Journal of Theoretical Physics

Founded and Edited by M. Apostol **140** (2006)

ISSN 1453-4428

Quantal transitions by change of parameters and in non-inertial motion (Lecture eight of the Course of Theoretical Physics)

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Abstract

It is shown that a quantal ensemble may suffer transitions when varying parameters which, in the adiabatic limit, give the Berry phase. Similarly, non-inertial motion may cause, in certain cases, quantal transitions for a quantal ensemble governed by Schrodinger's equation. For (non-uniform) translations or rotations of (interacting) ensembles of particles such transitions do not occur. The relevance of the gauge transformations is discussed in this context. The non-inertial quantal transitions may appear for particles in an external field, like electrons in the field of the nuclei in atoms and molecules. In the latter case, the non-inertial quantal transitions cause an additional width of the electronic spectral terms, beside the natural one produced by the motion of the nuclei. Such quantal transitions can lead to a thermal equilibrium, whose temperature is estimated and shown to be similar with the temperature associated with the Unruh effect. Similar results hold for quantal fields. The coupling of (non-relativistic) quantal ensembles to gravitational waves is presented, and shown to cause quantal transitions in certain cases.

Berry phase. Let us assume that the hamiltonian H, its eigenfunctions φ_k and the energy eigenvalues E_k depend on a parameter denoted generically by $\mathbf{R}$. We write explicitly this dependence in the eigenvalue equation

$$H(\mathbf{R})\varphi_k(\mathbf{R}) = E_k(\mathbf{R})\varphi_k(\mathbf{R}) \ . \tag{1}$$

In particular we are interested in a time dependence $\mathbf{R}(t)$ of the parameter $\mathbf{R}$, and write Schrodinger's equation as

$$i\hbar\partial\psi(t)/\partial t = H(\mathbf{R})\psi(t) \ . \tag{2}$$

In the adiabatic limit $\dot{\mathbf{R}} \to 0$ the original eigenstate $\varphi_n(\mathbf{R})$ is preserved during the temporal evolution, and a solution of equation (2) reads[1]

$$\psi_n(t) = \exp[-(i/\hbar)\int_0^t E_n(\mathbf{R}(t'))dt']e^{i\gamma_n(t)}\varphi_n(\mathbf{R}(t)) \ , \tag{3}$$

where $\gamma_n(t)$ is given by

$$\dot{\gamma}_n(t) = i\,(\varphi_n, \partial\varphi_n/\partial\mathbf{R})\,\dot{\mathbf{R}} \ . \tag{4}$$

For a circuit C described by parameter $\mathbf{R}$ it is Berry's geometric phase $\gamma_n(C)$.[2]

[1] The adiabatic limit should be taken both in the energy phase factor and phase γ.

[2] M. V. Berry, Proc. R. Soc. Lond. **A392** 45 (1984).

Transitions by change of parameters. This result implies that, in general, for non-vanishing $\dot{\mathbf{R}}$, the quantal ensemble may exhibit transitions between its various states. Indeed, the general solution of equation (2) can be written as

$$\psi(t) = \sum_k a_k(t) \exp[-(i/\hbar) \int_0^t E_k(\mathbf{R}(t'))dt'] \varphi_k(\mathbf{R}(t)) \ , \tag{5}$$

where the coefficients $a_k(t)$ obey the equation

$$\dot{a}_n = i \sum_k a_k \gamma_{nk}(t) \dot{\mathbf{R}} \exp[(i/\hbar) \int_0^t [E_n(\mathbf{R}(t')) - E_k(\mathbf{R}(t'))]dt'] \ , \tag{6}$$

and

$$\gamma_{nk}(t) = i \left(\varphi_n, \partial\varphi_k/\partial\mathbf{R}\right) \ . \tag{7}$$

This $\gamma_{nk}(t)$ is obviously a generalization of the Berry phase; the latter corresponds to

$$\gamma_n(t) = \int_0^t d\mathbf{R}(t')\gamma_{nn}(t') \ , \tag{8}$$

where the integration is performed along the path described by parameter $\mathbf{R}$ in its motion from $\mathbf{R}(t=0)$ to $\mathbf{R}(t)$. The $\gamma_{nk}(t)$ are the matrix elements of the operator $-\mathbf{P}/\hbar$, $\gamma_{nk} = -\mathbf{P}_{nk}/\hbar$, where $\mathbf{P}$ may be viewed formally as the momentum associated with parameter $\mathbf{R}$. Then, equation (6) gives the transition amplitudes caused by a perturbation $H_1 = \mathbf{VP}$, where $\mathbf{V} = \dot{\mathbf{R}}$ is the velocity of the parameter $\mathbf{R}$.

