to the case $p = 1$. Thus in some particular cases, Corollary 5.9 allows us to avoid certain constructions resulting from general and more intricate theorems (see /15/, p.20, Theorem 3.5).

6. *PERFECT MORSE FUNCTIONS ON COMPLEX GRASSMAN MANIFOLDS*

In this Chapter, the complex Grassman manifold $G_p(C^n)$ will be endowed with a differentiable structure (Proposition 6.2). A Morse function is constructed for which the Morse inequalities (5.19) are satisfied as equalities (Theorem 6.4). Using Corollary 5.9, the Betti numbers of the manifold $G_p(C^n)$ are found (Corollary 6.5). The Hartree-Fock problem in quantum mechanics is discussed in the end. We also show the physical signification of the quantum Hamiltonians for which the associated energy function is a perfect Morse function on the complex Grassman manifold (Proposition 6.6).

Definition 6.1. The complex Grassman manifold $G_p(C^n)$ is the set of linear subspaces of (complex) dimension p of C^n.

The case $p = 1$ corresponds to the complex projective space, while cases $p = 0$ and $p = n$ are trivial.

We will prove the following proposition.

Proposition 6.2. The space $G_p(C^n)$ admits a C^∞ manifold structure of dimension $2p(n-p)$, with respect to which it is diffeomorphic with the homogeneous space $U(n)/U(p) \times U(n-p)$.

$U(m)$ denotes the group of unitary $m \times m$ matrices.

Proof. It may be noted that $U(n)$ is a closed subspace in the Euclidean space C^{n^2}, identified to $\mathcal{M}(n)$. We admit the convention of topological identification of the complex Euclidean space C^m to the real Euclidean space R^{2m}. $U(n)$ is thus identified to a topological subspace of the space C^{n^2}. Now on the subspace $U(n)$ can be defined the topology induced from the space C^{n^2}. It follows that the topological space $U(n)$ is separate and has a countable base. Moreover, the space $U(m)$ is bounded ($Tr(UU^+) = n$, $U \in U(n)$) and closed, hence compact.

Let now $U(p) \times U(n-p)$ be the closed subgroup of the topological group $U(n)$ defined as

$$U(p) \times U(n-p) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \middle| \; U_1 \in U(p),\ U_2 \in U(n-p) \right\}. \qquad (6.1)$$

The homogeneous space in Proposition 6.2 will be denoted by

$$Q_{pn} = U(n)/U(p) \times U(n-p) \qquad (6.2)$$

Then, the topology of the space $U(n)$ generates a quotient space topology on the homogeneous space Q_{pn}. This topology is separate and of countable base, and the space Q_{pn} is compact.

A transitive continuous action $Q_{pn} \times U(n) \longrightarrow Q_{on}$, can be defined as $(\hat{U}, U') \longrightarrow \hat{U}U' = \widehat{UU'}$, where $U' \in U(n)$, while $\hat{U}$ is the coset

class of matrix $U \in U(n)$:

$$\hat{U} = \{U''U \mid U'' \in U(p) \times U(n-p)\} . \qquad (6.3)$$

Hence $U(n)$ acts as a group of homeomorphisms on the space Q_{pn}.

To introduce a differentiable structure on Q_{pn} in agreement with Definition 1.1, an atlas will be effectively constructed.

Let $\sigma : \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ be a permutation with the property that its restrictions to the subsets $\{1,2,\dots,p\}$ and $\{p+1,\dots,n\}$ are increasing. The σ permutations are called *Schubert symbols*. The set of C_n^p Schubert symbols is denoted by $S(p,n)$.

Let the notation:

$$\Delta^{\delta} = (\delta_{i\sigma(j)})_{1 \le i,j \le n} \in U(n), \quad \sigma \in S(p,n) . \qquad (6.4)$$

An open closure $\{\mathcal{V}_\sigma\}_{\sigma \in S(p,n)}$ of the space Q_{pn} is also introduced

$$\mathcal{V}_\sigma = \{\hat{U} \mid \hat{U} \in Q_{pn},\ U \in U(n),\ \det(U\Delta^\sigma)_{1 \le \alpha,\beta \le p} \ne 0\}. \qquad (6.5)$$

To see the correctness of the definition, it will be enough to note that in any unitary matrix the column vectors are linearly independent and that the set of matrices in $U(n)$ with nonzero minors consisting of the first p rows and p columns is open (any algebraic subset of a real Euclidean space is closed).

The homeomorphisms $h_\sigma : \mathcal{V}_\sigma \to \mathcal{M}(p,n-p)$ are also defined

$$h_\sigma(\hat{U}) = Z, \quad \hat{U} \in \mathcal{V}_\sigma, \quad Z \in \mathcal{M}(p,n-p), \quad \sigma \in S(p,n) \qquad (6.6)$$

where

$$U\Delta^\sigma = \begin{pmatrix} T & W \\ W' & T' \end{pmatrix} \in U(n), \quad Z = T^{-1}W \qquad (6.7)$$

and

$T \in \mathcal{T}(p)$, $T' \in \mathcal{T}(n-p)$; $W \in \mathcal{M}(p,n-p)$, $W' \in \mathcal{M}(n-p,p)$.

