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Fluids, Fluid Vortices and the Theory of Electricity and Magnetism (Lecture seven of the Course of Theoretical Physics)

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Abstract

The fluid dynamics is described, for ideal fluids; the dynamics of the viscous fluid is rather a physical kinetics. The fluid vortex is given an emphasis, and a new type of force is recognized in fluid dynamics, which is termed Euler's force. For vortices it acquires formally the expression of the Lorentz force felt by an electric current placed in a magnetic field. The formal analogy between fluid vortices and magnetostatics is analyzed, with emphasis on the profound difference between fluids and the theory of electricity and magnetism. In particular, Euler's force can produce the Biot-Savart force, the Ampere electrodynamic force acting between parallel currents, but also a Coulomb law of force acting beween vortices, or dipolar fields leading to a law of force which is not dipolar in general. The Euler force has a non-central component in general, introduces a net force and a non-vanishing torque acting upon a pair of vortices.

Interacting vortex filaments and vortex particles are discussed in this connection, and the separation between fields and particles is made on this basis. Two new types of matter are introduced, one consisting of vortex particles, another consisting of particles endowed with torroidal vorticities. They may be termed the vorticial and, respectively, dipolar liquids. In both cases, additional contributions are brought by Euler's force, in comparison with the laws of force in the theory of electricity and magnetism. Such liquids may have particular excitations, which may bring new features to known plasmas which they resemble. In particular, two-dimensional ensembles of parallel (antiparallel) vorticial fluid filaments are described, which can undergo a topological oder.

The tempting hypothesis of the electromagnetic field, electric charges and currents arising from vortices in a universal ethereal fluid is analyzed, and shown to be untenable. The electromagnetism is given as a distinct motion of separated, though inter-related, fields produced by charges and interacting with them. In contrast with the fluid dynamics the theory of electricity and magnetism starts with two different things, particles and fields, and ends up with one thing, the union of the two. This is the profound cause of the troubling "self-interacting infinities" in the theory of electricity and magnetism. This "double thinking" of the theory of electricity and magnetism is based on the distinction betwen electric and magnetic fields, the existence of the universal velocity of light, the retardation (or advanced) interaction, the basic separation of propagating fields from localized, point-like charges, etc. The way out of such difficulties, well-known in all theories of field, is noted, as provided by the quantal delocalization of relativistic particles over their existence.

Introduction. Around 1640 Descartes thought that swirling Planets move around the Sun like vortices in a universal fluid. His theory was soon superseded by Newton's universal attraction and theory of motion. In the 18th century, among many other physical and mathematical things, the fluid motion came into the attention of Bernoulli, Euler and Lagrange. Friction has been included in fluid motion through Navier-Stokes equations up to 1840. Around 1858 Helmholtz discovered a basic fact in fluid motion: the vortex. Fluid vortices are consistent swirling movements of fluids, which may possess their own dynamics. Possibly, Helmholtz had in mind the atmospheric cyclons. Soon, in 1865, another fluid was introduced by Maxwell, the electric and magnetic ethereal medium, and about the same year William Thomson (Lord Kelvin), based much on Helmholtz's work, revived the theory of vortices as ultimate matter constituents. Maxwell praised Kelvin's theory as an admirable advance coming from the interaction between mathematics and physics. Organized motion emerging from underlying motion caught much imagination. At the end of the 19th century Poincare synthesized the basic properties of vortices,[1] and there succeeded a great deal of work in the applied mathematics of the fluids.[2]

Meanwhile, the 20th century has witnessed the appearance of other *sui generis* fluids, like quantal wavefunctions, quantal fields, and even one more classical field, the gravitation, beside the electromagnetic one. In fact, fluids are (non-linear) fields of velocity. Emergent quantal or clasical objects, with their own dynamics, made out of an underlying (quantal or classical) swirling motion got a new interest, especially in connection with topological order in two dimensions, but, mainly, in relation to superconducting (and superfluid) vortices of Abrikosov.[3] Elementary particles made out of a non-linear dynamics of underlying fields caught the imagination of Einstein and Heisenberg in modern times, as well as this day, when quarks, leptons or gauge bosons, this "fantastic reality",[4] are brought into play. Of course, the vortices should be closely related to the "turbulent" problem of the turbulence.

Lagrange and Euler equations. Fluid velocity $\mathbf{v} = d\mathbf{r}/dt$ is a field $\mathbf{v}(t, \mathbf{r})$ of time t and position $\mathbf{r}$. The force acting upon the unit volume in a fluid is $-grad p$, where p is the pressure; adding an external force $\mathbf{f}$ per unit mass, Newton's equation reads $\rho d\mathbf{v}/dt = \rho\mathbf{f} - grad p$, where ρ is the fluid density, or

$$d\mathbf{v}/dt = \mathbf{f} - grad p/\rho \ . \tag{1}$$

This is Lagrange's equation of motion of the fluids.

The fluids flow continuously. This is why we prefer a fixed frame of coordinates. The continuity equation reads $d\rho = 0$, or $d\rho/dt = 0$, which gives

$$\partial\rho/\partial t + div(\rho\mathbf{v}) = 0 \ . \tag{2}$$

Lagrange's equation (1) can then be written as

$$\partial\mathbf{v}/\partial t + (\mathbf{v} grad)\mathbf{v} = \mathbf{f} - grad p/\rho \ . \tag{3}$$

This is Euler's equation of motion of the fluids. It is a non-linear equation in velocity. It can also be written as

$$\partial\mathbf{v}/\partial t - \mathbf{v} \times curl\mathbf{v} = \mathbf{f} - grad(v^2/2) - grad p/\rho \ , \tag{4}$$

since $grad(v^2/2) = (\mathbf{v} grad)\mathbf{v} + \mathbf{v} \times curl\mathbf{v}$.[5]

[1] H. Poincare, *Theorie des Tourbillons*, Gabay, Paris (1893).

[2] See, for instance, H. Lamb, *Hydrodynamics*, Dover, NY (1945); G. K. Batchelor, *An Introductuion to Fluid Dynamics*, Cambridge (1967); and references therein.

[3] A. A. Abrikosov, Sov. Phys. JETP **5** 1174 (1957).

[4] F. Wilczek, *Fantastic Realities*, WorldSci (2006).

[5] It can easily be checked by using the definition $(curlv)_i = \varepsilon_{ijk}\partial_j v_k$ of the curl and the definition $(a \times b)_i = \varepsilon_{ijk}a_j b_k$ of the vectorial product, where ε_{ijk} is the totally antisymmetric unit tensor of rank 3, $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

Static equilibrium. For static fluids $\mathbf{v} = 0$, the equlibrium is attained for $\mathbf{f} - gradp/\rho = 0$. This is the law of hydrostatics. It is the condition for mechanical equilibrium. It is worth noting that it may not have always a solution for pressure, since $\rho \mathbf{f} d\mathbf{r}$ is not a total differential in general. Very often, the external force $\mathbf{f}$ is conservative, *i.e.* it derives from a potential V (per unit mass), through $\mathbf{f} = -gradV$. If the fluid is incompressible, *i.e.* if $\rho = const$ everywhere at any time, the equation of the mechanical equilibrium can be integrated for pressure p, and we get for p a certain function of position, according to the external potential V. For instance, in the gravitational potential $V = gz$, the pressure is given by $p = -\rho gz$, where g is the gravitational acceleration and z denotes the altitude. This is the well-known Toricelli's barometric law for hydrostatic pressure. However, the fluid is usually at local equilibrium, so we have a local equation of state which relates the pressure to density and temperature T, through some function f: $p = f(\rho, T)$. That means that the temperature changes with position, so there is no global equilibrium. Then, we may have convection, exchange of heat and entropy, and the mechanical equilibrium may be unstable, under certain conditions. In order to have a global thermal equilibrium the temperature must be constant, and, of course, the fluid must be compressible. The Gibbs enthalpy Φ (per unit mass) can then be written as $d\Phi = -sdT + dp/\rho$, where s is the entropy per unit mass, so $d\Phi = dp/\rho$ is a total differential and mechanical quilibrium is reached for $V + \Phi = const$. The equation of state gives then ρ as a function of p, and the pressure (and density) can be obtained. For an ideal gas, for instance, $p = \rho T/m$, where m is the molecular mass, and $p \sim \exp(-mgz/T)$ in gravitational potential. The density has a similar dependence on the altitude z, according to the equation of state. The entropy also varies with the altitude in this case; it is higher at higher altitudes; it is much more order at the bottom. [6]

Ideal fluids. Friction and thermal conductivity are supposed to be absent in Euler's equation. That means the flow is adiabatic. The fluid is said to be ideal in this case. The entropy conserves locally, *i.e.* $ds = 0$, or $ds/dt = 0$, which makes

$$\partial s/\partial t + \mathbf{v} grads = 0 \ . \tag{5}$$

Together with the equation of continuity we get the conservation $\partial(\rho s)/\partial t + div(\rho s \mathbf{v}) = 0$ of the flow of entropy. Each infinitesimal bit of ideal fluid flows with its own constant entropy. The local equilibrium gives for the entropy s a function $s = g(\rho, T)$, which, together with the equation of state $p = f(\rho, T)$, can ve written as $s = h(\rho, p)$, where h is the function resulting from the combination of these two equations. The density and the pressure must be so as to satisfy the adiabatic equation (5).