Equation (6) is solved in the first order of the perturbation theory, with the initial conditions $a_n(0) = 1$, $a_k(0) = 0$, for $k \neq n$. We get the transition amplitudes

$$a_{kn}(t) = i \int_0^t d\mathbf{R}(t')\gamma_{kn}(t') \exp[(i/\hbar) \int_0^{t'} [E_k(\mathbf{R}(t'')) - E_n(\mathbf{R}(t''))]dt''] \ , \tag{9}$$

where an additional label k has been given to the coefficient a_n in order to indicate the transition from state n to state k. At the same time

$$a_{nn}(t) = 1 + i \int_0^t d\mathbf{R}(t')\gamma_{nn}(t') = 1 + i\gamma_n(t) \ . \tag{10}$$

From (9) and (10) one can see that in the adiabatic limit $\dot{\mathbf{R}} \to 0$ the Berry phase $a_{nn}(T) = e^{i\gamma_n(T)}$ is recovered for a circuit C, where $\gamma(T) = \gamma_n(C)$, T being the period during which the parameter $\mathbf{R}$ describes the circuit C.

In the first-order of the perturbation theory the $\mathbf{R}$-dependence of the matrix elements γ_{kn} and energy eigenvalues in the exponential factor in (9) may be neglected. The transition amplitudes can then be written as

$$a_{kn}(t) = -(i/\hbar) \int_0^t dt' \cdot \mathbf{V}(t')\mathbf{P}_{kn} \exp(i\omega_{kn}t') \ , \tag{11}$$

where $\omega_{kn}(t) = (E_k - E_n)/\hbar$.

First, we note that for a uniform change of parameters, *i.e.* for $\mathbf{V} = const$, the transition amplitudes are vanishing ($a_{kn}(t) = 0$, $k \neq n$). The diagonal amplitude $a_{nn}(t) = 1 - (i/\hbar)\mathbf{VP}_{nn}t \simeq \exp(-i\mathbf{VP}_{nn}t/\hbar)$ given by (10) contains the correction $\mathbf{VP}_{nn}$ to the energy of the state φ_n in the

first-order of the perturbation theory. The gauge transformation $\psi'_n = \exp(-i\mathbf{VP}_{nn}t/\hbar)\psi_n$ leaves Schrodinger's equation unchanged.

Let us assume that the velocity has a sudden variation from $\mathbf{V} = 0$ for $0 < t < t_0$ to $\mathbf{V} = const$ for $t_0 < t$, such that $\partial\mathbf{V}/\partial t = \mathbf{V}\delta(t - t_0)$. The transition amplitudes given by (11) become

$$a_{kn}(t) = -\mathbf{VP}_{kn}e^{i\omega_{kn}t}/(E_k - E_n) + [\mathbf{VP}_{kn}/(E_k - E_n)]e^{i(E_k - E_n)t_0/\hbar} \ . \tag{12}$$

The first term in the *rhs* of this equation corresponds to the change in the wavefunction under the action of the constant perturbation $\mathbf{VP}$ for $t > t_0$. The transition amplitude is given by the second term in the *rhs* of equation (12), so the transition probability is $w_{kn} = [\mathbf{VP}_{kn}/(E_k - E_n)]^2$.