$\mathcal{M}(m,m')$ denotes the space of the $m \times m'$ complex matrices (identified to $C^{m \times m'}$) and $\mathcal{T}(m)$ the subspace of the complex triangular matrices whose diagonal elements are strictly positive, $\mathcal{T}(p) \subset \mathcal{M}(p)$.

To justify the correctness of the definition the following theorems of matrix decomposition (see /8/, p.50 and 239) will be recalled .

i) Let $H \in \mathcal{H}(m)$, where $\mathcal{H}(m)$ is the space of the Hermitian $m \times m$ matrices positively defined. Then , there exists a unique matrix $T \in \mathcal{T}(m)$ such that

$$H = TT^{+} \tag{6.8}$$

and the elements of matrix T are differentiable functions of the elements of matrix H.

ii) Let $A \in GL(m,C)$, where $GL(m,C)$ is the group of the non-singular complex $m \times m$ matrices. Then, there exist the unique matrices $U \in U(m)$ and $T \in \mathcal{T}(m)$ such that

$$A = UT \tag{6.9}$$

and the elements of matrices U and T are differentiable functions of the elements of matrix A.

Let $U' \in \hat{U} \in \mathcal{V}_\sigma$. From property ii), for $m = p$ and $m = n-p$, it follows that there is a unique matrix $U'' \in U(p) \times U(n-p)$ such that $U' = U''U$, where U has the form given in (6.7). From property i) for $m = p$ and $m = n-p$, and from relation $UU^{+} = I_n$, the elements of matrices T, T', W' are uniquely determined as differentiable functions of the elements of matrix W. Moreover, matrices T, W, W', T' do not depend on the chosen representative $U' \in \hat{U}$. It follows that the elements of matrix Z in relation (6.7) are uniquely determined for $\hat{U} \in \mathcal{V}_\sigma$, and besides they are differentiable functions of the elements of matrix W. Hence, the mapping h_σ is

correctly defined and injective.

On the other hand, if $Z \in \mathcal{M}(p,n-p)$, then by using decomposition ii) for $m = p$, the equations

$$VT = (I_p + ZZ^{+})^{-1/2}, \tag{6.10}$$

$$VW = (I_p + ZZ^{+})^{-1/2} Z, \tag{6.11}$$

have unique solutions $V \in U(p)$, $T \in \mathcal{T}(p)$, $W \in \mathcal{M}(p,n-p)$, with matrix elements differentiable functions of the elements of matrix Z and

$$TT^{+} + WW^{+} = I_p, \tag{6.12}$$

and $Z = T^{-1}W$. Using (6.12), the matrix U is (uniquely) obtained from (6.7), having as elements differentiable functions of the elements of matrix Z. It follows that the mapping h_σ is surjective. In conclusion, h_σ is a bijection and a homeomorphism as all matrix parametrizations employed are continuous(even differentiable).

Let now the Schubert symbols σ, $\tau \in S(p,n)$ and $\hat{U} \in \mathcal{V}_\sigma \cap \mathcal{V}_\tau \neq \emptyset$. Let $Z = h_\sigma(\hat{U})$ and $Z' = h_\tau(\hat{U})$. From the differentiability of the matrix parametrizations earlier considered, it results that the elements of matrix Z' are differentiable functions of the elements of matrix Z and conversely. In fact, the family $\{(\mathcal{V}_\sigma, h_\sigma)\}_{\sigma \in S(p,n)}$ satisfies conditions 1.,2.,3. of Definition 1.1. and can be completed up to an atlas which determines on the space Q_{pn} a structure of differentiable manifold of dimension $2p(n-p)$.

To end the proof of the Proposition , it will be enough that a bijection of the homogeneous space Q_{pn} be established onto the Grassman manifold $G_p(C^n)$.

For each matrix $U = (u_{ij})_{1 \le i,j \le n} \in U(n)$, the vectors

$$u_i = (u_{i_1}, \dots, u_{i_n}) \in C^{n^2} \qquad 1 \le i \le n, \tag{6.13}$$

and the complex vector space $\langle u_1, \dots, u_p \rangle$ of orthonormal base $(u_1, \dots, u_p)$ are introduced.

