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Covariance, curved space, motion and quantization (Lecture ten of the Course of Theoretical Physics)

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Abstract

We analyze the effects of weak external forces and non-inertial motion upon the free motion, both relativistic and non-relativistic. Such effects amount to the free motion in a curved space, whose metric is established. Examples are given for translations and rotations. The Hamilton-Jacobi equation is derived for motion in such a curved space, and the effects of the curvature, *i.e.* of weak forces and non-inertial motion, upon the quantization are analyzed, starting from a generalization of the Klein-Gordon equation in curved spaces. It is shown that the quantization is actually destroyed, in general, by a non-inertial motion in the presence of forces, in the sense that such a motion may produce quantal transitions. Examples are given for a massive scalar field and for photons.

Newton's law. We start with Newton's law

$$m\frac{dv_\alpha}{dt}=f_\alpha \ \ , \tag{1}$$

for a particle of mass m, with usual notations. I wish to show here that it is equivalent with the motion of a free particle of mass m in a curved space, *i.e.* it is equivalent with

$$Du^i/ds=\frac{du^i}{ds}+\Gamma^i_{jk}u^ju^k=0 \ \ , \tag{2}$$

again with usual notations.[1]

[1]The geometry of the curved spaces originates probably with Gauss (~1830). It was given a sense by Riemann (*Uber die Hypothesen welche der Geometrie zugrunde liegen*, 1854), Grassman (1862), Christoffel (1869), thereafter Klein (*Erlanger Programm, Programm zum Eintritt in die philosophische Fakultat in Erlangen*, 1872), Ricci and Levi-Civita (1901). It was Einstein (1905,~1916), Poincare (1905), Minkowski (1907), Sommerfeld (1910), (Kottler, 1912), Weyl (*Raum, Zeit und Materie*, 1918), Hilbert (1917) who made the connection with the physical theories. It is based on point (local) coordinate transforms, cogredient (contravariant) and contragredient (covariant) tensors and the distance element. It is an absolute calculus, as it does not depend on the point, *i.e.* the reference frame. It may be divided into the motion of a particle, the motion of the fields, the motion of the gravitational field, and their applications, especially in cosmology and cosmogony. As the curved space is universal for gravitation, so it is for the non-inertial motion, which we focus upon here. The body which creates the gravitation and the corresponding curved space is here the moving observer for the non-inertial motion, beside forces. It could be very well that the world and the motion are absolute, but they depend on subjectivity, though it could be an universal subjectivity (inter-subjectivity). See W. Pauli, *Theory of Relativity*, Teubner, Leipzig, (1921).

Obviously, the spatial coordinates of equation (1) are euclidean, and equation (1) is a non-relativistic limit. It follows that the metric we should look for may read

$$ds^2 = (1+h)c^2dt^2 + 2cdtg_\alpha dx^\alpha - dx^\alpha dx^\alpha \ , \tag{3}$$

where functions $h, g_\alpha \ll 1$ are determined such that equation (2) goes into equation (1) in the non-relativistic limit $v_\alpha/c \ll 1$, and for a correspondingly weak force f_α. Such a metric, which recovers Newton's law in the non-relativistic limit, is not unique. The metric given by equation (3) can be written as

$$g_{ij} = \begin{pmatrix} 1+h & g_1 & g_2 & g_3 \\ g_1 & -1 & 0 & 0 \\ g_2 & 0 & -1 & 0 \\ g_3 & 0 & 0 & -1 \end{pmatrix} . \tag{4}$$

We perform the calculations up to the first order in h, g_α and v_α/c. The distance given by (3) becomes then $ds = cdt(1+h/2)$ and the velocities read

$$u^0 = dx^0/ds = 1 - h/2 \ , \ \ u^\alpha = dx^\alpha/ds = v_\alpha/c \ . \tag{5}$$

It is the Christoffel's symbols (affine connections)

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(\partial g_{mj}/\partial x^k + \partial g_{mk}/\partial x^j - \partial g_{jk}/\partial x^m) \tag{6}$$

which require more calculations. First, the contravariant metric is $g^{00} = 1-h$, $g^{0\alpha} = g^{\alpha 0} = g_\alpha$, $g^{\alpha\beta} = -\delta_{\alpha\beta}$, such that $g_{im}g^{mj} = g^{jm}g_{mi} = \delta^j_i$. By making use of (6) we get