If the velocity is periodic in time with frequency ω, $\mathbf{V}(t) = \mathbf{V}e^{i\omega t} + c.c.$, the transition probability per unit time is given by $w_{kn} = (2\pi/\hbar)(\mathbf{VP}_{kn})^2\delta(E_k - E_n \pm \hbar\omega)$, in the limit of the infinite time. The calculations are not restricted to the discrete spectrum, so we may also get the transition in the continuum spectrum, with inclusion of the density of states. Therefore, such changes of parameters can induce a disintegration (and a recombination) of the quantal ensemble. It is worth noting that frequencies ω in the variation of the parameter $\mathbf{R}$ must be comparatively high, of the order of the frequencies of the quantal ensemble, in order to produce quantal transitions.

A digression: a unitary transformation. The interaction $H_1 = \mathbf{VP}$ can be introduced explicitly in Schrodinger's equation (2),

$$i\hbar\partial\psi(t)/\partial t - \mathbf{VP}\psi(t) = H(\mathbf{R})\psi(t) \ . \tag{13}$$

A unitary transformation $\psi = \exp(-iS)\psi'$, where $S = (1/\hbar)\mathbf{RP}$, removes this interaction from the *lhs* of equation (13), but introduces a new interaction term in the hamiltonian, which becomes $H' = H + \mathbf{R}(\partial H/\partial\mathbf{R}) + ...$ One can show by direct calculation that the new interaction $\mathbf{R}(\partial H/\partial\mathbf{R})$ gives the same transition amplitudes in the first order of the perturbation theory as the ones given by equation (11), as expected. [3]

Unruh temperature. Thermal equilibrium can be reached through such transitions as those described above. Indeed, the master equation that governs the population of states N_k reads $\partial N_n/\partial t = \sum_k w_{nk}N_k - \sum_k w_{kn}N_n$. We may take an average transition probability w per unit time in this equation, and an average number ν of states coupled by such transition probabilities, such as $w\nu \sim 1/\tau$, where τ is a characteristic time of reaching the equilibrium. We seek an equilibrium solution $N_k \sim \exp(-\beta E_k)$, where $\beta = 1/T$ is the inverse of the temperature, and assume a generic quantal ensemble with $E_k \sim n\hbar\omega$, where ω is a characteristic frequency of the transitions and n is an integer. It is easy to see that the temperature can then be estimated as $T \sim \hbar\omega\nu$. For a change δT in temperature we get $\delta T \sim \hbar\omega\delta\nu \sim \hbar\omega(\delta w/w)$, where $\delta w = \delta(1/\tau)$. This result is particularly interesting, since $\omega \sim a/V$, where a is the average acceleration and V is the average velocity associated with such an excitation process. We get therefore $T \sim (\hbar a/V)\nu$, a result which is similar to the Unruh temperature.[4] If we assume that the frequencies of the quantal ensemble are sufficiently dense, then, for a frequency $\omega = 1GHz$, which is attainable on the macroscopic scale, we get a temperature $T \simeq 10^{-2}K$. [5]

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[3] In this calculation the relationship $(\varphi_k, (\partial H/\partial\mathbf{R})\varphi_n) = (E_n - E_k)\,(\varphi_k, \partial\varphi_n/\partial\mathbf{R}) + (\partial E_n/\partial\mathbf{R})\delta_{kn}$ is used. It is an extension of the well-known Feynman's "theorem" (R.P. Feynman, Phys. Rev. **56** 340 (1939)).

[4] W. G. Unruh, Phys. Rev. **D14** 870 (1976).

[5] Such transitions produce a width of the spectral lines, associated with a change of temperature, until the equilibrium is reached, which takes a time $\tau \sim w^{-1}$ of the order of the inverse of the average transition probability w. There exists, therefore, an uncertainty of temperature $\delta T \sim \hbar w$. (The width of a spectral line is $\Delta E_n = \sum w_{nk}\,|E_k - E_n|$).

Another digression: what we mean by measuring a quantal energy. If a quantal ensemble does not change energy with the environment, then it is in a state with a well-determined energy, say E_n, with a certain probability (stationary state). We make the ensemble to interact with a probe with all frequencies, and are able to identify all the emitted or absorbed energies. Thus we determine the energy spectrum of the ensemble. Among the absorptions, some result in destroying the ensemble, in the sense that its components become free. The lowest of these absorptions determines the origin of the energy scale (the zero of energy), so we have an "absolute" measurement of the spectrum. If the measurement repeats identically, then the ensemble was indeed in a certain energy state, with probability one. If the emission and absorption in the spectrum takes place with certain probabilities, then the original state was a superposition of stationary states (a state with "undetermined" energy). The intensity of the emission and absorption processes is given by the degeneracy (density of states) and the strength of the interacting probe.