The mapping $\mu : Q_{pn} \to G_p(C^n)$ is also introduced

$$\mu(\hat{U}) = \langle u_1, \dots, u_p \rangle, \quad \hat{U} \in Q_{pn}. \tag{6.14}$$

Mapping μ is defined correctly. Indeed, if the matrix $U \in \hat{U}$, then there exists a matrix $U'' \in U(p) \times U(n-p)$ such that $U' = U''U$. Hence,

$$u'_\alpha = \sum_{\beta=1}^{p} u''_{\alpha\beta} u_\beta \quad ; \quad 1 \le \alpha \le p, \tag{6.15}$$

where the matrix $U'' = (u''_{\alpha\beta})_{1 \le \alpha, \beta \le p} \in U(p)$. It follows that

$$\langle u'_1, \dots, u'_p \rangle = \langle u_1, \dots, u_p \rangle. \tag{6.16}$$

Mapping μ is injective. Indeed, let the matrices U, $U' \in U(n)$ such that $\mu(\hat{U}) = \mu(\hat{U}')$. Then

$$U'' = U' U^{-1} \in U(n)$$

$$u'_\alpha = \sum_{i=1}^{n} u''_{\alpha i} u_i \in \langle u_1, \dots, u_p \rangle \; ; \quad 1 \le \alpha \le p, \tag{6.17}$$

and it follows that $u''_{\alpha k} = u''_{k\alpha} = 0$, $1 \le \alpha \le p$; $p + 1 \le k \le n$. Hence $U'' \in U(p) \times U(n-p)$ and $\hat{U} = \hat{U}'$.

Mapping μ is surjective. Indeed, for each vector space $P \in G_p(C^n)$ there exists an orthonormal basis $(u_1, \dots, u_p)$ which can be completed up to an orthonormal basis $(u_1, \dots, u_n)$ of the vector space C^n. Then $P = \mu(\hat{U})$, where $U = (u_{ij})_{1 \le i,j \le n} \in U(n)$.

In conclusion, mapping μ is bijective. A topological space structure is introduced on $G_p(C^n)$, for which the set $U \subset G_p(C^n)$ is an open set, if and only if $\mu^{-1}(U) \subset Q_{pn}$ is an open set. Then μ is a homeomorphism of the space Q_{pn} onto the Grassman manifold $G_p(C^n)$.

In the following , a perfect Morse function will be explicitly constructed on the manifold $G_p(C^n)$. As a consequence, the Betti numbers of the manifold $G_p(C^n)$ will be found by using Morse theory itself.

Definition 6.3. A Morse function for which the Morse inequalities (5.19) are satisfied with the equal sign is called a *perfect Morse function.*

Further information on Morse functions construction on Grassman manifold is to be found in /24/.

Let now the numbers $c_i \in R$, $i = 1, \dots, n$, such that

$$c_1 < c_2 < \dots < c_n . \tag{6.18}$$

Let us introduce the function

$$f : G_p(C^n) \longrightarrow R, \tag{6.19}$$

$$f|_{\mu}(\mathcal{V}_\sigma) = f_\sigma \circ \mu^{-1}, \quad f_\sigma : \mathcal{V}_\sigma \to R, \quad \sigma \in S(p,n);$$

$$f_\sigma(\hat{U}) = \sum_{i=1}^{n} c_i \sum_{\alpha=1}^{p} |u_{\alpha i}|^2 , \tag{6.20}$$

where $U = (u_{ij})_{1 \le i,j \le n} \in U(n)$ and $\hat{U} \in \mathcal{V}_\sigma$.

We shall prove the following theorem.

Theorem 6.4. Function (6.19), (6.20) is a perfect Morse function on the Grassman manifold $G_p(C^n)$.

Proof. Let us introduce the diagonal matrices $A \in \mathcal{M}(p)$, $B \in \mathcal{M}(n-p)$:

$$a_{\alpha\beta} = c_{\sigma(\alpha)} \delta_{\alpha\beta} \quad ; \quad 1 \le \alpha, \beta \le p,$$

$$b_{kh} = c_{\sigma(k)} \delta_{kh}; \qquad p+1 \le k, h \le n. \tag{6.21}$$

Since the function f_σ is invariant under the action of the group $U(p)$,

$$f_\sigma(V\hat{U}) = f_\sigma(\hat{U}); \quad \hat{U} \in \mathcal{V}_\sigma , \quad V \in U(p),$$

the matrix U can be parametrized in the form (6.7) and relation (6.20) becomes

$$f_\sigma(\hat{U}) = \mathrm{Tr}(TAT^+ + WBW^+). \qquad (6.22)$$

Using eqs. (6.10) and (6.11), an equivalent expression is obtained for Definition (6.20)

$$f_\sigma(\hat{U}) = \mathrm{Tr}\left[(A + ZBZ^+)(I_p + ZZ^+)^{-1}\right]. \qquad (6.23)$$

Hence it follows that $\tilde{f}_\sigma = f_\sigma \circ h_\sigma^{-1} : C^{p(n-p)} \to R$ is a differentiable function correctly defined on the manifold $G_p(C^n)$.