$$\begin{gathered} \Gamma^0_{00} = (1/2c)\partial h/\partial t \ , \ \ \Gamma^0_{0\alpha} = \Gamma^0_{\alpha 0} = (1/2)\partial h/\partial x^\alpha \ , \\ \Gamma^0_{\alpha\beta} = \Gamma^0_{\beta\alpha} = (1/2)(\partial g_\alpha/\partial x^\beta + \partial g_\beta/\partial x^\alpha) \ , \\ \Gamma^\alpha_{\beta 0} = \Gamma^\alpha_{0\beta} = (1/2)(\partial g_\beta/\partial x^\alpha - \partial g_\alpha/\partial x^\beta) \ , \\ \Gamma^\alpha_{00} = (1/2)\partial h/\partial x^\alpha - (1/c)\partial g_\alpha/\partial t \ , \ \ \Gamma^\alpha_{\beta\gamma} = 0 \ . \end{gathered} \tag{7}$$

Now, the first equation in (2) has $du^0/ds = -(1/2c)\partial h/\partial t$ and $\Gamma^0_{jk}u^ju^k = (1/2c)\partial h/\partial t$ in its *rhs*, so it is satisfied identically in this approximation. The remaining equations in (2) read

$$\frac{dv_\alpha}{dt} = c^2(\frac{\partial g_\alpha}{c\partial t} - \frac{1}{2}\cdot\frac{\partial h}{\partial x^\alpha}) \ . \tag{8}$$

By comparing this with Newton's equation (1) we get the functions h and g_α as given by

$$\frac{\partial g_\alpha}{c\partial t} - \frac{1}{2}\cdot\frac{\partial h}{\partial x^\alpha} = f_\alpha/mc^2 \ . \tag{9}$$

As it is well-known, for a static gravitational potential Φ, the force is given by $f_\alpha = -m\partial\Phi/\partial x^\alpha$, so that $h = 2\Phi/c^2$ and $g_\alpha = const.$[2]

Translations. Suppose that the force $\mathbf{f}$ is given by a static potential φ, such that $\mathbf{f} = -\partial\varphi/\partial\mathbf{r}$. Then, $h = 2\varphi/mc^2$ and $\mathbf{g} = const.$

[2]With regard to equation (3), this was for the first time when Einstein "suspected the time" (1905).

Let us perform a translation

$$\mathbf{r}=\mathbf{r}'+\mathbf{R}(t')\ ,\ t=t'\ . \tag{10}$$

Then, Newton's equation $md\mathbf{v}/dt=\mathbf{f}$ given by (1) becomes

$$m\frac{d\mathbf{v}'}{dt'}=\mathbf{f}'-md\mathbf{V}/dt'\ , \tag{11}$$

where $\mathbf{f}'$ is the force in the new coordinates and $\mathbf{V}=d\mathbf{R}/dt'$ is the translation velocity. The inertial force $-md\mathbf{V}/dt'$ appearing in (11) is accounted by the $\mathbf{g}$ in the metric of the curved space. Indeed, equation (9) gives

$$\mathbf{g}=-\mathbf{V}/c\ , \tag{12}$$

up to a constant. The constant reflects the principle of inertia. We may put it equal to zero. The time-dependent $\mathbf{g}$ and $\mathbf{V}$ represent a non-inertial motion. Such a non-inertial motion is therefore equivalent with a free motion in a curved space, and the metric of this space is universal, like the gravitation, as it does not depend on the moving body. Of course, this statement is nothing else but the principle of equivalence, or the general principle of relativity. It is however noteworthy that the non-inertial curved space depends on the observer, through the velocity $\mathbf{V}$, by virtue of the reciprocity of the motion.

Coordinate transformations. The translation given by (10) corresponds to the local coordinate transformation

$$dx^\alpha=dx'^\alpha+V_\alpha dt'\ ,\ dx^0=cdt=cdt'=dx'^0\ . \tag{13}$$

It takes the square distance

$$ds^2=(1+h)c^2dt^2-dx^\alpha dx^\alpha \tag{14}$$

corresponding to $\mathbf{g}=0$ into

$$ds^2=(1+h-V^2/c^2)c^2dt'^2-2dt'V_\alpha dx'^\alpha-dx'^\alpha dx'^\alpha\ , \tag{15}$$

which, within our approximation, corresponds to (3) with $\mathbf{g}=-\mathbf{V}/c$. We recover, therefore, equation (12), as expected.