The adiabatic flow $ds = 0$ of the ideal fluids allows the equation $dw = dp/\rho + Tds$ for the heat function w to be written as $dw = dp/\rho$, which makes $gradp/\rho = gradw$. Then, for conservative forces $\mathbf{f} = -gradV$, Euler's equation (4) becomes

$$\partial \mathbf{v}/\partial t - \mathbf{v} \times curl\mathbf{v} = -grad(v^2/2 + \psi) \ , \tag{6}$$

where $\psi = w + V$. $d\psi$ is a total differential. equation (6) can also be written as $\partial(curl\mathbf{v})/\partial t - curl(\mathbf{v} \times curl\mathbf{v}) = 0$.

Now, we have three Euler's equations (6) for the three components of the velocity, one equation of continuity (2) and one equation (4) for the adiabatic flow; that makes five equations. And we have also three components of velocity, one density and one pressure; that makes five unknowns. It follows that the flow of the ideal fluids is solved completely; being given ρ and p all the other thermodynamic quantities are determined, by making use of their definition and of the equation of state.

[6] For an ideal classical fluid $s \sim \ln(T/\rho)$.

We may get the velocity from $\partial(curl\mathbf{v})/\partial t - curl(\mathbf{v} \times curl\mathbf{v}) = 0$, and then ψ and w from Euler's equation (6), providing the velocity is such as the *lhs* of this equation is a gradient. Then, we may get the density from the continuity equation, and try to solve $gradw = gradp/\rho$ for the pressure. In general, this integration of the pressure from the heat function is not possible. But the adiabatic equation ensures precisely such a solution.

For incompressible fluids $\rho = const$, the continuity equation gives $div\mathbf{v} = 0$, the equation for the adiabatic flow becomes $\partial p/\partial t + \mathbf{v}gradp = 0$ and $w = p/\rho$ (up to a constant internal energy per unit mass $\varepsilon = const$; indeed, $d\varepsilon = -pdV + Tds = (p/\rho^2)d\rho + Tds = 0$ for incompressible ideal fluids). We get then more equations than unknowns, and the solutions are over-determined; there may be no solution, in general.

In general, the adiabatic flow of the ideal fluids implies an entropy which depends of time and position. Again, each bit of fluid flows with its own constant entropy. The temperature may also depend on time and position, and we have no global thermal equilibrium, as expected. The thermal equilibrium does not allow for a macroscopic motion, and we do have such a motion in the flow of the fluids. Since there is no friction and no thermal conductivity in an ideal fluid, such a non-equilibrium movement is not surprising, though quite strange. [7]

We can recognize the energy per unit mass in the *rhs* of equation (6). The first term in equation (6) is the (local) acceleration, the second term can be viewed as a force,[8] so this equation tells that the variation of the energy in its *rhs* equals the (local) kinetic energy and the mechanical work. It is Newton's equation, as expected, for velocity fields.

It can easily be shown that the energy flows continuously, according to

$$\frac{\partial}{\partial t}(\rho v^2/2 + \rho\varepsilon) + div[\rho\mathbf{v}(v^2/2 + w)] = 0 \ . \tag{7}$$

where ε is the energy of the fluid per unit mass and $\rho\mathbf{v}(v^2/2 + w)$ is the energy flow. Similarly, the momentum flows continuously, according to

$$\partial(\rho v_i)/\partial t + \partial\Pi_{ik}/\partial x_k = 0 \ , \tag{8}$$

where $\Pi_{ik} = p\delta_{ik} + \rho v_i v_k$ is the momentum flow. Their integrals are conserved. The fluid carries momentum flow $p + \rho v^2$ along its velocity, and only the pressure p in the transverse direction (perpendicular to the velocity).

Streamlines. Bernoulli's law. Since $\mathbf{v} = d\mathbf{r}/dt$, the curve

$$dx/v_x = dy/v_y = dz/v_z = dt \tag{9}$$

is tangent to velocities. It is called the streamline. For a steady flow it does not change in time, and coincides with the path of the fluid particles. For a non-steady flow the path of the fluid particles differs from the streamline. The action $\int dt \cdot (v^2/2 - \psi)$ of the lagrangean along a path is an extremum, in view of Lagrange's equation (1). For a steady flow the energy per unit mass $v^2/2 + \psi$ is constant along a streamline, in view of the same equation of motion. The streamline does not move in this case. The constant, in general, differs from a streamline to another. The constancy of the energy along the streamlines in a steady flow is Bernoulli's law. It comes also from equation (6) by taking its projection along a steady-flow streamline.

Potential flow. Like any vector, the velocity can be represented as a sum $\mathbf{v} = \mathbf{v}_1 + \mathbf{v_2}$ of two vectors $\mathbf{v}_{1,2}$ enjoying the properties: $div\mathbf{v}_1 = 0$ and $curl\mathbf{v}_2 = 0$. That means that $\mathbf{v}_1$ comes from

[7] von Neumann and Feynman called such an ideal fluid the "dry water".

[8] It will shortly be termed the Euler force.

a curl, $\mathbf{v}_1 = curl\mathbf{A}$ (since $div \cdot curl\mathbf{A} = 0$) and v_2 comes from a gradient, $v_2 = grad\phi$ (since $curl \cdot grad\phi = 0$); $\mathbf{A}$ and ϕ are some vector and, respectively, scalar potentials.

Let us assume that $\mathbf{v}_1 = 0$, *i.e.* $\mathbf{v} = grad\phi$ (and $curl\mathbf{v} = 0$). This is called the irrotational motion or the potential flow. Euler's equation (6) leads then to $grad(\partial\phi/\partial t + v^2/2 + \psi) = 0$, *i.e.*

$$\partial\phi/\partial t + v^2/2 + \psi = const \tag{10}$$

everywhere (up to a function of time). This is the generalized Bernoulli's law, and a first integral of motion. For a steady flow it gets simplified to $v^2/2 + \psi = const$ everywhere.

A further simplification is achieved for an incompressible fluid, *i.e.* a fluid for which $\rho = const$ everywhere. The equation of continuity reads then $div\mathbf{v} = div \cdot grad\phi = \Delta\phi = 0$. With suitable boundary conditions this equation can be solved for ϕ, *i.e.* for $\mathbf{v}$, and from (10) we get the pressure p. The problem is then solved fully, providing the equation of adiabatic flow is satisfied.

It can easily be shown that if the velocity of the fluid is much smaller than the sound velocity in that fluid, $v \ll c$, then the flow may be viewed as being incompressible. If velocity changes appreciably in time τ over distances l, then the flow may be viewed as steady for $\tau \ll l/c$, *i.e.* for interaction propagating "instantly".

An example of potential flow is the small oscillations of a large body. Then, $\partial\mathbf{v}/\partial t \sim \omega v \sim v^2/a$, where ω is the frequency of oscillations and a is the amplitude; on the other hand, $(\mathbf{v}grad)\mathbf{v} \sim v^2/l$, where l is the size of the body; since $l \gg a$ we may neglect $(\mathbf{v}grad)\mathbf{v}$ in Euler's equation, which reads $\partial\mathbf{v}/\partial t = -grad(v^2/2 + \psi)$, or $\partial(curl\mathbf{v})/\partial t = 0$. However, the mean velocity in oscillatory motion vanishes, so that $curl\mathbf{v} = 0$, *i.e.* the motion is potential, in first approximation.