A few examples. We illustrate the above result by a particle of mass m moving in an infinite square potential well in one dimension. The eigenfunctions are $\varphi_n(x) = \sqrt{2/a}\sin(\pi nx/a)$ and the energy eigenvalues are given by $E_n = \pi^2\hbar^2n^2/2ma^2$, where $n = 1, 2, ...$ We take the width a of this potential well as parameter R, and consider that the wall placed at distance a from the origin is subjected to an oscillatory motion of frequency ω as described by $a = a_0 + \varepsilon\cos\omega t$, where $\varepsilon/a_0 \ll 1$. Making use of equation (11) we get the transition probabilities $w_{kn} = 2\pi\hbar\left[\varepsilon\omega kn/a_0(k^2-n^2)\right]^2\delta(E_k - E_n \pm \hbar\omega)$ per unit time, in the limit of the infinite time. We note that the diagonal matrix element γ_{nn} is vanishing in this case, $\gamma_{nn} = 0$.

Following Berry (*loc. cit.*), another example is provided by a spin $\mathbf{S}$ placed in a magnetic field $\mathbf{B}$. The hamiltonian reads $H = -g\mu\mathbf{BS}$, where g is the gyromagnetic factor and μ is the Bohr magneton. The energy eigenvalues are given by $E_n = -g\mu Bn$, where $n = -S, ...S$. In order to calculate the matrix elements entering equation (11) it is convenient to use the identity $(E_n - E_k)(\varphi_k, \partial\varphi_n/\partial\mathbf{R}) = (\varphi_k, (\partial H/\partial\mathbf{R})\varphi_n)$ for $k \neq n$. We write then $\mathbf{BS} = B(S_x\sin\theta\cos\phi + S_y\sin\theta\sin\phi + S_z\cos\theta)$, and take the angles θ and ϕ as parameters $\mathbf{R}$. First, let us set $\phi = 0$ and let θ describe a circuit according to $\theta = \omega t$, where $\omega \ll g\mu B/\hbar$. Making use of equation (9) we get transition probabilities $w_{kn} = (\pi\hbar\omega^2/8)\left[S(S+1) - n(n\pm1)\right]\delta_{k,n\pm1}\delta(E_n - E_k \pm \hbar\omega)$, in the limit of the infinite time. Since $\omega \ll g\mu B/\hbar$ these transition probabilities are vanishing, in fact, as we get by using equation (11). We may also set $\theta = const$ and let $\phi = \omega t$ describe a conical circuit of semiangle θ. The results are similar, the amplitudes being now proportional to $\sin\theta$. As it is well-known (Berry, *loc. cit.*), the Berry phase is given by $\gamma_n = -n\Omega(C)$, where $\Omega(C)$ is the solid angle subtended by the circuit C along which the parameter $\mathbf{R}$ moves. [6]

Another example is provided by the electronic terms of the molecules, which depend parametrically on the nuclear coordinates $\mathbf{R}$. The interaction $H_1 = \mathbf{VP}$ can easily be estimated as $H_1 \sim (m/M)E_{el}$, where E_{el} is a characteristic electronic term of the molecule and m/M is the ratio of the electron mass m to the nuclear mass M. It is of the same order of magnitude as the accuracy of the adiabatic decoupling of the electronic motion from the nuclear motion, so that it gives a natural width of the electronic terms in molecules. [7]

[6] For a circuit C, the integral in equation (8) can be transformed into a surface integral, such that $\gamma_n(C) = i\int d\mathbf{S}\cdot\sum_k'(\varphi_n, (\partial H/\partial\mathbf{R})\varphi_k)\times(\varphi_k, (\partial H/\partial\mathbf{R})\varphi_n)/(E_k - E_n)^2$, where $k = n$ does not contribute to the summation because $(\varphi_n, \partial\varphi_n/\partial\mathbf{R})$ is imaginary (Berry, *loc. cit.*).