In general, if x^i span a flat space with diagonal metric η_{ij} given by $\eta_{00}=1$ and the diagonal $\eta_{\alpha\alpha}=-1$, $\alpha=1,2,3$, we can consider local coordinate transformation $dx^i=a^i_jdx'^j$, such as $dx'^i=b^i_jdx^j$ and $dx_i=b^j_idx'_j$, where $a^i_kb^k_j=b^i_ka^k_j=\delta^i_j$. Then, the metric follows from $ds^2=\eta_{ij}dx^idx^j=\eta_{ij}a^i_ka^j_ldx'^kdx'^l$ as $g_{ij}=\eta_{lm}a^l_ia^m_j$ and $g^{ij}=\eta^{lm}b^i_lb^j_m$, where $\eta^{lm}=\eta_{lm}$. If we know the metric (g_{ij}), the coordinate transformation (a^i_j) is not uniquely deterrmined, in general. On the contrary, if we know the coordinate transformation (a^i_j) we can have the metric (g_{ij}).[3]

Within the approximation used here, we can look for a coordinate transformation of the form $a^i_j=\delta^i_j+A^i_j$, where $A^i_j\ll 1$. To the first order, the metric reads

$$g_{ij}=\begin{pmatrix} 1+2A^0_0 & A^0_1-A^1_0 & A^0_2-A^2_0 & A^0_3-A^3_0\\ A^0_1-A^1_0 & -1-2A^1_1 & -A^1_2-A^2_1 & -A^1_3-A^3_1\\ A^0_2-A^2_0 & -A^1_2-A^2_1 & -1-2A^2_2 & -A^2_3-A^3_2\\ A^0_3-A^3_0 & -A^1_3-A^3_1 & -A^2_3-A^3_2 & -1-2A^3_3 \end{pmatrix}\ . \tag{16}$$

Obviously, A^i_i are related to dilations, A^0_α, A^α_0 are asociated with translations, and A^β_α are associated with spatial rotations. The metric given by (3) is obtained for $A^0_0=h/2$, $A^\alpha_0=-A^0_\alpha=-g_\alpha/2$ and $A^\alpha_\beta=-A^\beta_\alpha$. This way, we are able to define a curved space for any weak force, including an

[3] In the usual fancy language the a^i_j are called tetrads or vierbeins.

inertial force produced by a non-uniform translation, in the non-relativistic limit, by local coordinate transformations. Force is therefore just a local coordinate transformation to a curved space, where the motion is free.

Curved space. It is easy to see that the relativistic flat distance given by $ds^2 = \eta_{ij}dx^i dx^j$ is taken into the curved distance given by equation (3) by the local coordinate transformation

$$dt = \frac{(1+h)dt' + (g+\beta\Delta)dx'/c}{\sqrt{(1+h)((1-\beta^2)}} \ , \qquad dx = \frac{c\beta(1+h)dt' + (\beta g+\Delta)dx'}{\sqrt{(1+h)((1-\beta^2)}} \ , \tag{17}$$

$dy = dy'\,, dz = dz'$, where $\Delta = \sqrt{1+h+g^2}$, $\mathbf{g}$ is along $dx = dx^1$, $\beta = V/c$ and the velocity V is $V = dx/dt$ for $dx' = 0$ ($dy = dx^2\,, dz = dx^3$). The inverse of this transformation is

$$dt' = \frac{g(\beta dt - dx/c) + \Delta(dt - \beta dx/c)}{\Delta\sqrt{(1+h)((1-\beta^2)}} \ , \qquad dx' = \sqrt{1+h}\cdot\frac{dx - c\beta dt}{\Delta\sqrt{1-\beta^2}} \ . \tag{18}$$

All the square roots in these equations must exist, which imposes certain restrictions upon h and β (reality conditions; in particular,$1+h>0$, $1-\beta^2>0$).

In the local transformations given above it is assumed that there exist global transformations $x^i(x')$ and $x'^i(x)$, where x, x' stand for all x^i and, respectively, x'^i, because the coefficients in these transformations are functions of x or, respectively, x'. This restricts appreciably the derivation of metrics by means of (global) coordinate transformations, because, in general, the 10 elements of a metric cannot be obtained by 4 functions $x^i(x')$. Conversely, we can diagonalize the curved metric at any point, such as to reduce it to a locally flat metric, [4] but the flat coordinates (axes) will not, in general, be the same for all the points; they depend, in general, on the point.