Helmholtz's law of circulation. Multiplying equation (1) by $d\mathbf{r}$ we get $(d\mathbf{v}/dt)d\mathbf{r} = -d\psi$. Integrating this equation along a closed curve we get

$$\oint (d\mathbf{v}/dt)d\mathbf{r} = -\oint d\psi = 0 \ . \tag{11}$$

The curve moves with the fluid, so we can write

$$\oint (d\mathbf{v}/dt)d\mathbf{r} = \frac{d}{dt}\oint \mathbf{v}d\mathbf{r} - \oint \mathbf{v}d\mathbf{v} = \frac{d}{dt}\oint \mathbf{v}d\mathbf{r} - \oint d(v^2/2) \ \ , \tag{12}$$

and, together with (11),

$$\frac{d}{dt}\oint \mathbf{v}d\mathbf{r} = \oint d(v^2/2 - \psi) = 0 \ \ , \tag{13}$$

or

$$\oint \mathbf{v}d\mathbf{r} = l = const \ . \tag{14}$$

This is the Helmholtz theorem: the circulation of the velocity along any closed fluid curve is constant. We may denote this constant by l, as it is reminiscent of the angular momentum. We emphasize that it holds only for ideal fluids.

Making use of the Stokes theorem, we get also

$$\oint \mathbf{v}d\mathbf{r} = \int d\mathbf{S} \cdot curl\mathbf{v} \ \ , \tag{15}$$

for any surface limited by the closed curve.

Suppose $curl\mathbf{v}=0$ at some instant on a streamline. It is preserved in time along that streamline, so the potential flow ($curl\mathbf{v}=0$) is preserved in time. Except for the streamlines ending up on a solid surface, like the surface of a body immersed in the flow. There, the velocity which is tangential to the surface is discontinuous, and we are not able to draw anymore a closed curve in the fluid encircling the streamline. An entire family of tangential discontinuities develops behind that body (in the wake of the flow), which makes the solution of the fluid flow non-uniquely determined. Actually, this holds only for ideal fluids. In real fluids, the friction makes the velocity continuous, and the boundary conditions determine the solution uniquely, together with the equations of motion.

In a potential flow the streamlines are open curves. Indeed, if they would be closed, the circulation of the velocity along them would not be zero, *i.e.* the flow would have a non-vanishing curl, *i.e.* it would not be a potential flow. This is however true for a simply-connected region. If the region is multiply-connected, the streamlines may be closed curves; the circulation of the velocity along such a streamline is non-vanishing, but it may be compensated by another closed streamline which together with the former may define a surface where the curl is vanishing. If the streamlines are open curves the circulation along them is not necessarily vanishing.

The vortex. Suppose a purely rotational motion, *i.e.* $\mathbf{v}=curl\mathbf{A}$, where $\mathbf{A}$ is a potential vector, such as $div\mathbf{v}=0$. We may choose $\mathbf{A}$ such as $div\mathbf{A}=0$, without any loss of generality. We denote $curl\mathbf{v}=2\omega$, and call ω (or $curl\mathbf{v}$) the vorticity. The analogy with the static magnetic field is obvious: the velocity $\mathbf{v}$ is the magnetic field $\mathbf{H}$ and the vorticity ω is $2\pi\mathbf{j}/c$, where $\mathbf{j}$ is the density of electric current and c is the velocity of light.

The curves

$$\frac{dx}{\omega_x}=\frac{dy}{\omega_y}=\frac{dz}{\omega_z} \tag{16}$$

are lines of vortex, and the surface $\Phi=const$ with $\omega grad\Phi=0$ is the surface of the vortex tube.

Since $curl\mathbf{v}=curl\cdot curl\mathbf{A}=-\Delta\mathbf{A}+grad\cdot div\mathbf{A}=-\Delta\mathbf{A}=2\omega$,[9] we get Poisson's equation $\Delta\mathbf{A}=-2\omega$, whose solution vanishing at infinity is given by

$$\mathbf{A}(\mathbf{r})=\frac{1}{2\pi}\int d\mathbf{r}'\cdot\frac{\omega(\mathbf{r}')}{R}\ , \tag{17}$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$. This is the law of Biot and Savart. The velocity $\mathbf{v}=curl\mathbf{A}$ is then given by

$$\mathbf{v}(\mathbf{r})=\frac{1}{2\pi}\int d\mathbf{r}'\cdot\frac{\omega(\mathbf{r}')\times\mathbf{R}}{R^3}\ . \tag{18}$$

The streamlines are closed, solenoidal curves, like the magnetic field, perpendicular to the vorticity ω and to the position vector $\mathbf{R}$. Their intensity decreases to infinity. This is the vortex.

Having determined the velocity, we use the equation of continuity (2) and Euler's equation (6) to get the density ρ and the heat function w. The equations are again over-determined, and ρ and p must satisfy in additon the equation (5) of adiabatic flow. For an incompressible fluid the equation of continuity is already satisfied ($div\mathbf{v}=div\cdot curl\mathbf{A})=0$), and we get directly the pressure from Euler's equation.

The vortices are consistent objects in an ideal fluid, in the sense that the circulation of the velocity along any closed curve is constant in time. They are open tubes, or closed tubes like torri, they may have their own motion, in particular the motion associated with their center-of-vorticity.

[9] An identity which can be checked by using again the tensor ε_{ijk}.

Special vortices. Let the fluid particles rotate with constant angular velocity $\omega = (0, 0, \omega)$ about an axis passing through some point. The velocity is given by $\mathbf{v} = \omega \times \mathbf{r} = (-\omega y, \omega x, 0)$, where $\mathbf{r}$ is the position with respect to the rotation axis. The circulation of the velocity along the circle of radius r is $2\pi\omega r^2$. The curl of velocity is $curl\mathbf{v} = 2\omega = (0, 0, 2\omega)$, and its integral over the surface limited by the circle is $2\omega\pi r^2$, *i.e.* the circulation of the velocity, as expected. This is a special vortex, whose vector potential is $A = (0, 0, -\omega r^2/2)$. It satisfies the Poisson equation $curl\mathbf{v} = 2\omega$, where ω is constant everywhere. The vector potential increases indefinitely with increasing radius r. The velocity also has a similar indefinite increase. The flow is steady, and Euler's equation $(\mathbf{v}grad)\mathbf{v} = -grad\psi$ gives $\psi = \omega^2 r^2/2$. For an incompressible fluid $div\mathbf{v} = 0$ is satisfied, and the pressure is given by $p = \rho\omega^2 r^2/2$. It increases toward the circumference of the rotating fluid, according to the centrifugal force. In general, the motion of a rotating fluid is described more conveniently in the rotating frame, where the centrifugal force must be added to Euler's equation. If the fluid has also a velocity relative to the moving coordinates, then the Coriolis force must also be added. This inertial force determines an inhomogeneity in the pressure, so that there exist inertial waves in a rotating fluid. Similar internal waves are due to gravity, beside surface waves also caused by gravity.

The equation of adiabatic flow reads $\mathbf{v}gradp = 0$ for a steady flow of an incompresible fluid; it is satisfied by this vortex.

A more complex vortex is given by velocities $\mathbf{v} = \omega \times \mathbf{r} \cdot f(r)$, where $\omega = (0, 0, \omega)$ is a constant vector and $f(r)$ is an arbitrary function of $r = (x^2+y^2)^{1/2}$. The velocity reads $\mathbf{v} = (-\omega y f, \omega x f, 0)$, and a circle of radius d must be used for the integration contour, beside the current circle of radius r, if function f is singular at the origin; in general, we must use multiply-connected regions where singularities are present, instead of simply-connected ones for finite velocities. The circulation of the velocity is therefore $2\pi\omega[r^2 f(r) - d^2 f(d)]$. The vorticity is given by $curl\mathbf{v} = (0, 0, 2\omega f + \omega r f')$, and one can check that the circulation is given by Stokes' theorem. The special case $f = 1/r^2$ gives a vanishing circulation. It is worth noting that the vorticity depends in general on radius r, *i.e.* it is not constant in the flowing fluid. Such a complex vortex looks like a superposition of elementary vortices. The forces are given by $-\partial\psi/\partial x = (\mathbf{v}grad)v_x = -2\omega^2 f^2 x$, etc, and, like the velocity, they may be singular at the origin. We get $\psi = 2\omega^2 \int rf^2 dr$, and the pressure $p = 2\rho\omega^2 \int rf^2 dr$ for an incompressible fluid. The continuity equation $div\mathbf{v} = 0$ is satisfied, and so is the equation $\mathbf{v}gradp = 0$ of the adiabatic flow.