[7] We could, in principle, obtain the exact molecular (or atomic) energy levels, by separating, for instance, the motion of the center of mass from the relative motion. However, such exact energy levels are only observable in long times τ, much longer than the characteristic nuclear times wich are of the order $\tau_n \sim (M/m)\tau_{el}$, where τ_{el} is the characteristic time of the electronic motion. Such times would imply very low transition rates ($w \sim 1/\tau$) and a weak energy coupling $\delta\varepsilon \sim \hbar/\tau$, which is much smaller than $(m/M)E_{el}$. The actual time of observation is of the order of the electronic time τ_{el} (according to the electromagnetic coupling strength), sufficient to resolve the electronic terms, so the effect of the nucler motion is, indeed, a parametric dependence of the electronic motion

A similar analysis regarding the transition probabilities can be done for the Aharonov-Bohm effect, making use of the typical experimental set-up discussed by Berry (*loc. cit.*). It is easy to see that the vector potential of the electromagnetic field does not give rise to transitions. However, the Aharanov-Bohm effect implies also a change of coordinates (the quantal ensemble is moved around a circuit). Consequently, it may be worth investigating the effect of the change of the coordinates upon the quantal ensemble.

Translations. Let us consider a translation $\mathbf{r} = \mathbf{r}' + \mathbf{R}(t')$, $t = t'$. In this case, the hamiltonian, its eigenfunctions and energy eigenvalues do not depend on the parameter $\mathbf{R}$. Schrodinger's equation (2) becomes

$$i\hbar\partial\psi(t', \mathbf{r}')/\partial t' = H(\mathbf{r}')\psi(t', \mathbf{r}') + i\hbar\mathbf{V}\partial\psi(t', r')/\partial\mathbf{r}' \ , \tag{14}$$

where $\mathbf{V} = \dot{\mathbf{R}}$. We may view the last term in the *rhs* of equation (14) as an interaction $H_1 = -\mathbf{V}\mathbf{p}$, where $p = -i\hbar\partial/\partial\mathbf{r}'$ is the momentum associated to coordinate $\mathbf{r}'$. According to the perturbation theory this term may produce transitions, whose amplitudes are given by

$$a_{kn}(t) = (i/\hbar)\int_0^t dt_1 \cdot \mathbf{V}(t_1)\mathbf{p}_{kn}e^{i\omega_{kn}t_1} \ , \tag{15}$$

and

$$a_{nn}(t) = 1 + i\int_0^t dt_1 \cdot \mathbf{V}(t_1)\mathbf{p}_{nn} \ . \tag{16}$$

For a free particle the transition amplitudes are vanishing, since $\mathbf{p}_{kn} = 0$ for $k \neq n$. Similarly, for an ensemble of (in general interacting) particles momentum $\mathbf{p}$ is the total momentum, *i.e.* the momentum of the center of mass of the ensemble, so there are no transitions in this case, as expected. The coefficient $a_{nn}(t)$ corresponds to a gauge transformation $\exp[i\int_0^t dt_1\mathbf{V}(t_1)\mathbf{p}_{nn}]$ of the n-state, which, in general has not a determined energy (it is not a stationary state, in general). For constant velocity $\mathbf{V} = const$, the phase of this gauge transform is the first-order correction to the energy of the n-state.[8] On the other hand, it is easy to check that the gauge transformation $\psi'(t, \mathbf{r}') = \exp[-(i/\hbar)(MV^2t/2 + M\mathbf{V}\mathbf{r})]\psi(t, \mathbf{r})$, where M is the mass of the ensemble, preserves Schrodinger's equation, in accordance with Galileo's principle of relativity.[9]

The situation is different for particles in an external field. There, in general, the off-diagonal matrix elements $\mathbf{p}_{kn}$ of the momentum of the particles are non-vanishing, and there may exist transitions. Suppose, for instance, that one or more particles in an ensemble of interacting particles acquire a large mass, so that they may be viewed as being at rest during the motion of the rest of particles. Their interaction with the rest of particles become now an external field for the latter, whose motion depend parametrically on the positions of the former. The coordinates of the heavy particles do not appear anymore in the momentum, so there may exist non-vanishing matrix elements of this momentum between states of the moving particles. It follows that non-inertial motion may give rise to quantal transitions for particles in an external field.

