Journal of Theoretical Physics

Founded and Edited by M. Apostol **117** (2005)

ISSN 1453-4428

Moon's Problem
A three-body problem
(Lecture five of the Course of Theoretical Physics)

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

A series expansion in powers of eccentricities is set up for the motion in central-field potentials, which connects such movements to anharmonic oscillators. The method is used to rederive the solution of Kepler's problem, describe the motion of a particle in a general central-field potential, and discuss the closed orbits and the first sign of "chaos". Similarly, Moon's problem is tackled by the same method, where the motion turns out to be described by triple series in powers of eccentricities, inclination against the ecliptic and the gravitational interaction originating in Sun. Newton's results are thus rederived in the first-order of the perturbation theory, regarding the "four Moons" and four periodicities, as known as early as from the classical Greeks. The method can also be applied to the Jupiter-Saturn couple, where their mutual gravitational interaction may be viewed as a perturbation. The missing integrals of motion, Poincare's "weak chaos" and trajectory "strong-chaotical" instabilities are also introduced. Finally, a new route of quantizing the motion in central-field potentials is presented, as based on the eccentricities expansion.

Introduction. The Moon's motion on the sky was recorded from ancient times. The four periodicities associated with this motion were known cca 3000 years ago, with one second accuracy (which means five decimals), and are the first historical measurements in Natural Sciences. Newton's Natural Philosophy was obviously motivated by the Planetary System of Celestial Bodies, like Earth, rotating around the Sun, and, firstly, the Moon's motion as the only one amenable to the most accurate empirical observations. Mathematical Physics was born from the Celestial Mechanics, and the great mathematical tools developed by the mathematicians like Euler, Lagrange, Hamilton, Jacobi, etc, of the 18th and 19th centuries, were occasioned by the three-body problem like Sun, Earth and Moon. It was this problem where Poincare noticed particular signs of complex motion which became known later as "chaos". The three-body problem, ergodic hypothesis and anharmonic oscillators are the three basic pillars that illustrate a chaotical behaviour. In particular, endless trajectories that never repeat, sensitivity upon the initial conditions, non-linearity, instabilities, bifurcation, fractal dimensions, etc, are all typical of "chaotical problems". Apollo program of aselenization was based on such calculations of the three-body complex Sun-Earth-Moon, and modern computers brought new insights into such complex movements.

Kepler's problem. Kepler's problem is the description of the motion of a particle of mass m in the gravitational potential $-\alpha/r$, where $\alpha > 0$. Like any other central-field potential, the gravitational potential conserves the angular momentum $\mathbf{L}$, so the motion is confined to a plane, and

$$L = mr^2\dot{\varphi} \ , \tag{1}$$

where φ is the angular coordinate. Equation (1) shows that the motion sweeps equal areas in equal times (Kepler's second law).

The energy of the motion reads

$$E = m\dot{r}^2/2 + mr^2\dot{\varphi}^2/2 - \alpha/r = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r \ , \tag{2}$$

as if the particle moves in an effective potential

$$U = L^2/2mr^2 - \alpha/r \ , \tag{3}$$

exhibiting the centrifugal $1/r^2$- energy. The orbits proceed between $r_1 = a(1-e)$ and $r_2 = a(1+e)$, where $a = \alpha/2\left|E\right|$ and

$$e = \sqrt{1 - 2L^2\left|E\right|/m\alpha^2} \tag{4}$$

is the eccentricity, for negative energies above $E_{min} = -m\alpha^2/2L^2$.

The effective potential (3) reaches its minimum value E_{min} for

$$r_0 = L^2/m\alpha \ , \tag{5}$$

where the eccentricity vanishes and the orbit is circular with radius r_0. By (4), the energy can also be represented as

$$\left|E\right| = \frac{\alpha}{2r_0}(1-e^2) \ , \tag{6}$$

or $r_0 = a(1-e^2)$.

The expansion of the effective potential U given by (3) around its minimum value gives

$$U = -\frac{\alpha}{2r_0} + \frac{\alpha}{2r_0^3}(r-r_0)^2 - \frac{\alpha}{r_0^4}(r-r_0)^3 + ..., \tag{7}$$

i.e. a small-oscillations expansion valid for $|r-r_0| \ll r_0$.