The cyclon. Filament. Suppose a thin, rectilinear distribution of vorticity given by $\omega(\mathbf{r}) = \omega a^2 \delta(\mathbf{r})$ and directed along the z-axis, where a is a characteristic thickness of this distribution and $\mathbf{r}$ is the position with respect to the z-axis. Such a distribution of vorticity is called a filament; it resembles very much a rectilinear, infinitely thin, electric current. The vector potential given by (17) can easily be calculated; it is given by

$$\mathbf{A}(r) = (\omega a^2/2\pi)\ln(z/r + \sqrt{z^2/r^2+1}\,|_{-L}^{+L} \simeq -(\omega a^2/\pi)\ln(r/2L) \ , \tag{19}$$

where $2L$ is the length of the filament and r is the distance to the filament. Indeed, $\Delta \ln r = 2\pi\delta(\mathbf{r})$. The velocities are given by $\mathbf{v} = (-\omega a^2 y/\pi r^2, \omega a^2 x/\pi r^2, 0) = (a^2/\pi)\omega \times grad(\ln r)$. One can check easily that $curl(\omega \times \mathbf{f}) = \omega div\mathbf{f} - (\omega grad)\mathbf{f}$, where $\mathbf{f}$ is some vector,[10] so $curl\mathbf{v} = (a^2/\pi)\omega\Delta(\ln r) = 2\omega a^2 \delta(\mathbf{r})$.

One can also check easily that $div\mathbf{v} = 0$, and, for an incompressible fluid, $p = -\rho\omega^2 a^4/2\pi^2 r^2$ and $\mathbf{v}gradp = 0$ (adiabatic flow), for a steady flow. This is quite suggestive of an atmospheric cyclon.

The force $-\mathbf{v} \times curl\mathbf{v} = -(2a^4/\pi)\omega^2\delta(\mathbf{r})grad(\ln r)$ brought by Euler's equation (per unit mass) vanishes everywhere. It follows that $v^2/2 + \psi = const$, *i.e.* the pressure changes such as to

[10] By using again the tensor ε_{ijk}.

accommodate the kinetic energy $v^2/2 = \omega^2 a^4/2\pi^2 r^2$, as found above. We note that the energy $\int d\mathbf{r} \cdot v^2/2$ is infinite, due to the vortex singularity. This is the vortex self-energy. Similarly, the self-force $-\mathbf{v} \times curl\mathbf{v} = -(2a^4/\pi)\omega^2\delta(\mathbf{r})grad(\ln r)$ found above is singular in origin. This is not as strange as it may look. If we want to create a finite vorticity along a line of vanishing thickness it is natural to give an infinite energy to the rotating fluid of vanishing mass localized on that line, and an infinite force to enable such a motion. Of course, the δ-function is not singular in fact, but it is some finite quantity localized on a small range. Therefore, we have not much to worry about the singularities displayed by localized vortices.

Apart from the free motion of the line of vorticity of a filament, we can consider the small deviations of this vorticity from a straight line. Then, we may have oscillations and waves of vorticity propagating along the filament.

Parallel filaments. Let us assume an ensemble of parallel filaments described by vorticity

$$\omega = a^2 \sum_i \omega_i \delta(\mathbf{r} - \mathbf{r_i}) \ , \tag{20}$$

where $\mathbf{r}_i$ are positions of these filaments in a plane perpendicular to their direction. We get the vector potential

$$\mathbf{A} = -(a^2/\pi) \sum_i \omega_i \ln |\mathbf{r} - \mathbf{r}_i| \tag{21}$$

and velocity

$$\mathbf{v} = curl\mathbf{A} = (a^2/\pi) \sum_i \omega_i \times grad(\ln |\mathbf{r} - \mathbf{r}_i|) = (a^2/\pi) \sum_i \frac{\omega_i \times \mathbf{R}_i}{R_i^2} \ , \tag{22}$$

where $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$.

Euler's equation brings a force (per unit mass)

$$\begin{aligned} -\mathbf{v} \times curl\mathbf{v} = 2\omega \times \mathbf{v} = -(2a^4/\pi) \textstyle\sum_{ij} \delta(\mathbf{r} - \mathbf{r}_i)\omega_i\omega_j grad(\ln |\mathbf{r} - \mathbf{r}_j|) = \\ = -(2a^4/\pi) \textstyle\sum_{ij} \delta(\mathbf{R}_i)\omega_i\omega_j \mathbf{R}_j / R_j^2 \end{aligned} \tag{23}$$

which vanishes everywhere, except the filaments. The $v^2/2$-contribution to the *rhs* of Euler's equation reads

$$v^2/2 = \frac{1}{2}(a^2/\pi)^2 \sum_{ij} \omega_i\omega_j \mathbf{R_i}\mathbf{R}_j / R_i^2 R_j^2 \ , \tag{24}$$

and its gradient is given by

$$grad(v^2/2) = (a^2/\pi)^2 \sum_{ij} \omega_i\omega_j \frac{R_i^2 - 2(\mathbf{R}_i\mathbf{R}_j)}{R_i^4 R_j^2} \mathbf{R}_i \ . \tag{25}$$

According to equation (22), we may look for a steady-flow solution, as we assume constant ω_i's; then, since the force given by (23) vanishes outside any $\mathbf{r}_i$, the *rhs* of Euler's equation vanishes for any $\mathbf{r} \neq \mathbf{r}_i$, so that the pressure is so as to accommodate the $v^2/2$, *i.e.* $p = -v^2/2$, for instance. In general, such a solution to Euler's equation may exist, though for an incompressible fluid it does not: the adiabatic equation $\mathbf{v}gradp = 0$ is not satisfied in that case, as it can be easily checked directly by making use of (22) and (25). However, such a solution does not exist in general for the points $\mathbf{r} = \mathbf{r_i}$, where the force $-\mathbf{v} \times curl\mathbf{v}$ is singular; such a solution will then be discontinuous in such points.

The singular points are dealt with by encircling any point $\mathbf{r}_i$ by a small circle of radius ε, integrating over its surface and letting then $\varepsilon \rightarrow a$. Doing so in (23), we get a force

$$\mathbf{f}_i = -(2a^2/\pi)\sum_j{}' \omega_i\omega_j\mathbf{R}_{ij}/R_{ij}^2 \ , \tag{26}$$

acting upon the i-th filament, as arising from $-\mathbf{v}\times curl\mathbf{v}$, where $\mathbf{R}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and the prime over the sum means $j \neq i$. We note that the singular self-force is not included, as if the filaments are line-like. The force the j-th filament acts upon the i-th filament is therefore

$$f_{ij} = -(2a^2/\pi)\omega_i\omega_j\mathbf{R}_{ij}/R_{ij}^2 \ , \tag{27}$$

while the force the i-th filament acts upon the j-th filament is

$$f_{ji} = -(2a^2/\pi)\omega_i\omega_j\mathbf{R}_{ji}/R_{ij}^2 = -f_{ij} \ . \tag{28}$$

Obviously, it is the analog of Ampere's electrodynamic force acting between two parallel electric currents, as $-\mathbf{v}\times curl\mathbf{v} \rightarrow \mathbf{j}\times\mathbf{H}$ is anlagous to Lorentz's force. We call $\mathbf{F} = -\mathbf{v}\times curl\mathbf{v} = 2\omega\times\mathbf{v}$ the Euler force. Formally it is analogous to the Lorentz force $\sim \mathbf{j}\times\mathbf{H}$ acting upon an electric current $\mathbf{j}$ placed in a magnetic field $\mathbf{H}$. However, as we shall see shortly there is a profound difference between Euler's force and the Lorentz force.