From eq. (6.23), it is found that

$$\frac{\partial \tilde{f}_\sigma}{\partial z_{\alpha k}} = \overline{\frac{\partial \tilde{f}_\sigma}{\partial \bar{z}_{\alpha k}}} = [-Z^+(I_p+ZZ^+)^{-1}(A+ZBZ^+)(I_p+ZZ^+)^{-1}+BZ^+(I_p+ZZ^+)^{-1}]_{k\alpha}$$

$$1 \leq \alpha \leq p < k \leq n. \qquad (6.24)$$

The critical point conditions

$$\frac{\partial \tilde{f}_\sigma}{\partial z_{\alpha k}} = 0, \qquad 1 \leq \alpha \leq p < k \leq n \qquad (6.25)$$

imply that

$$BZ^+ = Z^+(I_p + ZZ^+)^{-1}(A + ZBZ^+). \qquad (6.26)$$

By multiplying to the left both members of equality (6.26) by $(I_p+ZZ^+)Z$, it is obtained

$$BZ^+ = ZZ^+A. \qquad (6.27)$$

Introducing the value of the expression given by (6.27) into the equation (6.24) and taking into account condition (6.25) the following equations are found

$$(c_{\sigma(k)} - c_{\sigma(\alpha)})\bar{z}_{\alpha k} = 0, \qquad 1 \leq \alpha \leq p < k \leq n. \qquad (6.28)$$

But condition (6.18) implies that $c_{\sigma(k)} \neq c_{\sigma(\alpha)}$, if $1 \leq \alpha \leq p < k \leq n$, hence $Z = 0$ is the unique solution of system (6.28). By using (6.7), (6.10), (6.11), one gets $h_\sigma^{-1}(0) = (\hat{\Delta}^\sigma)$. It follows that

the only critical point of the function $f|_{\mu(\mathcal{V}_\sigma)}$ is $P_\sigma = \mu(\Delta^\sigma)$. Hence the function f admits C_n^p critical points P_σ, $\sigma \in S(p,n)$.

From (6.24) one readily obtains

$$\left.\frac{\partial^2 \tilde{f}_\sigma}{\partial z_{\alpha k}\, \partial \bar{z}_{\beta h}}\right|_{Z=0} = \delta_{hk}\, \delta_{\alpha\beta}(c_{\sigma(k)} - c_{\sigma(\alpha)}),$$

$$\left.\frac{\partial^2 \tilde{f}_\sigma}{\partial z_{\alpha k}\, \partial z_{\beta h}}\right|_{Z=0} = \left.\frac{\partial^2 \tilde{f}_\sigma}{\partial \bar{z}_{\alpha k}\, \partial \bar{z}_{\beta h}}\right|_{Z=0} = 0; \quad 1 \leq \alpha, \beta \leq p < h, k \leq n. \qquad (6.29)$$

Hence the Hessian matrix of the function $\tilde{f}_\sigma$ at the point $Z=0$ is diagonal and non-singular, and admits double degenerate eigenvalues $c_{\sigma(k)} - c_{\sigma(\alpha)}$; $1 \leq \alpha \leq p < k \leq n$. As a consequence, each $P_\sigma \in G_p(C^n), \sigma \in S(p,n)$, is a nondegenerate critical point of index

$$\lambda(\sigma) = 2\ \mathrm{card}\{(\alpha,k) \mid 1 \leq \alpha \leq p < k \leq n;\ \sigma(\alpha) > \sigma(k)\} \qquad (6.30)$$

From the preceding theorem, one readily infers the following consequence.

Corollary 6.5. The Betti numbers of the Grassman manifold $G_p(C^n)$ are

$$R_{2\lambda+1} = 0, \quad \lambda = 0,1,\ldots,p(n-p)-1,$$

$$R_{2\lambda} = \rho_\lambda(p,n), \quad \lambda = 0,1,\ldots,p(n-p), \qquad (6.31)$$

$$\rho_\lambda(p,n) = \mathrm{card}\{(\omega(1),\ldots,\omega(p)) \in (Z_+)^p \mid 0 \leq \omega(1) \leq \ldots \leq \omega(p) \leq n-p,$$
$$\omega(1) + \ldots + \omega(p) = \lambda\}.$$

Moreover

$$\chi(G_p(C^n)) = \sum_{\lambda=0}^{p(n-p)} R_{2\lambda} = C_n^p. \qquad (6.32)$$

Proof. Using the notations in the proof of Theorem 6.4 and relation (6.30) it is found that for any Schubert symbol $\sigma \in S(p,n)$, the index of the function f for the critical point P_σ is

$$\frac{1}{2}\lambda(\sigma) = \sigma(1) - 1 + \sigma(2) - 2 + \ldots + \sigma(p) - p, \qquad (6.33)$$

where obviously

$$0 \leq \sigma(1) - 1 \leq \ldots \leq \sigma(p) - p \leq n-p. \qquad (6.34)$$

The number of critical points of index 2μ of the function f is denoted by $C_{2\mu}$ and is equal to the number of partitions $\rho_\mu(n,p)$ of the number μ. Indeed, denoting

$$\omega(\alpha) = \sigma(\alpha) - \alpha, \qquad 1 \leq \alpha \leq p \qquad (6.35)$$

and using (6.33) with $2\mu = \lambda(\sigma)$ and (6.34), it is found that the numbers $\omega(\alpha)$ form an increasing sequence of integers

$$0 \leq \omega(1) \leq \omega(2) \leq \ldots \leq \omega(p) \leq n-p \qquad (6.36)$$

and $\rho_\mu(p,n)$ has the form from (6.31).