It is convenient to write $r - r_0 = Au$, where u is dimensionless, and cast the energy given by (2), (3) and (7) into the form

$$E = -\alpha/2r_0 + mA^2\dot{u}^2/2 + (\alpha A^2/2r_0^3)u^2 - (\alpha A^3/r_0^4)u^3 + ..., \tag{8}$$

or

$$E = -\alpha/2r_0 + mA^2[\dot{u}^2/2 + \omega^2u^2/2 - (A/r_0)\omega^2u^3 + ...], \tag{9}$$

where $\omega^2 = \alpha/mr_0^3$ and $A/r_0 = \varepsilon$ can be viewed as a small perturbation parameter. Equation (9) can also be written as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2u^2/2 - \varepsilon\omega^2u^3 + ...) \ , \tag{10}$$

which tells that the eccentricity e is related to the perturbation parameter ε. Equation (9) leads to the motion

$$\ddot{u} + \omega^2u - 3\varepsilon\omega^2u^2 + ... = 0 \tag{11}$$

of an anharmonic oscillator. Within the harmonic approximation the solution of equation (11) can be represented as $u^{(0)} = -\cos\omega t$, and

$$r^{(0)} = r_0 - A\cos\omega t \ . \tag{12}$$

The amplitude A can be derived from energy $E = -\alpha/2r_0 + mA^2\omega^2/2$ given by (9) or, equivalently, from equation (10). It leads to

$$\varepsilon = A/r_0 = e \ll 1 \ , \qquad (13)$$

i.e. the eccentricity e of the orbit is the ratio ε of the amplitude A of the harmonic oscillation to the original orbit radius r_0. The small-oscillations treatment is valid for small eccentricities.

Therefore, the solution of the motion given by (12) can be written as

$$r^{(0)} = r_0(1 - e\cos\omega t) \ , \qquad (14)$$

and, by (1),[1]

$$\varphi = \omega t + 2e\sin\omega t \ . \qquad (15)$$

It describes a circular motion, shifted by $r_0 e$. Indeed, $x = r_0 e + r^{(0)}\cos\varphi$ and $y = r^{(0)}\sin\varphi$, such that $x^2 + y^2 = r_0^2$ within the harmonic approximation. In addition, $\omega^2 = \alpha/mr_0^3$ shows that the square of the motion period is proportional to the third power of the linear size of the orbit (Kepler's third law).[2] By (15), $\omega t = \varphi - 2e\sin\varphi$.

The first-order cubic correction to equation (11) leads to

$$u = u^{(0)} + \varepsilon u^{(1)} = -\cos\omega t - \varepsilon\cos\omega t + \frac{\varepsilon}{2}(3 - \cos 2\omega t) \ , \qquad (16)$$

and equation (10) gives $\varepsilon = e(1 - e)$. The corresponding radius reads

$$r = r_0[1 - e\cos\omega t + \frac{e^2}{2}(3 - \cos 2\omega t)] \ , \qquad (17)$$

which, by (1), leads to

$$\varphi = \omega t + 2e\sin\omega t - \frac{e^2}{2}(3\omega t - \frac{5}{2}\sin 2\omega t)] \ . \qquad (18)$$

Equation (18) can easily be inverted to give

$$\omega t = \varphi - 2e\sin\varphi + \frac{3e^2}{2}(\varphi + \frac{1}{2}\sin 2\varphi) \ , \qquad (19)$$

which transforms (17) into

$$r = r_0(1 - e\cos\varphi + e^2\cos^2\varphi + ...) \ . \qquad (20)$$

Within this approximation, equation (20) describes an ellipse,

$$r/r_0 = 1 - e\cos\varphi + e^2\cos^2\varphi + ... = 1/(1 + e\cos\varphi) \ , \qquad (21)$$

with the semi-major axis $a = r_0/(1-e^2) = r_0(1+e^2+...)$, the semi-minor axis $b = r_0/(1-e^2)^{1/2} = r_0(1 + e^2/2 + ...)$ and the origin displaced by $ae = r_0 e + ...$ in the focus ae (Kepler's first law).[3]

According to equation (19) the period T of the motion is given by

$$\omega T = 2\pi(1 + 3e^2/2) \ , \qquad (22)$$

[1]Noteworthy, $L = \omega I$, where $I = mr_0^2$ is the moment of inertia.
[2]J. Kepler, *Harmonices Mundi*, Linz (1619)
[3]Indeed, from (21), $\cos\varphi = x/(r_0 - ex)$ and $\sin\varphi = y/(r_0 - ex)$, whence the ellipse equation.

which shows that the frequency ω is shifted to $\Omega = \omega(1 - 3e^2/2) = (\alpha/ma^3)^{1/2}$. The frequency shift $\Delta\omega/\omega = -3e^2/2$ ensures the cancellation of the resonant contributions to the second-order cubic correction and first-order quartic correction to the anharmonic motion.