It is easy to see that the force given by (28) derives from a potential

$$U_{ij} = (2a^2/\pi)\omega_i\omega_j \ln R_{ij} \ , \tag{29}$$

so the energy of the interacting filaments (per unit length) reads

$$U = (a^2/\pi)\sum_{i\neq j}\omega_i\omega_j \ln R_{ij} \ . \tag{30}$$

It is worth noting that the force total $\sum_{i\neq j} f_{ij}$ on the ensemble vanishes. We apply the same procedure to all the other contributions to Euler's equation. We find easily, by direct integration, that $(\mathbf{v})_i \rightarrow 0$, $(\partial\mathbf{v}/\partial t)_i \rightarrow 0$, and $(grad(v^2/2))_i \rightarrow 0$, where the subscript i denotes the result of integration over a circle of vanishing radius ε drawn around the i-th filament. It follows that Euler's equation requires the equilibrium condition $\mathbf{f}_i = -(grad\psi)_i$, and we may take any pressure that satisfies the equation of adiabatic flow; then external forces are needed to satisfy the equilibrium condition.

The fact that Euler's equation has no solution for the given distribution of vorticities (20) is not surprising: not any velocity makes a solution. Our distribution depends on the parameters $\mathbf{r}_i$, and in order to look for a solution, we have, at least, to give a dynamics to these parameters, *i.e.* to let the filaments move around. Indeed, the force $\mathbf{f}_i$ acting upon any i-th filament is not vanishing, so there is a motion of the filaments. We emphasize that a velocity $\mathbf{u}_i = \dot{\mathbf{r}}_i$ must be attributed to any i-th filament, as distinct from the fluid velocity $(\mathbf{v})_i$, and, similarly $(grad\psi)_i = 0$, as for line-like filaments. We get then a distinct dynamics of the filaments, as emerging from Euler's fluid equation, and coupled in a certain way to the fluid motion, as it will be seen shortly. First, we note the equation of motion

$$d\mathbf{u}_i/dt = \mathbf{f}_i = -(2a^2/\pi)\sum_j{}' \omega_i\omega_j\mathbf{R}_{ij}/R_{ij}^2 \tag{31}$$

for filaments. It is conceivable that the evolution of the positions $\mathbf{r}_i$ of the filaments might be so as to attain a minimum of the "mechanical", potential energy U given by (30), *i.e.* for vanishing forces on any filament,

$$\sum_j{}' \omega_j \mathbf{R}_{ij}/R_{ij}^2 = 0 \ . \tag{32}$$

However, such a solid state of filaments is not stable. This can be checked by direct calculation, though it would imply a long-range order which is not attainable in general in two dimensions. On the other hand, the motion toward such a state may not preserve the ensemble of vortices. The $v^2/2$ gives an energy per unit length

$$\begin{aligned} E = \int d\mathbf{r} \cdot v^2/2 = \int d\mathbf{r} \cdot \mathbf{v} curl \mathbf{A}/2 = \int d\mathbf{r} \cdot \mathbf{A} curl \mathbf{v}/2 = \\ = \int d\mathbf{r} \cdot \mathbf{A}\omega = -(a^4/\pi) \textstyle\sum_{i \neq j} \omega_i \omega_j \ln R_{ij} = -a^2 U \ . \end{aligned} \tag{33}$$

One can see that the "mechanical" energy U due to the forces acting between the filaments changes at the cost of the "field" energy E stored by the velocities, such as $E/a^2 + U = const$, as for magnetic fields. The analogy of $\mathbf{A}\omega$ with the interaction energy $\mathbf{jA}$ of a current in a magnetic field is obvious. This is the way the motion of the filaments is coupled to the motion of the velocity field.

During this evolution, Euler's equation may be satisfied for a non-steady flow of velocities, though it is not necessary for a given ensemble of vorticities. The problem of the continuity of the solution does not appear anymore, since the line-like (point-like) filaments are now removed from the fluid. This is the basic decomposition into particles and fields, as allowed by the vortex concept. The fields are produced by particles, the particles are localized fields, singularities in the fields, both the fields and the particles may have their own dynamics, and the fields interact with particles. This interaction relates the motion of the fields to the motion of the particles. We note that the kinetic energy E_{kin} of the particles, associated with velocities $\mathbf{u}_i$, is such as it conserves against the potential energy U, so it is a measure of the "field" energy E. This is distinct from electromagnetism. The particles are singularities of the fields, and as long as the self-interaction is avoided by the point-like (line-like) nature of the particles, there is no problem of infinities. The problem is however unavoidable in electromagnetism, where we start with two distinct things, point-like charges and fields, and ends up with one, the union of the two.

For an (unstable) equilibrium of the ensemble of the filaments, according to (32), which may look like an ordered, "decorated", two dimensional lattice of filaments, *i.e.* a lattice of filaments with opposite nearest-neighbours vorticities, the *rhs* of Euler's equation vanishes, and the pressure follows the $v^2/2$. Let the origin be on some i-th filament, and let ρ be the position vector with respect to this origin. For ρ close to the origin we have

$$\mathbf{v} = (a^2/\pi)\frac{\omega_i \times \rho}{\rho^2} + ..., \tag{34}$$

from (22), and

$$p = -v^2/2 = -\frac{1}{2}(a^2/\pi)^2 \omega_i^2/\rho^2 + ..., \tag{35}$$

for the dominant contributions, providing the equilibrium condition (32) is fulfilled. The filaments behave then mainly as independent, "one-particle", objects. It is easy then to see that the equation of adiabatic flow $\mathbf{v} grad p = 0$ is satisfied, even for incompressible fluids. It is not obvious that it is also satisfied for an incompresible fluid for finite ρ.

The ensemble of interacting vortex filaments introduced here can be viewed as a filaments liquid, or plasma. Obviously, the length a plays the role of a short-range cutoff, as length L plays the role of a long-range cutoff. Two filaments with opposite vorticities repel each other, so, according to (30), we may estimate the increase in energy associated with introducing such an anti-filament in an environment of nearest-neighbouring parallel filaments as $\delta U \sim -(a^2/\pi)\omega^2 \ln(a/L)$, for vorticities equal in absolute value. At the same time, the gain in entropy is of the order $\delta S \sim \ln(L/a)$. A phase transition is therefore possible at temperature $T \sim a^2\omega^2/\pi$. This is the well-known Berezinsky-Kosterlitz-Thouless transition in two dimensions from a high-temperature disordered liquid (plasma) to a low-temperature "condensed" phase, where filament-anti-filament pairs are present.[11] As it is well-known, it is a topologically ordered phase, not a long-range ordered one.

Oblique filaments. We may consider an ensemble of oblique filaments, but it is more complicate. Instead, we may restrain ourselves to a plane of oblique filaments, with vorticity given by (20). Let $\mathbf{r}_i$ be their positions in plane; then, the vector $\mathbf{r}$ can be written as $\mathbf{r} = \mathbf{r_i} + \mathbf{l}_i + \mathbf{d}_i$, where $\mathbf{l}_i = \omega_i[\omega_i(\mathbf{r}-\mathbf{r}_i)]/\omega_i^2$, and $d_i^2 = |\mathbf{r}-\mathbf{r}_i|^2 - [\omega_i(\mathbf{r}-\mathbf{r}_i)]^2/\omega_i^2$. It is this d_i which enters the vector potential (21). We can write again the "field" energy according to (33). However, the Euler force (27) is not in plane anymore, but it has also a component perpendicular to the plane. This tells us that the restriction to a plane is not a physically valid problem. As we shall see shortly, an ensemble of filaments of various orientations is not a stable ensemble.

Suppose that the integral in (19) is carried along the filament between $z-\varepsilon$ and $z+\varepsilon$, as for a small filament length 2ε. Then, we get $\mathbf{A} = (\omega a^2\varepsilon/\pi)\cdot 1/R$, where R is the distance to that small portion of filaments. Obviously, $\Delta\mathbf{A} = -4\omega a^2\varepsilon\delta(\mathbf{R})$, as if the vorticity is $\omega(\mathbf{R}) = 2\omega a^2\varepsilon\delta(\mathbf{R})$. We get therefore localized particles of vortices.