It can also be noted that for the function (6.19), (6.20)

$$C_{2\mu-1} = C_{2\mu+1} = 0. \qquad (6.37)$$

Then Corollary 5.9 ensures the equalities

$$C_{2\mu} = R_{2\mu} = \rho_\mu(p,n)$$
$$R_{2\mu+1} = R_{2\mu-1} = 0. \qquad (6.38)$$

Relation (6.32) results from equations (5.4), (6.38) and from counting all Schubert symbols.

An application of Morse theory to quantum mechanics will be described in the following. We will confine our presentation to the complex Grassman manifolds. For the sake of completeness, we shall first recall some notions on fermion algebra (see /13/

and references mentioned).

Following quantum mechanics notations, we shall introduce an associative algebra $\mathcal{A}(n)$ over C with the unity element e. Let

$$a^+_p,\ a_p,\ e; \qquad p = 1,\ldots,n, \qquad (6.39)$$

the algebra generators verifying the anticommutation relations

$$\{a^+_p, a_q\} = e\delta_{pq}, \qquad p,q = 1,\ldots,n;$$
$$\{a^+_p, a^+_q\} = \{a_p, a_q\} = 0 \qquad (6.40)$$

where

$$\{a,b\} = ab + ba.$$

The basis of the algebra consists of the elements

$$e,\ a^+_{q_1}\ldots a^+_{q_s} a_{p_1}\ldots a_{p_r}, \qquad 1 \leq p_1 < p_2 < \ldots < p_r \leq n,$$

$$1 \leq q_1 < q_2 < \ldots q_s \leq n; \quad 1 \leq r, s \leq n.$$

Let us construct a complex Hilbert space $\mathcal{H}$ with the scalar product $\langle\ |\ \rangle$, for which the algebra $\mathcal{A}(n)$ is realized as a subalgebra of the algebra of the linear operators that act on $\mathcal{H}$, so that the following three conditions be satisfied:

1. There exists a nonzero vector $|0\rangle$ having the properties that

$$a_p|0\rangle = 0, \qquad 1 \leq p \leq n; \qquad (6.41)$$
$$\langle 0|0\rangle = 1$$

2. The operator a^+_p is the adjoint of the operator a_p:

$$(a_p)^+ = a^+_p, \quad 1 \leq p \leq n \qquad (6.42)$$

3. The space $\mathcal{H}$ is minimal with respect to properties 1.,2.

Then we have the decomposition of the space $\mathcal{H}$ into an orthogonal sum

$$\mathcal{H} = \bigoplus_{r=0}^{n} \mathcal{H}_r, \tag{6.43}$$

where $\mathcal{H}_r$ is the subspace whose basis is formed by the vectors

$$a^+_{p_1} a^+_{p_2} \cdots a^+_{p_r} |0\rangle , \quad 1 \le p_1 < p_2 < \dots p_r \le n, \quad 0 < r \le n \tag{6.44}$$

and $|0\rangle$ for $r = 0$.

The space $\mathcal{H}$ is called the *fermion Fock space*. This realization of the $\mathcal{A}(n)$ algebra is called a fermion realization, and operators a^+_p, a_p are called fermion operators of creation and annihilation.

The vector $|0\rangle$ is called the *vacuum vector*, while the vectors that belong to the space $\mathcal{H}_r$ are called r-particle vectors. Another realization of the algebra $\mathcal{A}(n)$ is its realization as exterior algebra.

The operators

$$C_{pq} = a^+_p a_q, \quad p, q = 1,\dots,n \tag{6.45}$$

are introduced.

From the anticommutation relations (6.40) for the creation and annihilation operators, the following commutation relations are obtained

$$[C_{pq}, C_{rs}] = \delta_{qr} C_{ps} - \delta_{ps} C_{rq}; \quad 1 \le p,q,r,s, \le n. \tag{6.46}$$

These commutation relations are specific to the algebra $u(n)$ of the unitary group $U(n)$ (see /11/, p.446).

Let us denote

$$X = \sum_{p,q=1}^{n} x_{pq} C_{pq}, \quad x_{pq} = - \bar{x}_{qp}, \quad x_{pq} \in C, \; 1 \le p,q \le n. \tag{6.47}$$

The set $u'(n)$ of all operators X of the type (6.47) forms an algebra of antihermitian operators which is isomorphic to the Lie algebra $u(n)$ of the group $U(n)$.