One can see from (17) that in the flat limit $h,\ g \to 0$ the above transformations become the Lorentz transformations, as expected. Therefore, we may have corrections to the flat relativistic motion by first-order contributions of the parameters h and $\mathbf{g}$. Indeed, in this limit, the transformation (17) becomes

$$dt = \frac{(1+h/2)dt' + (g+\beta)dx'/c}{\sqrt{1-\beta^2}} \ , \qquad dx = \frac{c\beta(1+h/2)dt' + (g\beta+1)dx'}{\sqrt{1-\beta^2}} \ , \tag{19}$$

which include corrections to the Lorentz transformations, due to the curved space. If we choose $g = -\beta$, then we get the non-relativistic limit (since $g \ll 1$)

$$dt = (1+h/2)dt' \ , \qquad dx = dx' - cgdt' = dx' + Vdt' \tag{20}$$

of the translations given by (10) in the presence of a (weak) force. We note that such corrections affect the law of the composition of the velocities, but this law is now irrelevant since the velocities in the curved space are "curved". One can check easily that the light propagates with the same velocity c in both spaces.

[4]Usually called the tangent space in the sophisticated language.

The metric given by (3) provides the proper time

$$d\tau = \sqrt{1+h} \cdot dt \ , \tag{21}$$

corresponding to $dx^\alpha = 0$. The metric given by (3) can also be written as

$$ds^2 = c^2(1+h)[dt + \mathbf{g}d\mathbf{r}/c(1+h)]^2 - [d\mathbf{r}^2 + (\mathbf{g}d\mathbf{r})^2/(1+h)] \ , \tag{22}$$

hence the length given by

$$dl^2 = d\mathbf{r}^2 + (\mathbf{g}d\mathbf{r})^2/(1+h) \tag{23}$$

and the time

$$dt' = \sqrt{1+h} \cdot [dt + \mathbf{g}d\mathbf{r}/c(1+h)] \ , \tag{24}$$

corresponding to the length dl. The difference $\Delta t = \mathbf{g}d\mathbf{r}/c(1+h)$ between the two times, $dt_1 = d\tau/\sqrt{g_{00}} = dt$ in the proper time (21) and $dt_2 = dt'/\sqrt{g_{00}} = dt + \mathbf{g}d\mathbf{r}/c(1+h)$ in the time given by (24), gives the difference in the synchronization of two simultaneous events, infinitesimally separated. The difference in time depends on the path followed to reach a point starting from another point.

We note, by (22)-(24), that the light ray given by $ds = 0$ moves indeed with the velocity of light $dl/dt' = c$ in the curved space. In addition, when solving for the motion of the eikonal, we will have a shift in frequency (and a Doppler effect) and a bending of the light ray in the curved space.

We limit ourselves to the first order in h, $\mathbf{g}$, and put $\mathbf{g} = -\mathbf{V}/c$, in order to investigate corrections to the motion under the action of a weak force in a flat space moving with a non-uniform velocity $\mathbf{V}$ with respect to the observer. For the observer, such a motion is then a free motion in a curved space with metric (3). The proper time is then $d\tau = (1+h/2)dt$, the time given by (24) becomes $dt' = (1+h/2)dt + \mathbf{g}d\mathbf{r}/c$ and the length is given by $dl^2 = d\mathbf{r}^2$, as for a three-dimensional euclidean space.

Motion in a curved space. Let us assume that we have a particle moving freely in a flat space. We denote its contravariant momentum by $(P_0 = E_0/c, \mathbf{P})$ and the corresponding covariant momentum by $(P_0, -\mathbf{P})$, such that $P_0^2 - P^2 = m^2c^2$, where E_0 is the energy of the particle, and P_0, $\mathbf{P}$ are constant.

We can use the coordinate transformation given by (18) to get the momentum of the particle in the curved space. We prefer to write it down in its covariant form, using the metric (4). We get

$$\begin{aligned} p_0 &= (1+h)p^0 + gp^1 = \sqrt{1+h} \cdot \tfrac{P_0 - \beta P_1}{\sqrt{1-\beta^2}} \ , \\ p_1 &= gp^0 - p^1 = \tfrac{(g+\beta\Delta)P_0 - (g\beta+\Delta)P_1}{\sqrt{(1+h)(1-\beta^2)}} \ . \end{aligned} \tag{25}$$

Then, it seems that we would have already an integral of motion for the motion in the curved space, by using the definition $p_i = mcdu_i/ds$. However, this is not true, because the p_i are at point x' in the curved space, while the coefficients in the transformation (18) are at point x in the flat space. To know the global coordinate transformations $x(x')$ and $x'(x)$ would amount to solve in fact the equations of motion.