General central-field potential. Let $v(r)$ be an attractive central-field potential, such that the radial motion proceeds between $r_1$and r_2 given by[4]

$$L^2/2mr_{1,2}^2 + v(r_{1,2}) = E < 0 \ . \tag{23}$$

The effective potential $U(r) = L^2/2mr^2 + v(r)$ has a minimum value $-u_0 = r_0 v_1(1/2 + v_0/r_0 v_1) < 0$ for r_0 given by $L^2 = mr_0^3 v_1$, where v_0, v_1, v_2, ... denote the potential and, respectively, its derivatives for r_0. Making use of $r - r_0 = Au$ and $A/r_0 = \varepsilon$, the energy E can be written as

$$E = -u_0 + mA^2[\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\beta\omega^2 u^3 + \varepsilon^2\gamma\omega^2 u^4 ...] \ , \tag{24}$$

where $m\omega^2 = 3v_1/r_0 + v_2$, $\beta = (2v_1 - r_0^2 v_3/6)/(3v_1 + r_0 v_2)$ and $\gamma = (5v_1/2 + r_0^3 v_4/24)/(3v_1 + r_0 v_2)$. Making use of the eccentricity e defined by $e^2 = \delta(1 - |E|/u_0)$, where $\delta = -(v_1 + 2v_0/r_0)/(3v_1 + r_0 v_2)$, equation (24) can be rewritten as

$$e^2 = \frac{2\varepsilon^2}{\omega^2}(\dot{u}^2/2 + \omega^2 u^2/2 - \varepsilon\beta\omega^2 u^3 + \varepsilon^2\gamma\omega^2 u^4 + ...) \ . \tag{25}$$

The equation of motion given by (24) reads

$$\ddot{u} + \omega^2 u - 3\varepsilon\beta\omega^2 u^2 + 4\varepsilon^2\gamma\omega^2 u^3 ... = 0 \ , \tag{26}$$

and its solution is given by

$$r = r_0[1 - e\cos\omega t + \frac{\beta e^2}{2}(3 - \cos 2\omega t)] \ , \tag{27}$$

to the first-order of the cubic anharmonicity, where $e = \varepsilon(1 + \beta\varepsilon)$. Similarly, the angular variable is given by

$$\varphi = \sqrt{v_1/(3v_1 + r_0 v_2)}\{\omega t + 2e\sin\omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t]\} \ . \tag{28}$$

One can see that, in general, the trajectory of the motion is not closed, except for

$$\sqrt{v_1/(3v_1 + r_0 v_2)} = p/q \tag{29}$$

where p/q is a simple fraction. The gravitational potential $v(r) = -\alpha/r$ gives $p/q = 1$, while the spatial-oscillator potential $v(r) = const + \alpha r^2$ gives $p/q = 1/2$ ($\beta = 1/2$, $\gamma = 5/8$).

Denoting $1/\nu = \sqrt{v_1/(3v_1 + r_0 v_2)}$ and introducing the new phase $\chi = \nu\varphi$, equation (28) can be rewritten as

$$\chi = \omega t + 2e\sin\omega t - \frac{e^2}{2}[3(2\beta - 1)\omega t - \frac{2\beta + 3}{2}\sin 2\omega t] \ , \tag{30}$$

and it can easily be inverted to give

$$\omega t = \chi - 2e\sin\chi + \frac{e^2}{2}[3(2\beta - 1)\chi - \frac{2\beta - 5}{2}\sin 2\chi] \ . \tag{31}$$

[4]In order to avoid the fall on the centre the potential $v(r)$ must be less singular at the origin than $-L^2/2mr^2$.

Making use of (31) the equation of the trajectory (27) becomes

$$r = r_0'[1 - e\cos\chi + (2-\beta)e^2\cos^2\chi] \ , \tag{32}$$

where $r_0' = r_0[1-2(1-\beta)e^2]$. For the gravitational potential $\beta = 1$ and equation (21) is recovered from (32), while for the spatial oscillator $\beta = 1/2$, $\chi = 2\varphi$ and (32) becomes

$$r = r_0'[1 - e\cos 2\varphi + \frac{3e^2}{2}\cos^2 2\varphi] \ . \tag{33}$$

Since (33) is equivalent to $r^2 = r_0'^2/(1+2e\cos 2\varphi)$, it is easy to see that the corresponding trajectory is an ellipse centered at the origin. One can see from (30) that the spatial oscillator does not shift the frequency, but reduces it instead to $\omega/2$.