Another digression: quantization for non-inertial motion. Let $E = mv^2/2 = p^2/2m$ be the energy of a particle of mass m. The corresponding quantal motion is described by $\mathbf{p} \to -i\hbar\partial/\partial\mathbf{r}$

on the nuclear coordinates, in accordance with the adiabatic approximation, associated at the same time with a natural width of the spectral (electronic) lines, due to the nuclear motion. (This remark illustrates once again that "we are in theoretical physics, where the mathematical rigour is not only impossible, but it is nor desirable").

[8] It is worth noting the correction $\delta(p_{nn}^2/2M) = \mathbf{V}\mathbf{p}_{nn}$, where M is the mass of the ensemble and $\delta(\mathbf{p}_{nn}/M) = \mathbf{V}$, $V \ll p_{nn}/M$.

[9] The unitary transformation $\psi = \exp(-i\mathbf{R}\mathbf{p}/\hbar)\psi'$ takes the Schrodinger equation $i\hbar\partial\psi/\partial t = H\psi$ into $i\hbar\partial\psi'/\partial t = H\psi - \mathbf{V}\mathbf{p}\psi' + \mathbf{R}(\partial H/\partial\mathbf{r})\psi' + ...$ Making use of $(\varphi_k, (\partial H/\partial\mathbf{r})\varphi_n) = (E_n - E_k)(\varphi_k, \partial\varphi_n/\partial\mathbf{r})$ one can show by direct calculation that the additional interacting term in the hamiltonian has no relevance. Such a unitary transformation is different from the coordinate change.

and $E \to i\hbar\partial/\partial t$. Let the coordinate $\mathbf{r}$ changes like $\mathbf{r} \to \mathbf{r}' + \mathbf{R}$. Then, momentum changes like $\mathbf{p} \to \mathbf{p}' + m\mathbf{V}$, where $\mathbf{V} = \dot{\mathbf{R}}$. Energy reads $E = E' + \mathbf{V}\mathbf{p}' + mV^2/2$, where $E' = p'^2/2m$. The new momentum $\mathbf{p}'$ differs from the former $\mathbf{p}$ by a c-number $m\mathbf{V}$, so its quantization rests unchanged. It follows that $E' = p'^2/2m$ is quantized as previously. This means that the energy spectrum is not changed, as expected. However, the original state with energy E is not anymore identical to any new state with energy E', except for inertial motion $\mathbf{V} = const$, when their energies differ by an irrelevant constant. The new state has now an energy $E' = E - \mathbf{V}\mathbf{p}' - mV^2/2$, which contains precisely the interaction term $H_1 = -\mathbf{V}\mathbf{p}'$ as in equation (14), as expected. Even in the absence of transitions (as for free particle), the new state has not a well-determined energy in general, due to the non-inertial motion. This means that the quantal behaviour depends on the observer. This is so, as with other physical theories, because quantal mechanics is essentially a theory which describes measurable things, and, precisely, describes the results of such measurements.

Rotations. A similar result holds also for rotations. Let $r_i = \alpha_{ij}(t')r_j$, $t = t'$ be a change of coordinates $(i, j = 1, 2, 3)$, where α_{ij} is a rotation matrix of angle ϕ and angular velocity $\dot{\phi} = \mathbf{\Omega}$ about some axis, such as $r'_i = \alpha_{ji}(t)r_j$, $\alpha_{ji}\alpha_{jk} = \delta_{ik}$. Making use of $\alpha_{li}\dot{\alpha}_{lj} = \varepsilon_{ijk}\Omega_k$, where ε_{ijk} is the totally antisymmetric unit tensor, we get easily that an interaction $H_1 = \mathbf{\Omega}\mathbf{l}$ appears in hamiltonian, similar with the interaction given by (14), where $\mathbf{l}$ is the total (orbital) angular momentum. The discussion is similar with the one given above for translations. For a free particle, or an ensemble of interacting particles, the total angular momentum has not off-diagonal matrix elements. The coefficient a_{nn} given by (16) may generate a gauge transformation, which may reflect the non-stationarity of the rotating state. For uniform rotations, *i.e.* for $\mathbf{\Omega} = const$, the gauge transformation $\psi'(t, \mathbf{r}') = \exp[-(i/\hbar)(m\rho^2\Omega^2/2 - m\rho^2\mathbf{\Omega}\phi/\hbar)]\psi(t, \mathbf{r})$, where ρ is the distance of particles to the axis of rotation, leaves Schrodinger's equation unchanged, in accordance with its invariance under uniform rotations. In this gauge transformation $m\rho^2$ denotes the total momentum of inertia I and the first term in the phase is the kinetic energy $l^2/2I$.[10]