We can revert the above transformations for P_0 and P_1, and make use of $P_0^2 - P^2 = m^2c^2$, with $p_2 = -P_2$, $p_3 = -P_3$ for $g = -\beta$. We get

$$(p_0 + gp_1)^2 - \Delta^2(p^2 + m^2c^2) = 0 \ , \tag{26}$$

or

$$(E - c\mathbf{gp})^2 - c^2(1 + h + g^2)(p^2 + m^2c^2) = 0 \ , \tag{27}$$

where E is the energy of the particle and $\mathbf{p}$ denotes its three-dimensional momentum. This is the relation between energy and momentum for the motion in the curved space. It gives the Hamilton-Jacobi equation.

Equations of motion. The action of a free particle moving in the metric (3) is given by

$$S = -mc\int ds = -mc^2\int dt \cdot (1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2} = \int dt \cdot L \ , \tag{28}$$

where L is the lagrangian. By the principle of least action, we get the equations of motion

$$d\mathbf{p}/dt = \mathbf{F} \ , \tag{29}$$

where the momentum is given by

$$\mathbf{p} = \partial L/\partial \mathbf{v} = \frac{m\mathbf{v} - cm\mathbf{g}}{(1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2}} \tag{30}$$

and force

$$\mathbf{F} = \partial L/\partial \mathbf{r} = -(mc^2/2) \cdot \frac{\partial h/\partial \mathbf{r}}{(1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2}} \ . \tag{31}$$

In deriving the force we assume that h is a function of the coordinates only and $\mathbf{g}$ is a function of the time only. Similarly, the energy is given by

$$E = \mathbf{pv} - L = \frac{mc^2(1 + h) + mc\mathbf{vg}}{(1 + h + 2\mathbf{gv}/c - v^2/c^2)^{1/2}} \ . \tag{32}$$

We see that in the limit h, $\mathbf{g} \rightarrow 0$ the above equations become the equations of motion for a relativistic particle under the action of the force derived from the potential $\varphi = mc^2h/2$, subjected to a translation with velocity $\mathbf{V} = -c\mathbf{g}$. The product $\mathbf{vF}$ gives the variation in time of the kinetic energy dE_{kin}/dt, by definition, but the energy is not conserved, due to the $\mathbf{g}$-term in the lagrangian; indeed, $\partial E/\partial t = -\partial L/\partial t$.

Making use of $\mathbf{u} = \mathbf{v}/c - \mathbf{g}$ in the above equations we get straightforwardly the Hamilton-Jacobi equation (27).

Hamilton-Jacobi equation. The variation of ds in (28) is obtained by

$$\delta ds^2 = 2ds\delta ds = \delta(g_{ij}dx^i dx^j) = 2g_{ij}dx^i d\delta x^j + dx^i dx^j(\partial g_{ij}/\partial x^k)\delta x^k \ . \tag{33}$$

With a vanishing motion at the ends of the trajectory we get the equation of motion (2) from the principle of least action. In addition, the first term in (33) gives also the momentum p_i,

$$-\partial S/\partial x^i = mcu_i = p_i \ , \tag{34}$$

for the motion at the ends of the trajectory.[5]

On the other hand, by (28), we can derive the equations of motion in the form $d(\partial L/\partial \dot{q})/dt - \partial L/\partial q = 0$ for a generic coordinate q, the momentum $p = \partial L/\partial \dot{q} = \partial S/\partial q$ (the metric is now the flat one, and this p corresponds to the contravariant momentum) and introduce the hamiltonian

[5] The variation of the action at the ends of the trajectory was first introduced by Lagrange (~1780) and employed further by Hamilton (~1830) and Jacobi (~1880).

$H = p\dot{q} - L$, as we did before. By δH and the equations of motion we get the canonical equations of motion $\dot{q} = \partial H/\partial q$ and $\dot{p} = -\partial H/\partial q$, and, what is more important, from $dS/dt = L = \partial S/\partial t + \dot{q}(\partial S/\partial q) = \partial S/\partial t + p\dot{q}$, we get $\partial S/\partial t = -H$, *i.e.* the Hamilton-Jacobi equation, which defines the energy as $E = -\partial S/\partial t$, or $E/c = -\partial S/\partial x^0 = p_0$. It follows that

$$p_i = (E/c, -\mathbf{p}) \tag{35}$$

and E as a function of $\mathbf{p}$ and the coordinates is the hamiltonian.