Closed orbits and the first sign of "chaos". Higher-order contributions of the anharmonicities may lead, in general, to a shift in frequency, in order to avoid, at each step of the perturbation calculations, the resonant terms.[5] Equation (27) for the radius gets thereby a shifted frequency ω', and equation (1) for the phase motion reads now $\dot{\varphi} = (L/m\omega r^2)(\omega/\omega')\omega'$, which changes, in general, the prefactor $1/\nu$ in equation (28). However, the ratio ω/ω' is a series in e^2, and it can be absorbed in the $\chi-$, $\omega t-$ and $r-$series, with a renormalization of r_0'. Under these circumstances the condition $1/\nu = p/q$ given by (29) for closed orbits is left unchanged.[6] This is valid as long as the calculations are confined to finite orders of perturbation series, as for small oscillations and eccentricities, for instance. In the limit of the series summation the orbits are closed only for two power-law potentials: the gravitational potential $-\alpha/r$ and the spatial-oscillator potential $const + \alpha r^2$. Indeed, this can be seen easily on the equation of motion for the trajectory $r(\varphi)$, as given by (1) and (2), whose integration requires a quadratic form of the integrand, the only one able to lead to circular functions.[7] In general, the trajectories are closed provided the potentials are such as to cancel recurssively the frequency shifts in the formal perturbation series. However, for sufficiently large p and q, and a large number of cycles, the orbits are practically closed for any potential.[8] This is another illustration of the "ergodic hypoyhesis", and is viewed sometimes as the first sign of "chaos" and "chaotical" behaviour.

Moon's problem. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the positions of two bodies of mass m_1 (Earth, $m_1 \simeq 6\times 10^{24}Kg$) and, respectively, m_2 (Moon, $m_2 \simeq 7\times 10^{22}Kg$), subjected to gravitational potentials $-Gm_0m_1/r_1$, $-Gm_0m_2/r_2$ and interacting through $-Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right|$, where $G \simeq 6.7\times 10^{-11}m^3/Kg\cdot s^2$ is the gravitational constant. The body of mass m_0 (Sun, $m_0 \simeq 2\times 10^{30}Kg$) is at rest. The energy is given by

$$E = m_1\dot{\mathbf{r}}_1^2/2 + m_2\dot{\mathbf{r}}_2^2/2 - Gm_0m_1/r_1 - Gm_0m_2/r_2 - Gm_1m_2/\left|\mathbf{r}_1-\mathbf{r}_2\right| \ , \tag{34}$$

[5]Such terms are also called "secular terms", and the shift in frequency is also known as the Poincare-Lindstedt expansion (H. Poincare, *Les Methodes Nouvelles de la Mecanique Celeste*, Gauthier-Villars, Paris (1892); A. Lindstedt, *Uber die Integration einer fur die Storungstheorie wichtigen Differentialgleichung*, Astron. Nach. **103** 211 (1882)).

[6]Equation (29) is the first term of the series expansion of the well-known closure condition

$$\Delta\varphi/2\pi = (1/\pi)\int_{r_1}^{r_2} dr\cdot(L/r^2)/\sqrt{3m(E-v(r))-L^2/r^2} = p/q \ .$$

[7]Making use of the substitution $r = 1/u$ the equation for the trajectory $u(\varphi)$ reads $u'' + u = -(m/L^2)\partial v/\partial u$, whose solution is given by circular functions only for the gravitational potential $v \sim u$ and the spatial oscillator potential $v \sim 1/u^2$. This observation is called sometime "Bertrand's theorem" (J. Bertrand, *Mecanique Analytique*, Comptes Rendus, Acad. Sci. **77** 849 (1873)).

[8]To the extent to which an irrational number is approximated by a rational number.

and the angular momentum reads

$$\mathbf{L}_{tot} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 \ . \tag{35}$$

It is easy to see that $\mathbf{L}_{tot}$ is conserved. Making use of the center-of-mass coordinate $\mathbf{R} = m_1\mathbf{r}_1/M + m_2\mathbf{r}_2/M$, where $M = m_1 + m_2$, and the relative coordinate $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, the angular momentum becomes

$$\mathbf{L}_{tot} = M\mathbf{R} \times \dot{\mathbf{R}} + m\mathbf{r} \times \dot{\mathbf{r}} \ , \tag{36}$$

where $m = m_1m_2/M$ is the relative mass. Similarly, the energy can be written as

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - Gm_0m_1/\left|\mathbf{R} - m_2\mathbf{r}/M\right| - Gm_0m_2/\left|\mathbf{R} + m_1\mathbf{r}/M\right| - Gm_1m_2/r \, . \tag{37}$$

