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A few Notes on the Theory of Motion
(Lecture nine of the Course of Theoretical Physics)

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Abstract

The classical theory of motion emerging from the principle of least action and Euler-Lagrange equations is presented. The associated Hamilton theory is given, and the Hamilton-Jacobi classical equation for the action is derived. The corresponding theory of quantal motion, Schrodinger's&Co theory, is also given, as a natural extension of the classical, non-relativistic, theory of motion. Thereafter, Einstein's theory of relativistic motion is presented, and the corresponding Hamilton-Jacobi equation is derived. The Klein-Gordon equation (or Schrodinger's relativistic equation) is introduced as a natural quantal extension of the relativistic motion, and the Dirac theory of motion is also given. Finally, the generalization to the field motion is briefly discussed.

This text is dedicated to the memory of the late prof S Titeica, who aimed in his last years' lectures at the Hamilton-Jacobi equation.

The Theory of Euler, Lagrange, Hamilton and Jacobi. Let q be a coordinate (possibly multiple), $\dot{q} = dq/dt$ its velocity and t the time. Let $L(\dot{q}, q, t)$ be the **lagrangian** (or Lagrange's function) and

$$S = \int dt \cdot L(\dot{q}, q, t) \tag{1}$$

the **action**. The variation reads

$$\begin{aligned} \delta S = \int dt \cdot [(\partial L/\partial \dot{q})\delta \dot{q} + (\partial L/\partial q)\delta q] = \\ = \int dt \cdot [-\tfrac{d}{dt}(\partial L/\partial \dot{q}) + (\partial L/\partial q)]\delta q + [(\partial L/\partial \dot{q})\delta q] \quad , \end{aligned} \tag{2}$$

where the []-bracket takes care of the boundaries. Fixed at the boundaries, the motion obeys the **principle of least action**, *i.e.*

$$\frac{d}{dt}(\partial L/\partial \dot{q}) - (\partial L/\partial q) = 0 \ . \tag{3}$$

These are the **Euler-Lagrange equations of motion**. They epitomize **Newton's law**, reading: the time derivative of the **momentum** $p = \partial L/\partial \dot{q}$ equals the force $\partial L/\partial q$. For a force-free motion (uniform motion), as in the absolute space, the **momentum is conserved**. This is the **principle of inertia**. In addition, the motion proceeds such as the momentum is the coordinate derivative of the action,

$$p = \partial S/\partial q \quad , \tag{4}$$

as the boundary bracket in (2) tells.

Let's do the time change of the lagrangian, making use of Euler-Lagrange's equations of motion:

$$dL/dt = (\partial L/\partial \dot{q})d\dot{q}/dt + (\partial L/\partial q)\dot{q} + \partial L/\partial t =$$
$$= (\partial L/\partial \dot{q})d\dot{q}/dt + \tfrac{d}{dt}(\partial L/\partial \dot{q})\dot{q} + \partial L/\partial t = \tag{5}$$
$$= \tfrac{d}{dt}(p\dot{q}) + \partial(L - p\dot{q})/\partial t \ ,$$

or

$$\frac{d}{dt}(p\dot{q} - L) = \partial(p\dot{q} - L)/\partial t \ , \tag{6}$$

where

$$H(p,q,t) = p\dot{q} - L \tag{7}$$

is the **hamiltonian** (or Hamilton's function). Equation (7) defines a **Legendre transformation**. For an absolute time (conservative motion) $\partial L/\partial t = 0$, and $\partial H/\partial t = 0$, so the **hamiltonian is conserved**, $dH/dt = 0$. It defines the **energy**

$$E = H(p,q) \ . \tag{8}$$

Along the motion and during it, the lagrangian changes according to

$$dL = (\partial L/\partial \dot{q})d\dot{q} + (\partial L/\partial q)dq + (\partial L/\partial t)dt =$$
$$= pd\dot{q} + \dot{p}dq + (\partial L/\partial t)dt = \tag{9}$$
$$= d(p\dot{q}) - \dot{q}dp + \dot{p}dq + (\partial L/\partial t)dt \ ,$$

or

$$dH = \dot{q}dp - \dot{p}dq - (\partial L/\partial t)dt \ , \tag{10}$$

which establishes the **canonical (or Hamilton's) equations of motion**

$$\dot{q} = \partial H/\partial p \ , \ \dot{p} = -\partial H/\partial q \ , \ \partial H/\partial t = -\partial L/\partial t \ . \tag{11}$$

The later tells also that $\partial(p\dot{q})/\partial t = 0$.