Vortex particles. Vorticial liquid. Let $\omega(\mathbf{r}) = \omega a^3\delta(\mathbf{r})$ be the vorticity distribution of a vortex particle of vorticity ω and radius a. The vector potential given by (17) is

$$\mathbf{A}(\mathbf{r}) = (a^3/2\pi)\cdot\frac{\omega}{r} \ , \tag{36}$$

according to $\Delta\mathbf{A} = -2\omega(\mathbf{r})$. The velocity is given by

$$\mathbf{v} = -(a^3/2\pi)\cdot\omega\times grad(1/r) \ . \tag{37}$$

The Euler force $\mathbf{F} = -\mathbf{v}\times curl\mathbf{v} == 2\omega(\mathbf{r})\times v = -(a^6/\pi)\delta(\mathbf{r})\cdot\omega\times[\omega\times grad(1/r)]$ vanishes everywhere except for the particle, and Euler's equation is satisfied, as discussed above.

Let us assume an ensemble of vortex particles, with vorticity distribution

$$\omega(\mathbf{r}) = a^3\sum_i \omega_i\delta(\mathbf{r}-\mathbf{r}_i) \ . \tag{38}$$

The vector potential is

$$\mathbf{A}(\mathbf{r}) = (a^3/2\pi)\sum_i \frac{\omega_i}{|\mathbf{r}-\mathbf{r}_i|} \ , \tag{39}$$

and the velocity is given by

$$\mathbf{v} = -(a^3/2\pi)\cdot\sum_i \omega_i\times grad(1/|\mathbf{r}-\mathbf{r}_i|) \ . \tag{40}$$

[11] V. I. Berezinsky, Sov. Phys.-JEPT **32** 493 (1970); *ibid*, **34** 610 (1971); J. M. Kosterlitz and D. J. Thouless, J. Phys. **C6** 1181 (1973).

The Euler force (per unit mass)

$$\mathbf{F} = -\mathbf{v} \times curl\mathbf{v} = -(a^6/\pi)\sum_{ij}\delta(\mathbf{r}-\mathbf{r}_i)\omega_i \times [\omega_j \times grad(1/|\mathbf{r}-\mathbf{r}_j|)] \tag{41}$$

gives a force

$$\mathbf{f_i} = -(a^3/\pi)\sum_j{}' \omega_i \times [\omega_j \times grad_i(1/|\mathbf{r}_i-\mathbf{r}_j|)] \tag{42}$$

acting upon the i-th particle. It follows that the force the j-th particle acts upon the i-th particle is

$$\mathbf{f}_{ij} = (a^3/\pi)[\omega_j(\omega_i\mathbf{R}_{ij}) - (\omega_i\omega_j)\mathbf{R}_{ij}]/R_{ij}^3 \ , \tag{43}$$

where $\mathbf{R}_{ij} = \mathbf{r}_i - \mathbf{r_j}$, while the force the i-th particle acts upon the j-th particle is

$$\mathbf{f}_{ji} = -(a^3/\pi)[\omega_i(\omega_j\mathbf{R}_{ij}) - (\omega_i\omega_j)\mathbf{R}_{ij}]/R_{ij}^3 \ . \tag{44}$$

Obviously, these forces are not central, due to their first term. This term brings a net force actiong upon a pair of vortex particles, and a net torque. This is not surprising, in view of the non-linearly interacting velocities produced by localized vortices in a fluid. Parallel vortices ensure $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$. The analogy with the magnetostatics stops here. There is a profound difference in the theory of electricity and magnetism in comparison with fluid dynamics. It consists in a distinct origin of the electric current $\mathbf{j}$, which does not arise there from a magnetic field, as the vorticity arises in fluids from a curl of velocity. The electric current is related to the time derivative of the position vector, $\mathbf{j} \sim d\mathbf{R}_{ij}/dt$, which makes the first term in equation (43) to be proportional with a time derivative $d(1/R_{ij})/dt$. Consequently, this contribution vanishes when averaged over finite movements of the electric charges, as required for static fields. In addition, we note that the second term in the force given by equation (43), although of Coulomb-law type, implies vorticities but not scalar electric charges as in the theory of electricity and magnetism. Equation (43) gives a new type of force. We may call such an ensemble of particles, endowed with a pair force as given by (43), a vorticial liquid.

In general, a vorticial liquid is not at equilibrium; it possess internal movements. It may not be a potential ensemble even, *i.e.* one with a determined energy as a function of state, due to the non-central forces. An ordered combination of parallel and antiparallel vortices, *i.e.* an ordered combination of vortices and anti-vortices, may get equilibrium, but such an equilibrium is unstable. Indeed, one can check by direct calculation that force $\mathbf{f}_i$ may vanish for every particle, but the matrix of the force derivatives (the force matrix) is not positive definite.[12] Such a stable state would correspond to a latticial screening, which is not attainable neither for a purely-Colomb interacting ensemble of particles.

The second term in equation (43) arises from a potential $U_{ij} = -(a^3/\pi)(\omega_i\omega_j)/R_{ij}$. It can easily be shown by direct calculation that the "mechanical" potential energy $U = (1/2)\sum_{i\neq j} U_{ij}$ is counter-balanced by the "field" energy $E = \int d\mathbf{r} \cdot v^2/2 = \int d\mathbf{r} \cdot \omega\mathbf{A}$ correspponding to the field of velocities, *i.e.* $U + E/a^3 = const$.

One may imagine that the average of the non-central forces in (43) vanishes locally, so that we get then a classical Coulomb plasma consisting of vortices and anti-vortices. However, the first-term in (43) may bring particular local excitations, by creating a vortex-anti-vortex pair for instance, which may bring new features into such a plasma. Distributions of vorticial particles can also be imagined, as for instance filaments, tori, etc, and we get then other varieties of vorticial liquids.

[12] This checking was done by L. C. Cune.

Torroidal vorticities. Dipolar interaction. Suppose a vorticity $\omega = (-\omega\sin\theta,\omega\cos\theta,0)$, localized with thickness ε on a circle of radius a; it can be written as $\omega(\mathbf{r}) = \omega\varepsilon^2\delta(r-a)\delta(z)$, with the centre of the circle at the origin. In the limit $a \to 0$ equation (17) gives the vector potential

$$\mathbf{A} = \frac{\mathbf{m}\times\mathbf{r}}{r^3} \ , \tag{45}$$

where $\mathbf{m} = (\omega\varepsilon^2 a^2/2)(0,0,1)$ can be viewed as the momentum of vorticity. Again, the analogy with the magnetostatics is obvious: $\omega\varepsilon^2$ corresponds to the current i circulating through the circle ($\omega\varepsilon^2 = 2\pi j\varepsilon^2/c = 2\pi i/c$), and $i\pi a^2$ is indeed the magnetic moment of the torroidal circuit. The analogy may be pushed further, and look at ωa^2 like to an intrinsic angular momentum similar with the spin of a particle, which is related to its magnetic momentum. The velocity $\mathbf{v} = curl\mathbf{A}$ is given by[13]

$$\mathbf{v} = -\mathbf{m}/r^3 + 3\mathbf{r}(\mathbf{m}\mathbf{r})/r^5 \ . \tag{46}$$

It is easy to recognize here the dipole field, *i.e.* the magnetic or electric field produced by a magnetic or, respectively, an electric dipole. The Euler force $\mathbf{F} = -\mathbf{v}\times curl\mathbf{v} = 0$.