Since the space $\mathcal{H}$ is dimensionally finite, one may obtain a representation ρ of the group GL (n,C) onto the space $\mathcal{H}$, $\rho : GL(n,C) \to \mathcal{A}(n)$, and a representation ρ' of the Lie algebra $gl(n,C)$ of the group $GL(n,C)$, $\rho' : gl(n,C) \to \mathcal{A}(n)$:

$$\rho(e^{X'}) = \exp X, \qquad \rho'(X') = X,$$

$$X' = \sum_{p,q=1}^{n} x_{pq} E_{pq}, \quad X = \sum_{p,q=1}^{n} x_{pq} C_{pq}, \tag{6.48}$$

where $x_{pq} \in C$, $1 \le p,q \le n$, and the matrices $E_{pq} \in \mathcal{M}(n)$ are defined by

$$(E_{pq})_{ij} = \delta_{ip} \delta_{jp}, \quad 1 \le i,j \le n.$$

The restriction of the representation ρ to the group $U(n)$ is an unitary representation with $\rho'(u(n)) = u'(n)$. Of course $\rho'(E_{pq}) = C_{pq}$ and ρ is an isomorphism.

The restriction of the representation $\rho(U(n))$ to each $\mathcal{H}_r$ is irreducible.

We choose a subspace $\mathcal{H}_A$, $A \le n$ of $\mathcal{H}$ and let the vector

$$|\psi_\sigma\rangle = \prod_{i=1}^{A} a^+_{\sigma(i)} |0\rangle \in \mathcal{H}_A, \quad \sigma \in S(A,n). \tag{6.49}$$

We note that the anticommutation relations (6.40) and the definition (6.49) of the vector $|\psi_\sigma\rangle$, denoted by $|\psi_o\rangle$ when σ is the identical permutation, lead to the following relations

$$C_{ij} |\psi_o\rangle = |\psi_o\rangle \, \delta_{ij},$$

$$C_{im}|\psi_o\rangle = 0, \qquad 1 \leq i,j \leq A,$$
$$C_{mi}|\psi_o\rangle \neq 0, \qquad A+1 \leq m,m' \leq n, \tag{6.50}$$
$$C_{mm'}\,\psi_o\rangle = 0.$$

It appears that the unidimensional linear subspace generated by $|\psi_o\rangle$ is invariant under the action of the operators C_{ij} and $C_{mm'}$, $(1 \leq i, j \leq A;\ A+1 \leq m,m' \leq n)$, hence under the group $U(A) \times U(n-A)$.

One considers the orbit of the group $\rho(U(n))$

$$\mathcal{O} = \rho(U(n))|\psi_o\rangle \subset \mathcal{H}_A. \tag{6.51}$$

An equivalence relation is introduced on $\mathcal{O}$: the vectors $|\phi\rangle$, $|\phi'\rangle \in \mathcal{O}$ are equivalent provided there exist $\lambda \in C, |\lambda|=1$, such that $|\phi'\rangle = \lambda|\phi\rangle$. Let $\tilde{\mathcal{O}}$ denote the quotient space of the orbit with respect to the above equivalence relation. The representation ρ is inducing a transitive action $\tilde{\rho}$ of the group $U(n)$ onto the quotient space $\tilde{\mathcal{O}}, \tilde{\rho}(U)|\tilde{\phi}\rangle = \widetilde{\rho(U)|\phi\rangle}$, $U \in U(n)$, $|\tilde{\phi}\rangle \in \tilde{\mathcal{O}}$ (cf. /11/, p. 121, 124). Observing that the stationary group of the state $|\tilde{\psi}_o\rangle$ under the action $\tilde{\rho}$ is isomorphic to the product $U(A) \times U(n-A)$ one obtains a one-to-one correspondence between the elements (states) of $\tilde{\mathcal{O}}$ and the points of the homogeneous space $Q_{An} = U(n)/U(A) \times U(n-A) \approx G_A(C^n)$, the latter relation resulting from Proposition 6.2. In this way, a differentiable manifold structure isomorphic to that of the manifold $G_A(C^n)$ is induced onto $\tilde{\mathcal{O}}$.

In this context, a differentiable manifold structure is induced onto the space $\tilde{\mathcal{O}}$ by the bijective mapping $\omega: Q_{An} \to \tilde{\mathcal{O}}$:

$$\omega \circ h_\sigma^{-1}(Z) = |\widetilde{Z,\sigma}\rangle$$
$$|Z,\sigma\rangle = \rho(\Delta^\sigma)\,|\underline{Z}\rangle,$$
$$|\underline{Z}\rangle = \langle Z|Z\rangle^{-1/2}\,|Z\rangle, \tag{6.52}$$
$$|Z\rangle = U(Z)\,|\psi\rangle,$$
$$U(Z) = \exp\Big(\sum_{i=1}^{A}\ \sum_{m=A+1}^{n} \bar{z}_{im}\, C_{mi}\Big),$$