Since $r \ll R$ (Sun-Earth distance $r_1 \simeq 15 \times 10^7 Km$, Moon-Earth distance $r \simeq 380\,000 Km$) it is convenient to expand the gravitational potentials in (37) in powers of $\mathbf{rR}/R^2$. Keeping only the quadrupolar contribution the energy becomes

$$E = M\dot{\mathbf{R}}^2/2 + m\dot{\mathbf{r}}^2/2 - \alpha/R - \beta/r - \gamma[3(\mathbf{rR})^2/R^2 - r^2]/R^3 \ , \tag{38}$$

where $\alpha = Gm_0M$, $\beta = GmM$ and $\gamma = Gm_0m/2$, or

$$E = E_1 + E_2 + \gamma v \ , \tag{39}$$

where

$$E_1 = M\dot{\mathbf{R}}^2/2 - \alpha/R \ , \ \ E_2 = m\dot{\mathbf{r}}^2/2 - \beta/r \ , \tag{40}$$

and

$$v = -r^2(3\cos^2\chi - 1)/R^3 \ . \tag{41}$$

The angle χ in (41) is the angle between the two vectors $\mathbf{r}$ and $\mathbf{R}$. Since $r/R \sim 3 \times 10^{-3}$ for Moon-Earth-Sun (and $(r/R)^2 \sim 10^{-5}$) the interaction v may be viewed as a small perturbation, and γ in (39) may act as a formal perturbation parameter. It is convenient now to employ polar coordinates and rewrite (40) as

$$E_1 = M\dot{R}^2/2 + MR^2(\dot{\Theta}^2 + \dot{\Phi}^2\sin^2\Theta)/2 - \alpha/R \ , \tag{42}$$

and, similarly,

$$E_2 = m\dot{r}^2/2 + mr^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)/2 - \beta/r \ , \tag{43}$$

where $\cos\chi = \sin\Theta\sin\theta\cos(\Phi - \varphi) + \cos\Theta\cos\theta$ in (41). The angular momentum of the relative motion reads

$$\begin{gathered} l_x = -mr^2(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi) \ , \ \ l_y = mr^2(\dot{\theta}\cos\varphi - \dot{\varphi}\sin\theta\cos\theta\sin\varphi) \ , \\ l_z = mr^2\dot{\varphi}\sin^2\theta \ , \end{gathered} \tag{44}$$

or $l_r = 0$, $l_\theta = -mr^2\dot{\varphi}\sin\theta$, $l_\varphi = mr^2\dot{\theta}$. Similar expressions hold for the angular momentum $\mathbf{L}$ of the center of mass, and $\mathbf{L}_{tot} = \mathbf{L} + \mathbf{l}$.[9]

Perturbation theory. Leaving aside for the moment the interaction v, equations (36), (42) and (43) describe two independent motions, each in its own gravitational potential, *i.e.* two independent Kepler's problems. Denoting by superscript (0) their relevant coordinates, the solutions of

[9] In view of the great disparity between m_1 and m_2, the center of mass is located practically on the first body (Earth, $M \simeq m_1$), and the relative motion corresponds practically to the second body (Moon, $m \simeq m_2$).

these problems can readily be written. Indeed, choosing $\Theta^{(0)} = \pi/2$, the radius $R^{(0)}$ and the phase $\Phi^{(0)}$ are given by

$$R^{(0)} = R_0[1 - e_1 \cos \Omega t + \frac{e_1^2}{2}(3 - \cos 2\Omega t) + ...] \ , \tag{45}$$

and, respectively,

$$\Phi^{(0)} = \Omega t + 2e_1 \sin \Omega t - \frac{e_1^2}{2}(3\Omega t - \frac{5}{2} \sin 2\Omega t)...] \ , \tag{46}$$

or

$$R^{(0)} = R_0(1 - e_1 \cos \Phi^{(0)} + e_1^2 \cos^2 \Phi^{(0)} + ...) = R_0/(1 + e_1 \cos \Phi^{(0)}) \ , \tag{47}$$

where $R_0 = L^{(0)2}/M\alpha$, $\Omega^2 = \alpha/MR_0^3$ and $e_1 = (1 - 2R_0 |E_1| /\alpha)^{1/2}$ is the eccentricity of the elliptical orbit.[10] For vanishing eccentricities (Earth's orbit eccentricity is $e_1 \simeq 0.017$) the solutions given above read $R^{(0)} = R_0$ and $\Phi^{(0)} = \Omega t$, *i.e.* a circular trajectory (and $E_1 \simeq -\alpha/2R_0$). A similar solution holds for the relative motion, though it must be written for a tilted reference frame. Making use of (43) and (44), the radius of the circular orbit is given by $r_0 = l^{(0)2}/m\beta$, and small oscillations around this position of equilibrium can also be considered (like in (45)), with a characteristic frequency given by $\omega^2 = \beta/mr_0^3$ ($\omega \gg \Omega$).[11] Since the eccentricity $e_2 = (1 - 2r_0 |E_2| /\beta)^{1/2}$ is small (Moon's orbit eccentricity is $e_2 \simeq 0.055$), the orbit can again be approximated by a circle of radius $r^{(0)} = r_0$.