We can also write the vorticity as $\omega(\mathbf{r}) = (2/a^3)(\mathbf{m}\times\mathbf{r})\delta(r-a)\delta(z)$, and we can consider a distribution

$$\omega(\mathbf{r}) = (2/a^3)\sum_i (\mathbf{m}_i\times\mathbf{R}_i)\delta(R_i-a)\delta(z-z_i) \tag{47}$$

of torroidal vorticities with centres at $\mathbf{r}_i$, where $\mathbf{R}_i = \mathbf{r}-\mathbf{r}_i$. It gives a vector potential

$$\mathbf{A}(\mathbf{r}) = \sum_j \frac{\mathbf{m}_j\times\mathbf{R}_j}{R_j^3} \tag{48}$$

and a velocity

$$\mathbf{v}(\mathbf{r}) = \sum_j [-\mathbf{m}_j/R_j^3 + 3\mathbf{R}_j(\mathbf{m}_j\mathbf{R}_j)/R_j^5] \ , \tag{49}$$

where $\mathbf{R}_j = \mathbf{R}_{ij}+\mathbf{R}_i$. We can now compute the Euler's force $\mathbf{F} = -\mathbf{v}\times curl\mathbf{v} = 2\omega(\mathbf{r})\times\mathbf{v}(\mathbf{r})$ in the limit $a\to 0$, and get the force (up to an $a^2\varepsilon^2$-factor)

$$\mathbf{f}_{ij} = 12\pi\mathbf{m}_i(\mathbf{m}_j\mathbf{R}_{ij})/R_{ij}^5 + 4\pi grad[-\mathbf{m}_i\mathbf{m}_j/R_{ij}^3 + 3(\mathbf{m}_i\mathbf{R}_{ij})(\mathbf{m}_j\mathbf{R}_{ij})/R_{ij}^5] \tag{50}$$

the j-the torus acts upon the i-th torus. Again, its first term is a non-central force, and $f_{ij} \neq -f_{ji}$ in general, as due to this non-central contribution; the pair of torri is acted by a non-vanishing force and torque. In the second term of the force given by equation (50) we recognize the dipolar force acting between two electric or magnetic dipoles, as arising from the dipolar potential energy.

Again, the analogy with the magnetostatics is lost. The difference consists in the electric current $\mathbf{j}$ which enters the magnetic moment $\mathbf{m}_j$ in equation (50), which is $d\mathbf{R}_{ij}/dt$ in the theory of electricity and magnetism. The first term in equation (50) is then proportional to the time derivative $d((1/R_{ij}^3)$ which vanishes when averaged over finite movements of electric charges.

It is easy to check by direct calculation[14] that the "field" energy $E = \int d\mathbf{r}\cdot v^2/2$ of the velocity field counter-balances the "mechanical" potential energy of the torri distribution

$$U = 2\pi\sum_{i\neq j}[-\mathbf{m}_i\mathbf{m}_j/R_{ij}^3 + 3(\mathbf{m}_i\mathbf{R}_{ij})(\mathbf{m}_j\mathbf{R}_{ij})/R_{ij}^5] \ , \tag{51}$$

[13] It can be obtained by using again the tensor ε_{ijk}.

[14] In performing the integrals over the localized torri, we have to note that non-vanishing contributions appear from the $\mathbf{R}_j = \mathbf{R}_{ij} + \rho_i$, where ρ_i is the vector radius (of length a) of the i-th torrus.

i.e. $E+U=const.$

The dipolar liquid given by the distribution (47) has no ordered state. It may be viewed as a disordered plasma, with special types of excitations, as discussed for the vorticial liquid.

Electric charge. It is tempting to construct a theory of electricity and magnetism as based on fluid vortices. However, such a construction is untenable.

For instance, one may view the magnetic field $\mathbf{H}$ as a field of velocities whose curl is proportional to the density $\mathbf{j}$ of electric currents. We get then the right laws of magnetostatics. Indeed, puting $\mathbf{v}\rightarrow\mathbf{H}$ and $curl\mathbf{H}=4\pi\mathbf{j}/c$, we have already Ampere's law. With $\mathbf{H}=curl\mathbf{A}$ and $div\mathbf{A}=0$, we get Green's law $div\mathbf{H}=0$, and Biot and Savart's solution $\mathbf{A}=\int\mathbf{j}/R$, $\mathbf{H}=-(1/c)\int\mathbf{j}\times grad(1/R)$ for Poisson equation $\Delta\mathbf{A}=-4\pi\mathbf{j}/c$. Moreover, Euler's force $\mathbf{F}=-\mathbf{v}\times curl\mathbf{v}$ in Euler's equation becomes Lorentz's $\mathbf{j}\times\mathbf{H}$ force, and its mechanical work may compensate the variation of the field energy $\mathbf{H}^2/2$. However, the things stop here. It is impossible to get localized vortices obeying the laws of electricity and magnetism, as shown by the vorticial and dipolar liquids described above. In addition, the time dependence of the magnetic and electric fields are not supported anymore by the analogy with vortices and fluid dynamics. The basic difference between fluid dynamics and electricity and magnetism arises from the existence of localized electric charge.

Maxwell equations. Newton's laws of motion arise from a basic perception of motion in space and time which is the inertial motion. A similar perception would be satisfactory for electricity and magnetism.

The basic fact known about electricity and magnetism is that some force fields do occur, are produced, by an electric charge with density, say, $\rho(\mathbf{r})$. To produce something, we have two means: use a gradient $grad\varphi$ of some scalar potential φ, and use a time derivative $\partial\mathbf{A}/\partial t$ of some vector potential $\mathbf{A}$. Indeed, something is produced by changes of something. It is worth noting that such a production is not a flow $\partial/\partial t+(\mathbf{v}grad)$ with velocity $\mathbf{v}$ of something. Putting together the two changes $grad\varphi$ and $\partial\mathbf{A}/\partial t$ requires obviously a universal velocity c; it is the light velocity. Then, we may define an electric field through

$$\mathbf{E}=-grad\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\ . \tag{52}$$

This is the basic law of the theory of electricity and magnetism. Obviously, it is a consistent law for inertial frames, if we view the passing from a frame to another moving with a constant velocity $\mathbf{v}$ as a rotation in the four-dimensional space defined by $(\mathbf{r},ict)$. Then, we have at once the principle of relativity, the Lorentz transformations and the theory of relativity.[15]

Equation (52) requires the existence of another field $\mathbf{H}$, defined by $\mathbf{H}=curl\mathbf{A}$. Indeed, if we take the curl in both sides of (52) we get $curl\mathbf{E}=(1/c)[d(curl\mathbf{A})/dt]$, since $curl\cdot grad=0$. We have then immediately the Maxwell equation

$$curl\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \tag{53}$$

for the Faraday and Lenz law of induction. In addition, since $\mathbf{H}=curl\mathbf{A}$, we get the Maxwell equation

$$div\mathbf{H}=0 \tag{54}$$

for Green's law of the rotational magnetic field.

[15] The relativistic invariance requires $(\mathbf{p},iE/c)$ be a negative invariant for a particle; it gives Einstein's relativistic energy $E=\sqrt{m^2c^4+c^2p^2}$, where E is the energy of the particle, $\mathbf{p}$ is its momentum and m is its mass.

We take now the divergence of equation (52): $div\mathbf{E} = -\Delta\varphi - (1/c)[\partial(div\mathbf{A})/\partial t]$; it is easy to see that we can arrive at once to a wave equation with sources and a law of producing electric fields from such sources by imposing the Lorentz gauge

$$div\mathbf{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 \ ; \tag{55}$$

Then we get the wave equation

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho \tag{56}$$

for the scalar potential generated by electric charges, and the Maxwell equation

$$div\mathbf{E} = 4\pi\rho \tag{57}$$

for the Gauss law of the electric fields.

It remains to take the time derivative of equation (52), use the Lorentz gauge: $\partial\mathbf{E}/\partial t = cgrad \cdot div\mathbf{A} - (1/c)(\partial^2\mathbf{A}/\partial t^2)$, and use in addition $curl\mathbf{H} = curl \cdot curl\mathbf{A} = -\Delta\mathbf{A} + grad \cdot div\mathbf{A}$; we get $(1/c)(\partial\mathbf{E}/\partial t) - curl\mathbf{H} = \Delta\mathbf{A} - (1/c^2)(\partial^2\mathbf{A}/\partial t^2)$. In order to get a new wave equation for the vector potential produced by electric currents $\mathbf{j}$, and a production of magnetic field by electric currents, we put

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j} \tag{58}$$

and get the Maxwell equation

$$curl\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \tag{59}$$

for Ampere's law of producing magnetic fields through electric currents. $(1/4\pi)\partial\mathbf{E}/\partial t$ is Maxwell's displacement current. In addition, taking the time derivative of (57) and using (59) we get the continuity equation

$$\frac{\partial\rho}{\partial t} + div\mathbf{j} = 0 \ \ , \tag{60}$$

which shows that electric curents are moving electric charges,

$$\mathbf{j} = \rho\mathbf{v} \ . \tag{61}$$

With this, the derivation of the Maxwell equations of electricity and magnetism is completed. By adequate notations, it is easy to show that they are relativistically invariant. We note that they are linear equations in fields. They follow the principle of superposition. In addition, in the non-relativistic limit $v/c \rightarrow 0$ the vector potential vanishes, so the magnetic field is a purely relativistic effect. Moreover, they contain singularities for point-like charges and currents, like Euler's equation of fluids for point-like vortices.