where the elements of the matrix Z are local coordinates for $Z \in \mathcal{M}(A, n-A)$ and the neighborhood $\mu(\mathcal{V}_\sigma) \subset G_A(C^n)$ for each $\sigma \in S(A,n)$. This construction is justified observing that

$$\widetilde{\rho(U\Delta^{\sigma+})^+|\psi_o\rangle} = |\widetilde{Z,\sigma}\rangle,$$

where

$$U = U_1(Z)\ U_2^+(Y)\ U_2(Z),$$
$$U_1(Z) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$
$$U_2(Z) = \begin{pmatrix} I_A & Z \\ 0 & I_{n-A} \end{pmatrix}, \tag{6.53}$$
$$A = (I_A + ZZ^+)^{-1/2},$$
$$B = (I_{n-A} + Z^+Z)^{1/2},$$
$$Y^+ = -Z^+(I_A + ZZ^+)^{-1}.$$

From relation (6.53) it follows that $UU^+ = I_n$, hence $U \in U(n)$. Moreover, $\widehat{U\Delta^\sigma} = h_\sigma^{-1}(Z)$, as it follows from relations (6.7), (6.10) and (6.11) where $VT = A$, $VW = AZ$.

On the other hand, by using relations (6.50) it is found that

$$\rho\begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}|\psi_o\rangle = \det C|\psi_o\rangle; \quad C \in GL(A;C),\ D \in GL(n-A,C),$$
$$\rho(U_2(Z))|\psi_o\rangle = |\psi_o\rangle; \quad \rho(U_2^+(Z)) = U(Z). \tag{6.54}$$

From these relations, formulae (6.52) are obtained and besides, the relation

$$\langle Z' | Z \rangle = \det(I_A + Z'Z^{+}), \tag{6.55}$$

where $Z, Z' \in \mathcal{M}(A, n-A)$.

To each operator $H \in \mathcal{A}(n)$ a differentiable function f_H on the complex Grassman manifold is associated :

$$f_H : G_A(C^n) \to C,$$

$$f_H^{\sigma} |_{\mu(\mathcal{V}_\sigma)} = f_H \circ h_\sigma \circ \mu^{-1}, \quad \sigma \in S(A,n)$$

$$f_H^{\sigma} : \mathcal{M}(A, n-A) \to C, \tag{6.56}$$

$$f_H^{\sigma}(Z) = \langle Z|Z\rangle^{-1} \langle Z| \rho(\Delta^{\sigma+}) H \rho(\Delta^{\sigma}) | Z \rangle.$$

Of course,

$$f_H(P) = \langle Z,\sigma | H | Z,\sigma \rangle \ ; \ P = (\mu \circ h^{-1})(Z).$$

If H is a self-adjoint operator, then f_H is real. A (quantum) Hamiltonian H is considered , given by a polynomial self-adjoint operator in the operators C_{pq}. Such a model was first introduced by Jordan. f_H is called energy function and is interpreted, after Dirac's idea, as a classical Hamiltonian. If P is a critical point of the function f_H, then $\omega \circ \mu^{-1}(P)$ is called a *Hartree-Fock state*. The Hartree-Fock problem for the Hamiltonian H consists in studying the Hamiltonian function f_H on the phase space $G_A(C^n)$. A first stage of the study consists in determining the critical points and their index in order to study the topology and differentiability of the Hamiltonian trajectories. A first outcome of the problem is that the minimum number of Hartree-Fock states for an energy function with nondegenerate critical points is C_n^A (see /22/). This is a

direct consequence of earlier considerations on the Hartree-Fock states, corollary 6.5 , equation (6.32) and Morse inequelities (5.19).

We will prove in the following that perfect Morse functions are associated to Hamiltonian linear in C_{pq} and of non-degenerate spectrum (see /3/, /4/). By means of a unitary transformation such a Hamiltonian turns into a linear combination of operators C_{ii}, $i = 1,\dots,n$.

Proposition 6.6. The energy function f_H (6.56) associated to the Hamiltonian

$$H = \sum_{i=1}^{n} c_i C_{ii}, \quad c_1 < c_2 < \dots < c_n, \tag{6.57}$$

coincides to the perfect Morse function defined by relation (6.19) for $p = A$.

Proof. From relations (6.56) , (6.57) and the remark that

$$\rho(\Delta^{\sigma}) C_{pq} \rho(\Delta^{\sigma})^{+} = C_{\sigma(p)\sigma(q)}, \quad 1 \le p,q \le n,$$

it follows that

$$f_h^{\sigma}(Z) = \langle Z|Z\rangle^{-1} \langle Z| \sum_{i=1}^{n} c_{\sigma(i)} C_{ii} |Z\rangle \tag{6.58}$$

where the variable $Z \in \mathcal{M}(A, n-A)$.