Obviously, for a free particle, $p_i p^i$ is a constant, and we put $p_i p^i = m^2c^2$. Therefore, the Hamilton-Jacobi equation reads

$$g^{ij}p_ip_j = m^2c^2 \ , \tag{36}$$

with $p_i = -\partial S/\partial x^i$. The contravariant metric g^{ij} (the inverse matrix with respect to g_{ij}) is given by

$$g^{ij} = \frac{1}{\Delta^2}\begin{pmatrix} 1 & g_1 & g_2 & g_3 \\ g_1 & -\Delta^2 + g_1^2 & g_1g_2 & g_1g_3 \\ g_2 & g_1g_2 & -\Delta^2 + g_2^2 & g_2g_3 \\ g_3 & g_3g_1 & g_3g_2 & -\Delta^2 + g_3^2 \end{pmatrix} \ . \tag{37}$$

Making use of this metric, one can check easily that equation (36) is the same as the Hamilton-Jacobi equation (27). With $-\partial S/\partial x^i = p_i$, *i.e.* $E = -\partial S/\partial t$ and $\mathbf{p} = \partial S/\partial \mathbf{r}$, it reads

$$(\partial S/\partial t + c\mathbf{g}\partial S/\partial \mathbf{r})^2 - c^2(1 + h + g^2)[(\partial S/\partial \mathbf{r})^2 + m^2c^2] = 0 \ . \tag{38}$$

In the limit $h = 2\varphi/mc^2 \to 0$ and $\mathbf{g} = -\mathbf{V}/c \to 0$ it describes the relativistic motion of a particle under the action of the (weak) force $\mathbf{f} = -\partial\varphi/\partial\mathbf{r}$ and for an observer moving with a (small) velocity $\mathbf{V}$. One can check directly that the coordinate transformations given by equation (20) takes the free Hamilton-Jacobi equation $(\partial S/\partial t)^2 - c^2[(\partial S/\partial \mathbf{r})^2 + m^2c^2]$ into the "interacting" Hamilton-Jacobi equation (38), as expected.

The eikonal equation. A wave moves through $k_idx^i = -d\Phi$, where $k_i = -\partial\Phi/\partial x^i = (\omega/c, -\mathbf{k})$, ω is the frequency, $\mathbf{k}$ is the wavevector and Φ is called the eikonal.[6] In a flat space k_i are constant, and the wave propagates along a straight line, such that $k_ik^i = 0$, *i.e.* $\omega^2/c^2 - k^2 = 0$ and $\Phi = -\omega t + \mathbf{kr}$. This is a light ray. In a curved space $k_ik^i = 0$ reads $g^{ij}k_ik_j = 0$, and for g^{ij} slightly departing from the flat metric we have the geometric approximation to the wave propagation. It is governed by the eikonal equation $g^{ij}(\partial\Phi/\partial x^i)(\partial\Phi/\partial x^j) = 0$, or

$$(\partial\Phi/c\partial t + \mathbf{g}\partial\Phi/\partial\mathbf{r})^2 - (1 + h + g^2)(\partial\Phi/\partial\mathbf{r})^2 = 0 \ , \tag{39}$$

which is the Hamilton-Jacobi equation (38) for $m = 0$.

We neglect the g^2-contributions to this equation and notice that the first term may not depend on the time (h is a function of the coordinates only). It follows then that the first term in the above equation can be put equal to ω_0/c,

$$\partial\Phi/c\partial t + \mathbf{g}\partial\Phi/\partial\mathbf{r} = -\omega_0/c \ , \tag{40}$$

where ω_0 is the frequency of the wave in the flat space, and

$$(\partial\Phi/\partial\mathbf{r})^2 = k^2 = \frac{1}{1+h}\cdot(\omega_0/c)^2 = \frac{1}{1+h}\cdot k_0^2 \ , \tag{41}$$

[6]It follows that the propagation of light goes by an extremum of the eikonal written as $\int d\lambda \cdot \sqrt{g_{ij}(dx^i/d\lambda(dx^j/d\lambda)}$, where λ is an arbitrary parameter. Equations of motion (2) are then obtained with ds replaced by $d\lambda$.

where $\mathbf{k}_0$ is the wavevector in the flat space. Within our approximation equation (40) becomes

$$\partial\Phi/c\partial t = -\omega_0/c - \mathbf{g}\mathbf{k}_0 \ . \tag{42}$$

We measure the frequency ω corresponding to the proper time, *i.e.* $\omega/c = -\partial\Phi/c\partial\tau$, where $d\tau = \sqrt{1+h}dt$ for our metric, so the measured frequency of the wave is given by

$$\omega/c = -\partial\Phi/c\partial\tau = -\frac{1}{\sqrt{1+h}} \cdot \partial\Phi/c\partial t = \frac{1}{\sqrt{1+h}} \cdot \omega_0/c + \mathbf{g}\mathbf{k}_0 \ . \tag{43}$$

There exists, therefore, a shift in frequency

$$\Delta\omega/\omega_0 = -h/2 + c\mathbf{g}\mathbf{k}_0/\omega_0 \ . \tag{44}$$

The first term in equation (44) is due to the static forces (like the gravitational potential, for instance), while the second term is analogous to the (longitudinal) Doppler effect, for $\mathbf{g} = -\mathbf{V}/c$.