For particles in an external field the angular momentum may have non-vanishing off-diagonal matrix elements, so non-uniform (accelerated) rotations may induce quantal transitions.

Conclusion. The main conclusion of the results described herein is that non-inertial motion may cause quantal transitions for quantal ensembles in external fields. That means that an observer who is set in non-inertial motion may record such quantal transitions. Such quantal transitions are, in general, associated with non-uniform (accelerated) changes in parameters of the motion. These non-inertial quantal transitions do not occur for ensembles of interacting particles, as they are associated with the motion of the center of mass, or with the motion of the ensemble as a whole.

The analysis can be extended to fields. The field equations are solved for the eigenmodes of frequency ω, let $\varphi_\omega(\mathbf{r})$ be the solutions. The fields are then quantized by writing them as a superposition of eigenmodes $a_\omega e^{i\omega t}\varphi_\omega(\mathbf{r})$, where a_ω are destruction or creation operators (according to particles or antiparticles). Coordinate transformations generate interacting terms in the hamiltonian, which may lead to transitions, *i.e.* to creation and destruction of field quanta, according to the time dependence of the coordinate transformations. For free, or interacting, fields, as well as for non-accelerated coordinate transformations, such transitions do not appear. It is essential to have spatially non-uniform eigenmodes, as for fields in "external fields", for such transitions to occur. More general coordinate transformations can be imagined, like the ones which are spatially non-uniform, which may lead to transitions. Such local coordinate transformations are, essentially, similar with a gravitational field.

[10] In general, a gauge transformation of the form $H' = \exp[-i\chi(\mathbf{r})]H\exp[i\chi(\mathbf{r})]$, where $\chi(\mathbf{r}) = -m\mathbf{V}\mathbf{r}/\hbar$, leads to $H' = H + mV^2/2 - \mathbf{V}\mathbf{p}$, which amounts to a redefinition of the momentum ($\mathbf{p} \to \mathbf{p} - m\mathbf{V}$). Similarly, $\chi(r) = m\rho^2\mathbf{\Omega}\phi/\hbar$ gives $H' = H + m\rho^2\Omega^2/2 + \mathbf{\Omega}\mathbf{l}$, which means a redefinition of the angular momentum.

The time dependence of the change of coordinates must be fast enough in order to match the quanta of the fields (the excitation spectrum), and so have transitions. The non-inertial quantal transitions requires frequencies in the non-inertial motion of the observer comparable with the quantal frequencies.

Indeed, the latter point is worth stressing in attempting to device an experiment whereby such non-inertial quantal effects would be observable. Let us suppose a complex, statistical, quantal ensemble, like condensed matter for instance, which may have quantal levels in the range of macroscopically attainable frequencies, say, of order of GHz's. We set an observer to check the spectral thermal equilibrium of the ensemble, and compare the results with those recorded by the observer placed in an oscillatory motion with freqency, say, $1GHz$. In the latter case the observer will record an increase in the temperature of the ensembe in this spectral range, *i.e.* for $1GHz \sim 10^{-2}K$. This is similar with Unruh black-body radiation of gravitational black holes.

We may also imagine a sudden jolt imposed upon a quantal ensemble, like, for instance, ions traversing a limited portion of space where a high (static) electric field is present. In this case, the non-inertial motion may have a richer spectrum, and quantal transitions may occur between the ionic states, though their spectral content is rather low. Suppose, at the same time, that the observer records spectral lines of this ionic gas, and compare them with those recorded in the absence of the electric field. The former will exhibit an additional, small width, in comparison with the latter, due to the non-inertial motion.