Finally, the integral in equation (2) can be estimated as follows:

$$\int dt \cdot [-\frac{d}{dt}(\partial L/\partial \dot{q}) + (\partial L/\partial q)]\delta q = -p\delta q + L\delta t = -H\delta t \ , \tag{12}$$

so the variation of the action reads

$$\delta S = -H\delta t + p\delta q \ . \tag{13}$$

Hence, the **Hamilton-Jacobi equation**

$$\partial S/\partial t + H(\partial S/\partial q, q, t) = 0 \ . \tag{14}$$

It is worth noting that q and t are here independent variables, and the action is a function $S(q,t)$ (the velocity does not enter, because S is supposed to be a functional of $q(t)$, though not necessarily the one for an actual motion, so that $\dot{q} = dq/dt$). Indeed, then, according to (1) and (2),

$$dS/dt = \partial S/\partial t + (\partial S/\partial q)\dot{q} = \partial S/\partial t + p\dot{q} = L \ , \tag{15}$$

hence again the Hamilton-Jacobi equation (14). For a conservative motion, $S = -Et + S_1(q)$, for a uniform motion $S = pq + S_2(t)$.

It is worth noting that the Hamilton-Jacobi equation has no relevance at all upon the original motion. It enlarges considerably the original lagrangian formulation of the theory of motion, and, actually, reveals a different aspect of the action function, not included in the original formulation. It consists in assuming the action as a function of two independent variables, the coordinate q and the time t. The derivation of the Hamiltonian-Jacobi equation as given in (12) and (13) does not imply any equations of motion, nor the principle of least action, but only definitions and the assumption that S depends on q and t. For actual motion, the action defined by (1) is a function of time (the end time of the trajectory). This is an appreciable generalization, which remains to be given a sense for any actual motions. The Hamilton-Jacobi equation is a general description of the motion, based solely upon the distinction between time and coordinates.

Indeed, the original motion can be recovered from the Hamilton-Jacobi equation (14) as follows. The solution of the Hamilton-Jacobi equation depends on constants of integration α; it reads $S(q,t;\alpha)$. We take the derivative of the Hamilton-Jacobi equation with respect to α:

$$\partial S'/\partial t + (\partial H/\partial p)(\partial S'/\partial q) = 0 \ , \tag{16}$$

where $S' = \partial S/\partial \alpha$. Now, we employ the equation of motion $\dot{q} = \partial H/\partial p$, as for the actual motion we wish to describe. Equation (16) becomes

$$\partial S'/\partial t + \dot{q}(\partial S'/\partial q) = dS'/dt = 0 \ , \tag{17}$$

which tells that $S' = const$. The motion is therefore described by

$$\partial S(q,t;\alpha)/\partial \alpha = const \ . \tag{18}$$

It gives the trajectory $q(t)$. Equation (18) is only the form the equations of motion take in terms of the action function. For instance, the free motion of a particle of mass m has the solution $S = -Et + \sqrt{2mE}q$, according to the Hamilton-Jacobi equation (14). Equation (18) gives $-t + \sqrt{m/2E}q = const$, which indeed is the trajectory of a free paricle with velocity $v = \sqrt{2E/m}$.

We emphasize that it is only through the Euler-Lagrange, or the canonical equations of motion ($\dot{q} = \partial H/\partial t$ used in deriving (17)), that we have given here a meaning to the Hamilton-Jacobi equation. This meaning is called the **classical motion**.