Equations of motion for $\theta^{(0)}$ and $\varphi^{(0)}$ can easily be obtained from the lagrangean (43),

$$d(mr^2\dot{\theta})^{(0)}/dt - (mr^2\dot{\varphi}^2 \sin \theta \cos \theta)^{(0)} = 0 \ , \ d(mr^2\dot{\varphi} \sin^2 \theta)^{(0)}/dt = 0 \ , \tag{48}$$

whose solutions can be written straightforwadly for $r = r_0$, as

$$\theta^{(0)} = \pi/2 + \theta_0 \sin \omega t + ... \ , \ \varphi^{(0)} = \omega t + ..., \tag{49}$$

where $\theta_0 \simeq 5^\circ = \pi/36$ is the inclination of Moon's orbit against the ecliptic.[12] It can be viewed as another perturbation parameter. There is also a frequency shift given by $\omega' = \omega\sqrt{1 - \theta_0^2}$ which is neglected in (49), within this order of perturbation computations.

Equations of motion corresponding to the lagrangean given by (39) to (43) can easily be written. For Φ and Θ they read

$$d(MR^2\dot{\Phi} \sin^2 \Theta)/dt - \gamma\partial v/\partial\Phi = 0 \ , \tag{50}$$

and, respectively,

$$d(MR^2\dot{\Theta})/dt - MR^2\dot{\Phi}^2 \sin \Theta \cos \Theta - \gamma\partial v/\partial\Theta = 0 \ . \tag{51}$$

The solutions of these equations are written as $\Phi = \Phi^{(0)} + \gamma\Phi^{(1)} + ...$, $\Theta = \Theta^{(0)} + \gamma\Theta^{(1)} + ...$ and $R = R^{(0)} + \gamma R^{(1)} + ...$ To the first order in γ, equations (50) and (51) give $\Phi^{(1)} = -(3r_0^2/4M\omega^2R_0^5) \sin 2\omega t$, $\Theta^{(1)} = 0$ and $R^{(1)} = 0$, *i.e.* Earth's orbit does not change within this approximation ($R = R_0$, $\Theta = \pi/2$), except for a small additional contribution to the angular phase, as given by[13]

$$\Phi = \Omega t - \gamma(3r_0^2/4M\omega^2R_0^5) \sin 2\omega t \ . \tag{52}$$

It is also worth writing down the angular momentum

$$L = L_z = L_z^{(0)} - \gamma(3r_0^2/2\omega R_0^3) \cos 2\omega t \ , \tag{53}$$

[10]From $\Omega^2 = \alpha/MR_0^3$ one can check easily the Earth's year ~ 365 days.

[11]Or $\omega = l^{(0)2}/mr_0^2$; Moon's period ~ 27 days is checked from $\omega^2 = \beta/mr_0^3$.

[12]Actually, $\theta_0 = l_\perp^{(0)}/l^{(0)}$, where $l^{(0)}$ is the angular momentum and $l_\perp^{(0)}$ is the transversal component of the angular momentum ($l^{(0)2} = l_\perp^{(0)2} + l_z^{(0)2}$) of the relative motion.

[13]Since $\omega \gg \Omega$, the frequency Ω has been droped out wherever it was irrelevant.

where $L_z^{(0)} = MR_0^2\dot{\Phi}^{(0)} = MR_0^2\Omega$, and "energy"

$$E_1 = E_1^{(0)} - \gamma(3r_0^2\Omega/2\omega R_0^3)\cos 2\omega t \ , \tag{54}$$

where $E_1^{(0)} = -\alpha/2R_0$ within this approximation.

The relative motion is described by a set of equations of motion similar with those given by (50) and (51), whose solutions are given by $r = r_0$, $\theta = \pi/2 + \theta_0 \sin\omega t$ and

$$\varphi = \omega t + \gamma(3/4m\omega^2 R_0^3)\sin 2\omega t \tag{55}$$

within this approximation. The corresponding angular momentum is given by

$$l_z = l_z^{(0)} + \gamma(3r_0^2/2\omega R_0^3)\cos 2\omega t \ , \tag{56}$$

and one can see (by comparying it with (53)) that the total angular momentum $\mathbf{L}_{tot}$ is conserved. "Energy" E_2 becomes

$$E_2 = E_2^{(0)} + \gamma(3r_0^2/2R_0^3)\cos 2\omega t \ , \tag{57}$$

and energy $E = E_1 + E_2 + \gamma v$ is conserved within this order of approximation.[14]