However, there is a profound difference between Euler's equation for fluids and Maxwell equations. Euler's equation has only one object, the field of velocities, and particular forms of this field, localized vortices, can be consistently separated from the velocity field, providing thereby an image for particles and fields. The two objects thus separated have their own dynamics, even possibly inconsistent in general, and posssibly inter-related in some particular cases. At least in simple cases like one vortex such a separation is fully consistent.

This is not so with Maxwell equations. The theory of electricity and magnetism recognizes two separate, distinct entities, fields and point-like charges, but inter-relates them through Maxwell equations, which amounts to identify them as one and the same object. This is a logic fault, albeit supported by the natural empirical situation. It has troubling consequences, as expected, in the well-known "self-interacting infinities". The unavoidable logical consequence of such a self-contradictory position, is that such a situation, as described by the theory of electricity and magnetism, does not exist, under certain conditions. This is implied by the quantal delocalization of the point-like charges during their relativistic existence.

The wave equations (56) and (58) suggest that fields are coupled to matter through an action reading

$$S_{fm} = \int dt \cdot (-q\varphi + \mathbf{i}\mathbf{A}/c) \ , \tag{62}$$

where q is a point-like charge and $\mathbf{i} = q\mathbf{v}$ is the corresponding electric current. We get

$$-qgrad\varphi - (q/c)(\mathbf{A}grad)\mathbf{v} \tag{63}$$

for the variation of this action with respect to particle coordinate $\mathbf{r}$. Now we use $grad(\mathbf{ab}) = (\mathbf{a}grad)\mathbf{b} + (\mathbf{b}grad)\mathbf{a} + \mathbf{a} \times curl\mathbf{b} + \mathbf{b} \times curl\mathbf{a}$ for $\mathbf{v}$ and $\mathbf{A}$,[16] and get

$$\begin{gathered} -qgrad\varphi + (q/c)(\mathbf{v}grad)\mathbf{A} + (q/c)\mathbf{v} \times curl\mathbf{A} = \\ = -qgrad\varphi - (q/c)\partial\mathbf{A}/\partial t + (q/c)d\mathbf{A}/dt + (1/c)\mathbf{i} \times \mathbf{H} = \\ = q\mathbf{E} + (1/c)\mathbf{i} \times \mathbf{H} + (q/c)d\mathbf{A}/dt \end{gathered} \tag{64}$$

for this variation. On the other hand, the variation of the free-particle action is $-d\mathbf{p}/dt$, and from (62) we get an additional term $-(q/c)d\mathbf{A}/dt$, so that $\mathbf{P} = \mathbf{p} + q\mathbf{A}/c$ is the canonical momentum of a particle in the field.[17] Equating then $d(\mathbf{p} + q\mathbf{A}/c)$ with equation (64) we get the Lorentz force

$$d\mathbf{p}/dt = q\mathbf{E} + (1/c)\mathbf{i} \times \mathbf{H} \tag{65}$$

acting upon a charged particle moving in an electromagnetic field. The magnetic field brings no contribution to the rate $\mathbf{v}d\mathbf{p}/dt$ of change of the kinetic energy E_{kin} of a particle.

The electromagnetic waves propagate with the velocity of light c, as described by equations (56) and (58). It is easy to show[18] that the field energy $E_f = (1/4\pi)(E^2/2 + H^2/2)$ propagates according to $\partial E_f/\partial t = -\mathbf{jE} - div\mathbf{S}$, where $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ is the Poynting vector. Integrating over space, $\mathbf{jE}$ leads to the rate $\mathbf{v}d\mathbf{p}/dt$ of change of the kinetic energy E_{kin} of the charges, so that $\partial(E_f + E_{kin})/\partial t + div\mathbf{S} = 0$, *i.e.* the rate of change of the field energy plus the kinetic energy of the charges is equal to the flow of the field energy. This is quite different from the field of velocities and the motion of fluid vortices.

The action of the field is $\int dtdr \cdot (E^2 - H^2)/8\pi$. Its variation (including the field-matter part S_{fm}) leads to Maxwell equations. Its flows lead to the momentum conservation. The Poynting vector carries a momentum density $\mathbf{S}/c$, while the field contains the Maxwell stress $(E_iE_j + H_iH_j)/4\pi - \delta_{ij}E_f$. The momentum conservation looks similar to the motion of an elastic, rigid, ethereal medium, rather than a fluid.

Particles and fields. Infinities. Electric charges are point-like. Consequently, their electromagnetic energy is infinite. The momentum carried by the electromagnetic field produced by a moving point-like charge gives an "electromagnetic mass" of the charge, which again is infinite,

[16] which can be checked by using again the tensor ε_{ijk} for the definition of the curl and of the vectorial product.
[17] and its hamiltonian reads $H = p^2/2m + q\varphi = (\mathbf{P} - q\mathbf{A}/c)^2/2m + q\varphi$.
[18] Since $div(\mathbf{a} \times \mathbf{b}) = \mathbf{b}curl\mathbf{b} - \mathbf{a}curl\mathbf{b}$, by using again the tensor ε_{ijk}.

due to the vanishing size of the particle. The rate of change in time of such an electromagetic momentum gives a force. It acts upon the particle itself, and it is Lorentz's damping force. It is infinite, too, because of the vanishing size of the particle. The charge interacts with its own field, since the electromagnetic field has its own wave-like motion with the velocity of light, which is different from the velocity of the moving charge. In the non-relativistic limit $v/c \to 0$, there is no such a damping force. The retarded electromagnetic interaction looks like the field is produced by the particle now, propagates, and particle meets it again in its motion, after a while.

These infinities are the origin of the well-known ultraviolet divergencies in quantal field theory. A cutoff should be practized in such computations, arising from a finite size of particle. It is the distance over which the light propagates in the time uncertainty corresponding to the rest energy of the particle, *i.e.* the Compton length $c \cdot \hbar/mc^2 = \hbar/mc$, where $\hbar$ is Planck's constant. It means that we should do perturbation calculations as long as the particles do exist. The renormalization technique does this job.

Summary. Let us summarize. The fluids may support vortices. This was shown by Helmholtz. Such vortices may be localized, either, for instance, as filaments or as vortex particles, or as particles endowed with torroidal vorticities. Such a localization may be viewed as singular, giving the idea of particles made up of fields, in particular fields of velocity. Then a problem appears: the singularities in Euler's equation. We may get rid of them by assuming that particles have their own dynamics, related to the dynamics of fields by the conservation of energy in some cases. This is satisfactory. On the other hand, such a picture is quite similar with static magnetic fields, though Euler's force may bring appreciable differences. The differences arise from the non-linear nature of the fluid dynamics arising from one single object - the velocity field -, in contrast with the linear Maxwell equations. It is also related to the definition of the electric currents as being proportional to the velocity of the moving charges, in contrast to the fluid vorticity, which arises precisely from the curl of its field, *i.e.* the curl of the velocity. Nevertheless, it is tempting to liken the currents with vortices, and magnetic fields to fields of velocity in some ethereal fluid. Unfortunately, such a picture is untenable for electric fields and charges, and the time behaviour of the electric and magnetic fields has no connection to any fluid motion. Moreover, the electromagnetic field seems to require an ethereal rigid solid rather than a fluid. The electromagnetism is fundamentally different from the fluid dynamics, though it admits the distinction between particles and fields. This admission is not consistently kept however in Maxwell equations, which end up with removing it in fact. This logical inconsistency of the theory of electricity and magnetism leads to well-known difficulties with the infinities, which the relativity and the quantal motion may deal with.

The fluid motion is by far richer than the motion of the electricity and magnetism, as a consequence of the non-linear Euler's force. It allows the contemplation of new types of matter, like the vorticial and dipolar liquids. The theory of electricity and magnetism, with its logical inconsistencies, makes unavoidable conclusions like quantal and relativistic behaviour, and, especially, the limited existence of the substantial matter.