The Theory of Schrodinger&Co. Equation (13) gives the change of a phase. It is one of the most general motion. Then, it seems reasonable to introduce a **wavefunction**

$$\psi(q,t) = e^{iS(q,t)/\hbar} \ , \tag{19}$$

where $\hbar$ is Planck's constant of action and action S is allowed to assume imaginary values too. By $S = -i\hbar \ln \psi$, the Hamilton-Jacobi equation (14) reads

$$i\hbar \partial\psi/\partial t = H\psi \ , \tag{20}$$

where $p\psi = -i\hbar\partial\psi/\partial q$ and $E\psi = i\hbar\partial\psi/\partial t$. This is **Schrodinger's eigenvalues equation**, and motion can now have a meaning. This meaning is the **quantal motion**. Noteworthy, the classical trajectory is lost.

The wavefunctions may linearly be superposed, and we get interference, as for waves. The physical quantities are operators, like energy and momentum, for the wavefunctions. Any measurement gives an eigenvalue, which may be continuous or discrete, and we may have a reduction of the original wavefunction. The measurements are therefore statistical, and we have mean values and deviations. The probability of localization is $|\psi|^2$ for instance, and it must be normalized. Those operators that do not commute produce measurements whose deviations obey Heisenberg's uncertainty principle. The operators are matrices for eigenfunctions. Writing $\psi = A\exp(iS/\hbar)$ with S the real action, we recover the Hamilton-Jacobi equation (14) in the classical limit $\hbar \rightarrow 0$, and, in addition, we get also the conservation of the probability flow for amplitude A. This is practically all of the philosophy of the quantal motion. It is derived by giving a wavelike sense to the Hamilton-Jacobi equation (14).

It is worth noting that an independent principle of least action can also be formulated for Schrodinger's equation, which implies (and originates in) the stationarity of the energy eigenfunctions.

Relativistic particle. Einstein's theory. The **principle of relativity** imposes a certain restriction on the motion. This restriction consists in the invariance under linear transformations of the **world-line element** squared $ds^2 = c^2d\tau^2 = c^2dt^2 - d\mathbf{r}^2$, where c is the velocity of light, τ is the proper time of the particle at rest and $\mathbf{v} = d\mathbf{r}/dt$ is the velocity of the moving particle. For $\mathbf{v} = const$ the equations of motion (as well as the equations of the electromagnetic field) are invariant under the corresponding **Lorentz transformations**, thus satisfying the Galilean principle of relativity for **inertial frames**. The principle is extended by the **theory of gravitation**, by requiring the invariance under local linear transformations of the general world-line element squared $ds^2 = g_{ij}dx^idx^j$, where g_{ij}, $i, j = 0, 1, 2, 3$, $dx^0 = cdt$, $dx^{1,2,3} = d\mathbf{r}$, is the **metric tensor** of the curved space-time. The equations of motion are invariant in this case, providing they are corespondingly modified. For a "flat" space, $g_{ij} = \eta_i\delta_{ij}$, where $\eta_i = (+ - --)$ is the signature of the metric tensor. The corresponding contravariant metric tensor is given by $g^{ij} = \eta^i\delta^{ij}$, such that $g^{ik}g_{kj} = \delta^i_j$, as for a general metric tensor. The covariant displacement is $dx_i = (cdt, -d\mathbf{r})$ and the contravariant one is $dx^i = (cdt, d\mathbf{r})$. The labels are lowered or raised as $g_{ij}A^j$ or $g^{ij}A_j$, for any tensor A, and the scalar product is A_iA^i, etc. Another form of relativistic constraint is seen in the fact that energy E and mometum $\mathbf{p}$ are related by $E = \sqrt{m^2c^4 + c^2p^2}$, where m is the mass of the particle.