The calculations can be carried through further orders of perturbation theory (at least in principle), and the method can also be applied to other situations of three bodies interacting through gravitational potentials, like, for instance, two bodies gravitating around a third one (Jupiter and Saturn, for instance, where a natural perturbation is just their own interaction, since their mass is much lighter than Sun's mass, and they do not get too close to each other).[15]

The "four Moons" and four periodicities. As one can see from the results presented above, the motion is described, within the first order of the perturbation theory, by three frequencies. The first one is Ω, the second is $\omega' = \omega\sqrt{1-\theta_0^2}$ and the third is ω. They may correspond to the sideral year (*i.e.* with respect to the fixed stars), the "sideral Moon" and the "nodal Moon" (since the intersections of Moon's orbit with the ecliptic are called Moon's nodes). These frequencies were known to the Greeks, within one second accuracy: the sideral year is cca 365 days, the sideral month is cca 27.32 days and the nodal month is cca 27.21 days.[16] One can check easily the difference ~ 0.1 days between the sideral month and the nodal month as coming from the correction factor $\simeq \theta_0^2/2 \sim (\pi/36)^2/2$. The correction factor $3\gamma/2m\omega R_0^3 = 3Gm_0/4\omega R_0^3 = 3\Omega^2/4\omega$ appearing in the angular frequencies as given by equation (55) was known to Newton. It leads to $4\omega/3\Omega \simeq 18$ years a period of Moon's retrograde motion, and the correction factor $3\Omega^2/4\omega^2 \sim 0.004$ applied to the sideral Moon gives ~ 0.1 days, which is half of the difference between the sideral Moon and another, "anomalous Moon" of cca 27.5 days.

The γ-correction factor can be viewed as another frequency, which means that the motion is described in fact by four frequencies, in agreement with the empirical periodicities and with the four constants of motion (energy E and angular momentum $\mathbf{L}_{tot}$). It seems unlikely to exist additional constants of motion, as to match the six degrees of freedom, at least with analytical functions (and series), so the three-body problem is not "integrable".[17] However, non-analytical behaviour may

[14] Proper definition is needed for the dependence of v on coordinates Φ and φ in order to ensure the conservation of energy.

[15] An analytical series expansion in terms of known functions for the coordinates of the Planets was suggested by Weierstrass as a problem in the contest held around 1890 in honor of Sweden's King. Poincare won the contest without solving the problem, though pointed out possible instabilities.

[16] Correction $\Omega/\omega \sim 1/13.5$ to the sideral month gives the fourth, synodal Moon $\simeq 29.5$ days. "Second" accuracy amounts to five decimals.

[17] This is sometime referred to as a Bruns-Poincare theorem.

exist, as, for instance, an infinite phase velocity $\dot{\varphi}$ for a vanishing polar angle θ. This may imply an abrupt change in the trajectory (for instance, instead of rotating very fast around the pole, the trajectory may take suddenly a longitudinal circle). Apart from particular initial conditions, such chaotic behaviour of the three-body problem would require an external perturbation, usually time dependent, a situation sometime referred to as the "Moon's problem", where Earth's coordinates act like time-dependent external fields. Nevertheless, the motion described above, very likely, by four fundamental frequencies (as well as by the corresponding "combined" frequencies and their higher harmonics), may look already very complicated to warrant the adjective "erratic", or "chaotic", though over very small scale of magnitude.[18] It is sometime called Poincare's "weak chaos", in contrast to trajectory instabilities that are termed "strong chaos".[19]

Laborious contributions to Newton's γ-correction were brought by d'Alembert and Clairaut around 1750, while Delauney set about to compute about 500 perturbation terms around 1840 (published in about 2000 print pages). Hill ($\sim$1880) and Poincare ($\sim$1890-1900) developed further insights into such a complicated mechanical behaviour of the 3-body problem, while modern computers (employed especially in connection with aselenization plans) brought additional insights. The difficulties reside in the slow convergence, resonant terms, a required accuracy rather high (a small error on the Earth may result in a big failure on the Moon!), and computing algorithms. Meanwhile, chaotical behaviour was left to be looked for in the quantal behaviour, where quantization is attempted for erratic classical trajectories.