A last digression: waves. Suppose that we have a wave $\psi(t, \mathbf{r})$, characterized by a phase velocity $\mathbf{v}$ (not necessarily the light velocity, *i.e.* it may be associated with substance), which obeys consequently the wave equation

$$\partial^2\psi/\partial t^2 - v^2\partial^2\psi/\partial \mathbf{r}^2 = 0 \ . \tag{17}$$

The solution of this equation is a function of phase $\omega t - \mathbf{kr}$, where frequency ω and wavevector $\mathbf{k}$ are related through $\omega^2 = v^2k^2$. We perfom a translation $\mathbf{r} = \mathbf{r}' + \mathbf{R}(t')$, and $t = t'$. Obviously, the wave equation is not invariant under this transform, nor even for a uniform translation $\mathbf{V} = \dot{\mathbf{R}} = const.$[11] A convenient change of the wave function ψ ensures such an invariance. The phase changes into $\omega t' - \mathbf{kr}' - \mathbf{kR}$, which may be viewed either as a (generalized) Doppler effect by changing correpondingly the frequency, or as a corresponding change in the (instantaneous) phase velocity. A similar situation holds for rotations.

Gravitational waves. Suppose that we have a flat space with metric $\eta_{\mu\nu} = (+, -, -, -)$, where we write Schrodinger's equation for an ensemble of interacting particles, or particles in an external field. Suppose further that this ensemble moves in a weak gravitational field with a metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is small with respect to unity. In general, it is equivalent with a coordinate transform $dx^\mu = a^\mu_\nu dx'^\nu$, such as $dx'^\mu = b^\mu_\nu dx^\nu$, $dx_\mu = b^\nu_\mu dx'_\nu$, where b is the inverse of the matrix a. The metric tensor becomes $g_{\mu\nu} = \eta_{\rho\sigma}a^\rho_\mu a^\sigma_\nu$. We assume $a^\mu_\nu = \delta^\mu_\nu + A^\mu_\nu$, where A^μ_ν is determined by $h_{\mu\nu}$.

We perform such a coordinate transform in Schrodinger's equation, and look for transitions. If we want transitions, the metric tensor must depend on time. In these conditions, we chose the metric tensor corresponding to gravitational waves. As it is well-known, well-determined gravitational waves, *i.e.* the ones with determined energy, propagate in one direction, say $x^1 = x$. Then, the metric tensor is determined by $h_1 = h_{23} = -2A^2_3$ and $h_2 = h_{22} = -h_{33} = -2A^2_2 = 2A^3_3$, where we assume the matrix A^μ_ν symmetrical.

[11] Galileo's invariance is preserved for $\mathbf{V} = \dot{\mathbf{R}} = const.$ The necessity for Einstein's invariance under Lorentz transformations arises from the fact that Maxwell equations are not invariant under Galileo's transformations (starting with the second-order contributions in v/c, where c is the velocity of light).

In ordinary coordinates this coordinate transform reads $t = t'$, $x = x'$, $y = (1 - h_2/2)y' - (h_1/2)z'$, $z = (1 + h_2/2)z' - (h_1/2)y'$. It corresponds to the presence of the gravitational wave. It gives an interaction term

$$H_1 = (\dot{h}_1/2)(zp_y - yp_z) + (\dot{h}_2/2)(yp_y - zp_z) + h_2(p_y^2 - p_z^2)/2m + h_1 p_y p_z/m \tag{18}$$

in the hamiltonian, where the primes have been omitted. This interaction term may generate quantal transitions, It plays the role of the interaction between particles and gravitational waves.

A similar analysis can be extended to fields, with similar conclusions. Herewith, the question of quantization in the gravitational field may get an answer. It is meaningfull for free gravitational fields to be quantized only for small amounts of quantal action; this means gravitational waves, which are quantized into gravitons. Quantal states of particles or fields, gravitons included, moving into a gravitational field may suffer quantal transitions by interacting with the gravitational fields.