As it is well-known, the consequences of the principle of relativity are the **contraction of lengths, dilation of time, Doppler effect, aberration of light and the light drag, and the energy of inertia** (or **inertia of energy**) mc^2**.**

The invariant action of a particle of mass m and electric charge e subjected to an **electromagnetic field of potentials** $A_i = (\varphi, -\mathbf{A})$, where φ is the electrostatic potential and $\mathbf{A}$ is the vector potential, reads

$$S = \int [-mcds - (e/c)A_idx^i] \ , \tag{21}$$

where $ds = \sqrt{dx_idx^i}$, $dx_i = g_{ij}dx^i$, $g_{ij} = \eta_i\delta_{ij}$. Making use of the **relativistic velocity** $u_i = dx_i/ds$, $u_iu^i = 1$, and $\delta ds = dx_id(\delta x^i)/ds = u_id(\delta x^i)$, its variation reads

$$\delta S = \int ds \cdot \left\{-mcdu_i/ds + (e/c)[\partial A_i/\partial x^k - \partial A_k/\partial x^i]u^k\right\}\delta x^i - \left[[mcu_i + (e/c)A_i]\delta x^i\right] \ , \tag{22}$$

where the []-bracket takes care of the boundaries of the motion. Hence, the Euler-Lagrange equations of motion

$$mc\frac{du_i}{ds} = (e/c)F_{ij}u^j \ , \tag{23}$$

for a relativistic particle, where

$$F_{ij} = \partial A_i/\partial x^j - \partial A_j/\partial x^i = \partial_j A_i - \partial_i A_j \tag{24}$$

is the (antisymmetric) **tensor of fields**, and

$$P_i = mcu_i + (e/c)A_i = -\partial S/\partial x^i \tag{25}$$

are the **relativistic momenta** of motion.

The Euler-Lagrange equations of motion given by (23) are generalized, according to **Newton's law of motion**, to

$$mc\frac{du_i}{ds} = g_i \ , \tag{26}$$

where g_i are forces. Moreover, $p_i = mcu_i$ are the **momenta of motion**, as distinct from the relativistic momenta P_i, so the equations of motion, in their most general form, read

$$\frac{dp_i}{ds} = g_i \ . \tag{27}$$

Making use of

$$u_i = \left(\frac{1}{\sqrt{1-\beta^2}}, -\frac{\mathbf{v}/c}{\sqrt{1-\beta^2}}\right) \ , \tag{28}$$

where $\beta = v/c$, and introducing the energy through

$$p_i = (E/c, -\mathbf{p}) = \left(\frac{mc}{\sqrt{1-\beta^2}}, -\frac{m\mathbf{v}}{\sqrt{1-\beta^2}}\right) \ , \tag{29}$$

we get

$$p_i p^i = E^2/c^2 - p^2 = m^2c^2 \ , \tag{30}$$

which is the **relativistic energy**. Similarly, from (25), we get

$$p_i = P_i - (e/c)A_i = ((E - e\varphi)/c, -\mathbf{P} + (e/c)\mathbf{A}) \ , \tag{31}$$

and

$$(E - e\varphi)^2/c^2 - [\mathbf{P} - (e/c)\mathbf{A}]^2 = m^2c^2 \ . \tag{32}$$

According to (25), $E = -\partial S/\partial t$ and $\mathbf{P} = \partial S/\partial \mathbf{r}$, so

$$(\partial S/\partial t + e\varphi)^2/c^2 - [\partial S/\partial \mathbf{r} - (e/c)\mathbf{A}]^2 = m^2c^2 \ . \tag{33}$$

This is the **Hamilton-Jacobi equation** for a relativistic particle in an electromagnetic field. With $S = S' - mc^2t$ and $c \to \infty$ we get its non-relativistic limit

$$\partial S'/\partial t + H(\partial S'/\partial \mathbf{r}, \mathbf{r}) = 0 \ , \tag{34}$$

where the hamiltonian is given by

$$H = [\mathbf{P} - (e/c)\mathbf{A}]^2/2m + e\varphi \ , \tag{35}$$

and $\mathbf{P} = \partial S'/\partial \mathbf{r}$. It is worth noting that the energy in equation (33) may get negative values too, corresponding to $S = S' + mc^2t$ and $e \to -e$. These **antiparticles** move backwards in time.