A particular motion in Coulomb potential. The energy in Coulomb (or gravitational) potential $-\alpha/r$ reads

$$E = m\dot{r}^2/2 + L^2/2mr^2 - \alpha/r \ , \tag{58}$$

where m is the particle mass and L is the angular momentum. Let $L = 0$ for the moment, and $E = -\alpha/r_0$. The particle will pass through the origin up to the second r_0, then will return to the former r_0, in a periodic movement. Formally, it can be viewed as oscillating between $r = 0$ and $r = r_0$, around $r_0/2$. By (58), it is easy to get

$$m\dot{r}^2/2 + \alpha(r - r_0)/rr_0 = 0 \ , \tag{59}$$

or, by $r = r_0/2 + \rho$,

$$m\dot{\rho}^2/2 + \alpha(\rho - r_0/2)/r_0(\rho + r_0/2) = 0 \ . \tag{60}$$

It is also convenient to use $\rho = r_0u/2$, so (60) becomes

$$\dot{u}^2 + \omega^2(u - 1)/(u + 1) = 0 \ , \tag{61}$$

where

$$\omega^2 = 8\alpha/mr_0^3 \ . \tag{62}$$

It is easy to integrate equation (61). One obtains

$$2 \arcsin \sqrt{(1 - u)/2} + \sqrt{1 - u^2} = \omega t \ , \tag{63}$$

and the solution must be periodically extended to any t. It describes a periodic motion with period $T = 4\pi/\omega$, as if the frequency would be $\omega/2$.

Quantization. Equation (60) gives also

$$m\dot{\rho}^2/2 + m\omega^2\rho^2/2 + ... - \alpha/r_0 = 0 \tag{64}$$

[18]In particular, the corresponding orbits are not closed (or "periodic") anymore.

[19]The latter are similar with well-known parametric resonance, or non-linear resonance, in the theory of the linear oscillators, driven by an external control parameter.

by expansion in powers of ρ, which describes a linear harmonic oscillator. It follows

$$\hbar\omega(n+1/2)/2 = \alpha/r_0 \ , \tag{65}$$

where frequency $\omega/2$ is used (as for the complete motion) and $n = 0, 1, 2, ...$ According to the quantal hypothesis, one can also write it as

$$\hbar\omega\delta n/2 = |E|_q \ , \tag{66}$$

where $\delta n = n = 1, 2, 3...$and E_q is the quantized energy. Making use of $\omega = (8\,|E|_q^3\,/m\alpha^2)^{1/2}$ one gets

$$|E|_q = \frac{m\alpha^2}{2\hbar^2 n^2} \ , \tag{67}$$

i.e. the quantized energy of the Hydrogen atom. The anharmonic corrections to (64) do not contribute, as it can be seen from the variation equation (66). The corresponding approximate wavefunctions of the linear oscillator must be displaced so as to be peaked on the origin.

Similarly, for $L \neq 0$, the effective potential in (58) has a minimum value for

$$r_0 = L^2/m\alpha \ , \tag{68}$$

and energy reads

$$E = m\dot{r}^2/2 + (\alpha/2r_0^3)(r - r_0)^2 + ... - \alpha/2r_0 \ . \tag{69}$$

The frequency is given by $m\omega^2 = \alpha/r_0^3$, and

$$\hbar\omega(n+1/2)/2 = \alpha/2r_0 + E \ . \tag{70}$$

Since $L^2/2I = L^2/2mr_0^2 = \alpha/2r_0 = |E|_q$, it is easy to see that (70) leads to

$$\sqrt{2\hbar^2\,|E|_q^3\,/m\alpha^2}\delta n = |E|_q \ , \tag{71}$$

which is again the quantal energy (67) of the Hydrogen atom, for $\delta n = n = 1, 2, 3....$ The corresponding approximate wavefunctions of linear harmonic oscillator are now peaked on r_0.

The method can be generalized to any central-field potential $v(r)$. The minimum of the effective potential is reached for r_0 given by

$$-L^2/mr_0^3 + v_1 = 0 \tag{72}$$

where $v_1$is the first derivative of v for r_0. The energy expansion reads

$$E = m\dot{r}^2/2 + (3v_1/r_0 + v_2)(r - r_0)^2/2 + ... + L^2/2mr_0^2 + v_0 \ , \tag{73}$$

where v_0 is the potential function for r_0, v_2 is the second derivative of v for r_0, and frequency is given by

$$\omega^2 = 3v_1/mr_0 + v_2/m \ . \tag{74}$$

The quantization relation reads

$$\hbar\omega(n+1/2)/2 = E - L^2/2mr_0^2 - v_0 \ . \tag{75}$$

Making use of (72) the energy can be related to r_0 by $|E|_q = L^2/2mr_0^2 = v_1r_0/2$. It follows the quantized energy is given by

$$\hbar^2[3v_1(|E|_q)/mr_0(|E|_q) + v_2(|E|_q)/m]n^2/4 = |E|_q^2 \ , \tag{76}$$

where $n = 1, 2, 3....$