The generating function $\varphi : R^n \to R$ is introduced

$$\varphi(x) = \langle Z| \exp\left(\sum_{i=1}^{n} x^i C_{ii}\right) |Z\rangle, \quad x = (x^1,\dots,x^n) \in R^n. \tag{6.59}$$

Using equations (6.53) and (6.55) it is obtained

$$\varphi(x) = \det(e^{X_1} + Z e^{X_2} Z^{+}). \tag{6.60}$$

Here the matrices $X_1 \in \mathcal{M}(A)$, $X_2 \in \mathcal{M}(n-A)$,

$$(X_1)_{ij} = x_i\, \delta_{ij}; \quad 1 \leq i,j \leq A,$$

$$(X_2)_{mm'} = x_m \delta_{mm'}; \quad A+1 \leq m,m' \leq n. \tag{6.61}$$

From the relation

$$\langle Z |\, C_{ii} \,|\, Z \rangle = -\frac{\partial \varphi(x)}{\partial x^i}\Big|_{x=0} \; ; \quad 1 \leq i \leq n, \tag{6.62}$$

it is found that

$$\langle Z \,| C_{ii} \,| Z \rangle = (I_A + ZZ^+)^{-1}_{ii} \; ; \quad 1 \leq i \leq A,$$

$$\langle Z \,| C_{mm} \,| Z \rangle = \left[Z^+ (I_A + ZZ^+)^{-1} Z\right]_{mm} \; ; \; A+1 \leq m \leq n \tag{6.63}$$

From equations (6.62) and (6.55) it follows that

$$f_H^{\sigma}(Z) = \tilde{f}_{\sigma}(Z).$$

where the function f_σ is defined by relation (6.23).

It can be seen that the critical points $\mu(\Delta^\sigma)$ of the function f_H are mapped by $\omega \circ \mu^{-1}$ into the Hartree-Fock states $|\widetilde{\psi_\sigma}\rangle$, $\sigma \in S(A,n)$.

BIBLIOGRAPHY

/1/ R.Abraham, J.Marsden, Fundations of mechanics 2nd ed., Addison-Wesley, Benjamin, Cummings, Reading, Mass, 1978.

/2/ M.I.Atiyah, R.Bott, Phil. Trans. Soc. London A308 , 523 1982.

/3/ S.Berceanu, A.Gheorghe, Perfect Morse functions on manifolds of coherent states, to be published.

/4/ S.Berceanu, A.Gheorghe, Perfect Morse functions on the manifold of Slater determinants, to be published.

/5/ F.Berezin, Comm. Math. Phys. 63, 131, 1978.

/6/ R.Bott, in : Differential Geometrical Methods in Mathematical Physics, Lectures Notes in Mathematics, Springer - Verlag, Berlin, New-York, 1980.

/7/ R.Bott, Bull. Amer. Math. Soc. 7, 331, 1982.

/8/ F.P.Grantmacher, Teoria matriţ 3, Moscova, 1966.

/9/ W.Greub , S.Halperin, R.Vanstone, De Rham Cohomology of Manifolds and Vector Bundles, Academic Press, New York, London, 1972.

/10/ V.Guillemin, S.Sterenberg, Geometric asymptotic A.M.S. Providence, R.I. 1977.

/11/ S.Helgasson, Differential Geometry, Lie Groups and symmetric spaces, Academic Press, 1978.

/12/ I.D.Ion, M.Radu, Algebra, Bucureşti, 1981.

/13/ P.Kramer, M.Sarceno, Geometry of time dependent variational principle in quantum mechanics, Lectures notes in Physics, Vol. 140, Springer Verlag, Berlin, New York, 1981.

/14/ J.Leray, Analyse lagrangienne et mécanique quantique, Séminnaire du Collège de France, 1976-1977, R.C.P. 25, Strasbourg, 1978.

/15/ J.Milnor, Morse Theory, Princeton, New Jersey, 1963

/16/ J.Milnor, Lectures on h-cobordism theorem , New Jersey, 1965.

/17/ M.Morse, The calculus of variations in the large, New York, 1934.

/18/ J.R.Munkers, Elementary differential topology, Princeton New Jersey, 1963.

/19/ S.P.Novikov, U.M.N. 37, 3, 1983.

/20/ C.S.Pontreaghin , Introduction to combinatorial topology , Bucharest, 1948.

/21/ M.M.Postnikov, Vvedenie v teoria Morsa, Moscva, 1971.

/22/ C.Rosensteel, E.Ihring, J.Math. Phys., 211 (8), 2297 1980.

/23/ S.Sterenberg, Lekţii po diferentialnoi geometrii, Moscva, 1970.

/24/ T.T.Sugawara, Mem. Fac. Sci.Kyushu Univ., A. Math., XXXIV, 107, 1980.

/25/ C.Teleman, Elementi topologhii i diferentiruemîe mnogobrazia, Moscva , 1967.

/26/ C.Teleman, Geometrie diferenţială, locală şi globală, Ed. Tehnică, Bucureşti, 1974.

/27/ E.Witten, J.Diff. Geom. 17, 661 (1982).