By (41) we have

$$(\partial\Phi/\partial\mathbf{r})^2 = (1-h)k_0^2 \ . \tag{45}$$

We assume that h depends only on the radius r, and write the above equation in spherical coordinates; Φ does not depend on θ, and we put $\theta = \pi/2$;

$$(\partial\Phi/\partial r)^2 + (1/r^2)(\partial\Phi/\partial\varphi)^2 = (1-h)k_0^2 \ ; \tag{46}$$

the solution is of the form

$$\Phi = \Phi_r(r) + M\varphi \ , \tag{47}$$

where M is a constant and

$$\Phi_r(r) = \int_\infty^r dr \cdot \sqrt{(1-h)k_0^2 - M^2/r^2} \ ; \tag{48}$$

the trajectory is given by $\partial\Phi/\partial M = const$,[7] hence

$$\varphi = -\int_\infty^r dr \cdot \frac{M/r^2}{\sqrt{(1-h)k_0^2 - M^2/r^2}} \ . \tag{49}$$

For $h = 0$ we get $r\sin\varphi = M/k_0$, which is a straight line passing at distance M/k_0 from the centre. The deviation angle is

$$\Delta\varphi = -(k_0^2/2)\int_\infty^r dr \cdot \frac{h \cdot M/r^2}{(k_0^2 - M^2/r^2)^{3/2}} \ . \tag{50}$$

Therefore, the light ray is bent by the static forces in a curved space. [8] One can also define the refractive index $\mathbf{n}$ of the curved space, by $\mathbf{k} = \mathbf{n}(\omega/c)$. Its magnitude is related to $\mathbf{g}\mathbf{k}_0$, while its direction is associated to the inhomogeneity h of the space.

It is worth noting, by (42), that the time-dependent part of the eikonal is given by

$$\Phi_t(t) = -\omega_0 t + \mathbf{k_0}\mathbf{R}(t) \tag{51}$$

[7]Constant M is a generalized coordinate which moves freely; therefore, the force acting upon it vanishes, $\partial L/\partial M = 0$, or $d(\partial S/\partial M)/dt = 0$, *i.e.* $\partial S/\partial M = const$.

[8]The metric given by (3) for $h = 2\varphi/c^2$ differs from the metric created by a gravitational point mass m with $\varphi = Gm/r$; they coincide only in the non-relativistic limit. The deviation angle given by (50) for a gravitational potential is smaller by a factor of 4 than the deviation angle in the gravitational potential of a point mass.

for $\mathbf{g} = -\mathbf{V}/c$, *i.e.* the eikonal corresponding to a translation, as expected.

Quantization. Suppose that we have a free motion. Then we know its solution, *i.e.* the dependence of the coordinates, say some x, on some parameter, which may be called some time t. Suppose further that we have a motion under the action of some forces. Then, we know the dependence of its coordinates, say some x', on some parameter, which may be the same t as in the former case. Then, we may establish a correspondence between x and x', *i.e.* a global coordinate transformation. It follows that the motion under the action of the forces is a global coordinate transformation applied to the free motion. Similarly, two distinct motions are put in relation to each other by such global coordinate transformations.

This line of thought, due to Einstein, lies at the basis of both the special theory of relativity and the general theory of relativity.

Indeed, it has beeen noticed that the equations of the electromagnetic field are invariant under Lorentz transformations of the coordinates, which leave the distance given by $s^2 = c^2t^2 - \mathbf{r}^2$ invariant. These transformations are an expression of the principle of inertia, and this invariance is the principle of relativity.[9] As such, the Lorentz transformations are applicable to the motion of particles, starting, for instance, from a particle at rest. Let $x = c\beta\tau/\sqrt{1-\beta^2}$, $t = \tau/\sqrt{1-\beta^2}$ be these Lorentz transformations where τ is the time of the particle at rest. We may apply these transformations to the momentum $\mathbf{p} = \partial S/\partial \mathbf{r}$ and $p_0 = -\partial S/c\partial t = E/c$, where E is the energy of the particle. Then, we get immediately $\mathbf{p} = \mathbf{v}E/c^2$ and $E = E_0/\sqrt{1-\beta^2}$. The non-relativistic limit is recovered for $E_0 = mc^2$, the "inertia of the energy". The equations of motion are $d\mathbf{p}/dt = \mathbf{f}$, and we can see that indeed, there appear additional, "dynamic forces", depending on relativistic v^2/c^2-terms, in comparison with Newton's law. In adition, we get the Hamilton-Jacobi equation $E^2 - c^2(p^2 + m^2c^2) = 0$. This is the whole theory of special relativity.