Making use of the definition of the **electric field** $\mathbf{E} = -\partial\varphi/\partial\mathbf{r} - (1/c)\partial\mathbf{A}/\partial t$ and of the **magnetic field** $\mathbf{H} = curl\mathbf{A}$, we get the field tensor

$$F_{ij} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -H_3 & H_2 \\ -E_2 & H_3 & 0 & -H_1 \\ -E_3 & -H_2 & H_1 & 0 \end{pmatrix} \quad , \tag{36}$$

and its contravariant version $F^{ij} = g^{ik}g^{jl}F_{kl}$, which differs from the covariant one by the sign of the electric field. One can check easily that the equations of motion (23) are the **Lorentz law of motion**

$$d\mathbf{p}/dt = e\mathbf{E} + (e/c)\mathbf{v} \times \mathbf{H} \tag{37}$$

and the rate of change

$$dE_{kin}/dt = e\mathbf{vE} \tag{38}$$

of the kinetic energy $E_{kin} = cp_0 = mc^2/\sqrt{1-\beta^2} = c\sqrt{m^2c^2+p^2}$, which is equal, according to the *rhs* of equation (38), to the mechanical work done by the field per unit time, as the Lorentz law of motion tells.

All these can be obtained, naturally, if we start with the action (21) in the form

$$S = \int dt \cdot L = \int dt \cdot [-mc^2\sqrt{1-\beta^2} - e\varphi + (e/c)\mathbf{vA}] \ . \tag{39}$$

The momentum is then given by

$$\mathbf{P} = \partial L/\partial\mathbf{v} = m\mathbf{v}/\sqrt{1-\beta^2} + (e/c)\mathbf{A} = \mathbf{p} + (e/c)\mathbf{A} = \partial S/\partial\mathbf{r} \ \ , \tag{40}$$

and the equations of motion are the Euler-Lagrange equations

$$\frac{d}{dt}(\partial L/\partial\mathbf{v}) - \partial L/\partial\mathbf{r} = 0 \ . \tag{41}$$

The space derivative of the lagrangian reads

$$\partial L/\partial\mathbf{r} = (e/c)v_i\partial A_i/\partial\mathbf{r} - e \cdot grad\varphi \ \ , \tag{42}$$

where $i = 1, 2, 3$, and it is easy to see that

$$v_i\partial A_i/\partial\mathbf{r} = (\mathbf{v}grad)\mathbf{A} + \mathbf{v} \times curl\mathbf{A} \tag{43}$$

by direct calculation. We have therefore

$$\partial L/\partial\mathbf{r} = (e/c)(\mathbf{v}grad)\mathbf{A} + (e/c)\mathbf{v} \times \mathbf{H} - e \cdot grad\varphi \ \ , \tag{44}$$

and

$$\frac{d}{dt}[\mathbf{p} + (e/c)\mathbf{A}] = (e/c)(\mathbf{v}grad)\mathbf{A} + (e/c)\mathbf{v} \times \mathbf{H} - e \cdot grad\varphi \ . \tag{45}$$

On the other hand,

$$\frac{d}{dt}\mathbf{A} = \partial\mathbf{A}/\partial t + (\mathbf{v}grad)\mathbf{A} \ \ , \tag{46}$$

so that we get the Lorentz law of motion

$$\frac{d}{dt}\mathbf{p} = e\mathbf{E} + (e/c)\mathbf{v} \times \mathbf{H} \ . \tag{47}$$

Equation (40) serves to extract $\mathbf{v}$ as function of $\mathbf{P}$. We get easily the hamiltonian

$$H = \mathbf{P}\mathbf{v} - L = c\sqrt{m^2c^2 + [\mathbf{P} - (e/c)\mathbf{A}]^2} + e\varphi \ , \tag{48}$$

which is constant in time, the corresponding canonical equations of motion $\mathbf{v} = \partial H/\partial \mathbf{P}$, $d\mathbf{P}/dt = -\partial H/\partial \mathbf{r}$, and $H = -\partial S/\partial t$. The derivations go perfectly similar with the classical theory of Euler, Lagrange, Hamilton and Jacobi given above. The Hamilton-Jacobi equation reads $\partial S/\partial t + H(\partial S/\partial \mathbf{r}, \mathbf{r}) = 0$, which agrees with the one given by equation (33), up to the squared energy.

Klein-Gordon equation. The natural quantal extension of the Hamilton-Jacobi equation (32) or (33) is

$$(i\hbar\partial/\partial t + e\varphi)^2\psi - c^2[i\hbar\partial/\partial \mathbf{r} + (e/c)\mathbf{A}]^2\psi = m^2c^4\psi \ . \tag{49}$$

It is the **Klein-Gordon equation** for a relativistic particle in en electromagnetic field (or **relativistic Schrodinger equation**, as derived first by Schrodinger). It gives approximately relativistic corrections to the Hydrogen spectrum, known as the **fine-structure corrections**, in terms of powers of the fine-structure constant $\alpha = e^2/\hbar c = 1/137$ (contributions of order $1/c^2$). It was soon, however, realized, that, obviously, it does not include **the spin** of the electron, as suggested by atomic spectra.

Moreover, the non-relativistic Schrodinger equation (20) conserves probability $|\psi|^2$, through the continuity equation

$$\partial |\psi|^2 /\partial t - (i\hbar/2m)div(\psi^* grad\psi - \psi grad\psi^*) = 0 \ , \tag{50}$$

so the space integral of $|\psi|^2$ is constant in time. The Klein-Gordon equation leads to $\partial\rho/\partial t + div\mathbf{j} = 0$ with $\rho = Im[\psi^*(\partial/\partial t - ie\varphi/\hbar)\psi]$ and $\mathbf{j} = c^2 Im[\psi^*(\partial/\partial \mathbf{r} - ie\mathbf{A}/\hbar c)\psi]$, and density ρ, with or without field, is not always positive. Such a motion would not make therefore any sense.

Dirac's theory of motion. The original idea of Dirac was to arrive at the quadratic Klein-Gordon, or Schrodinger relativistic equation, by iterating a first-order differential equation. The iteration of differential operators for linear sets of wavefunctions is essential for quantal motion. Consequently, he started with

$$i\hbar\partial\psi/\partial t = H\psi \tag{51}$$

for a free relativistic particle and a linear hamiltonian

$$H = \alpha c\mathbf{p} + \beta mc^2 \ , \tag{52}$$

where α and β remain to be determined. Iterating this equation one gets

$$(i\hbar\partial/\partial t)^2\psi = \alpha_i\alpha_j c^2 p_i p_j + (\alpha_i\beta + \beta\alpha_i)mc^3 p_i + \beta^2 m^2 c^4 \ , \tag{53}$$

whence one can see that α and β must be matrices satisfying

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij} \ , \ \alpha_i\beta + \beta\alpha_i = 0 \ , \ \beta^2 = 1 \ , \tag{54}$$

$i, j = 1, 2, 3$, and ψ must be a column wavefunction with several components. For instance,

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \ , \ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{55}$$

where σ are Pauli's matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{56}$$

and ψ is a four-component column. The iterated equation (53) takes then the form of the Klein-Gordon eqation for a free relativistic particle. Moreover, Dirac's equation given by (51) and (52) reads

$$(\gamma^0 \partial/\partial x^0 + \gamma^i \partial/\partial x^i)\psi = (mc/\hbar)\psi \ , \tag{57}$$

where $\gamma^0 = i\beta$, $\gamma = i\beta\alpha$, and one can check directly that γ^i, $i = 0, 1, 2, 3$ is a vector under linear transformations which preserve the world-line element squared, *i.e.* Dirac's equation is **relativistically invariant**. If Λ is such a linear transform, then $\gamma' = S(\Lambda)\gamma S^{-1}(\Lambda)$ and $\psi' = S(\Lambda)\psi$, where S is the corresponding unitary transform.

For a relativistic particle in an electromagnetic field an obvious extension of Dirac's equation is

$$(i\hbar\partial/\partial t + e\varphi)\psi = c[\mathbf{p} - (e/c)\mathbf{A}]\alpha\psi + mc^2\beta\psi \ , \tag{58}$$

where $p = -i\hbar\partial/\partial\mathbf{r}$. Obviously, it preserves the relativistic invariance. However, by iteration, it does not reproduce exactly the Schrodinger equation (Klein-Gordon equation) (49) for a relativistic particle in an electromagnetic field. It contains two new terms, describing the interaction of the spin with the magnetic field (Zeeman interaction, **Pauli equation**, correction of the order $1/c$) and an interaction with the electric field (of the same order $1/c$). These new terms appear as a consequence of the noncommutativity of the Dirac matrices and of the action of the momentum differential operator on the electromagnetic field. This quadratic equation reads

$$(i\hbar\partial/\partial t + e\varphi)^2\psi = c^2[\mathbf{p} - (e/c)\mathbf{A}]^2\psi - ec\hbar\mathbf{\Sigma}\mathbf{H}\psi + iec\hbar\alpha\mathbf{E}\psi + m^2c^4\psi \ , \tag{59}$$

where

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} . \tag{60}$$

(Noteworthy, charge e is $-|e|$for an electron). In a central field, equation (58) shows that the conservation of the angular momentum is, in fact,

$$[H, \mathbf{r} \times \mathbf{p} + \hbar\Sigma/2] = 0 \ , \tag{61}$$

which tells that the particle described by Dirac's equation has indeed a **one-half spin**. In addition, Dirac's equation (58) satisfies the continuity equation $\partial\rho/\partial t + div\mathbf{j}$ for probability with $\rho = \psi^*\psi$ and $j = c\psi^*\alpha\psi$. Contributions of the order of $1/c^2$ are also present in the quadratic form of the Dirac equation, which account for the fine-structure of atomic spectra (spin-orbit interaction and a contact interaction).

However, **spin-spin interaction** (which is also a $1/c^2$-contribution to the fine-structure, as well as the **hyperfine interaction** both betwen electrons and electrons and nuclei) would require an equation for many particles. In addition, the **Lamb shift** seen in the atomic spectra (as well as the "anomalous" magnetic moment) is beyond Dirac's equation. Both would require a fundamental change in the philosophy of the motion, corresponding to interacting fields.

Concluding remarks. Fields. Dirac's equation raises another important, related difficulty, which consists in its predicting **states of negative energy** (as expected). Though originally solved by Dirac by the filled sea of negative-energy levels, according to **Pauli exclusion principle**, such negative states appear also for relativistic bosons (as expected), where the exclusion principle does not work. The notion of fields is once more needed; they solve this difficulty by the assumption of the **antiparticles**, by the concept of **elementary excitations** and, of course, by viewing the interacting motion as an act of **scattering**. Basically for fields is the undefined number of particles.

The notion of elementary excitations reveals the importance of the **vacuum**. **Bound states** and **composite particles** may be viewed as new vacua appearing through **symmetry braeking** and

macroscopic occupation. It they are non-relativistic, they have a (quantal) structure. If they are relativistic, they do not extend beyond their **Compton length**, and the question of structure becomes meaningless. Indeed, a bound motion of frequency ω is such that $\omega \sim c/a$, where a is the linear size, hence the energy $mc^2 \sim \hbar\omega \sim \hbar c/a$, and the Compton length $a \sim \hbar/mc$; below a there is no structure. This is why the **confined quarks** are not observable. It is a profound aspect of the **positivist view**, according to which only what is observable is meaningful and lends itself to a theory.

In this respect, the divergencies and infinities experienced by the field theories, and their **renormalization** cure reflect the same inadequacy of our desires to the unobservability of the Natural World in the limit of the high energies and small distances. Working fields are therefore sort of low-energy **"effective theories"**.

Also in this connection, the **strong and weak interactions** introduced new fields by **non-commutative gauge** invariance, as required by the "observability" (or the **"covariance"**) principle. We may note that only bilinear forms in fields (or composites of bilinear forms) are meaningful in this respect, and this is why the **gravitational field** is not amenable to a quantal treatment.

Final note. I wrote this text out of my conviction that we are in a permanent need of reconstructing logically our Physical Theories.