The situation is similar in the general theory of relativity, except for the fact that in a curved space we have not the global coordinate transformations, in general, as in a flat space. However, the Hamilton-Jacobi equation gives access to the action function, which may provide a relationship between some integrals of motion. Action S depends on some constants of integration, say M. Then, these constants can be viewed as freely-moving generalized coordinates, so $\partial S/\partial M = const$, because the force $\partial L/\partial M = d(\partial S/\partial M)/dt$ vanishes. Equation $\partial S/\partial M = const$ provides the equation of the trajectory. Of course, this is based upon the assumption that the motion is classical, *i.e.* non-quantal, in the sense that there exists a trajectory.[10] For instance, the solution of the Hamilton-Jacobi equation for a free particle is $S = -Et + \mathbf{pr}$, where E and $\mathbf{p}$ are constants such that $E = \sqrt{m^2c^4 + c^2p^2}$. By $\partial S/\partial E = const$ we get $-t + (E/c^2p)(\mathbf{pr}/p) = const$, which is the trajectory of a free particle.

For a classical motion it is useless to attempt to solve the motion in a curved space produced by a non-inertial motion, like non-uniform translations, because it is much simpler to solve the motion in the absence of the non-inertial motion and then get the solution by a coordinate transformation, like a non-uniform translation for instance. For a quantal motion, however, the things change appreciably.

The Hamilton-Jacobi equation admits another kind of motion too, the quantal motion. Obviously, for a free particle, the classical action given above is the phase of a wave. Then, it is natural to introduce a wavefunction ψ through $S = -i\hbar \ln \psi$, where $\hbar$ turns out to be Planck's constant. The classical motion is recovered in the limit $\hbar \rightarrow 0$, $Re\psi = finite$ and $Im\psi \rightarrow \infty$, such that

[9]This was noticed by Einstein (1905).

[10]This procedure, applied to the Hamilton-Jacobi equation (27) with $\mathbf{g} = 0$ led to the precession in Mercury's perihelia in the gravitational field of the Sun (Einstein, 1915).

$S = finite$. With this transformation we have $\mathbf{p} = -i\hbar(\partial\psi/\partial\mathbf{r})/\psi$ and $E = i\hbar(\partial\psi/\partial t)/\psi$, which means that momentum and energy are eigenvalues of their corresponding operators, $-i\hbar\partial/\partial\mathbf{r}$ and $i\hbar\partial/\partial t$, respectively.[11] It follows that the physical quantities have not well-defined values anymore, in contrast to the classical motion. In particular, there is no trajectory of the motion. Instead, they have mean values and deviations, *i.e.* they have a statistical meaning, and the measurement process has to be defined in such terms. It turns out that the wavefunction squared is just the density of probability for the motion to be in some quantal state, and for a defined motion this probability must be conserved.

Klein-Gordon equation. With the substitution $E \rightarrow i\hbar\partial/\partial t$ and $\mathbf{p} \rightarrow -i\hbar\partial/\partial\mathbf{r}$ in the Hamilton-Jacobi equation in the flat space we get the Klein-Gordon equation

$$\partial^2\psi/\partial t^2 - c^2\partial^2\psi/\partial\mathbf{r}^2 + (m^2c^4/\hbar^2)\psi = 0 \ . \tag{52}$$

A similar quantization for the Hamilton-Jacobi equation given by (38) encounters difficulties, since the operators $1 + h + g^2$ and $p^2 + m^2c^2$ do not commute with each other, nor with the operator $E - c\mathbf{gp}$ (we recall that h is a function of the coordinates only, $h(\mathbf{r})$, and $\mathbf{g}$ is a function of the time only, $\mathbf{g}(t)$). We may neglect the g^2-term in $1 + h + g^2$, and write the Hamilton-Jacobi equation (38) as

$$\frac{1}{1+h}(E - c\mathbf{gp})^2 = c^2(p^2 + m^2c^2) \ , \tag{53}$$

where the two operators in the *lhs* of this equation commute now, up to quantities of the order of hg (or higher), which we neglect. With these approximations, the quantization rules can now be applied, and we get an equation which can be written as

$$(\partial/\partial t + c\mathbf{g}\partial/\partial\mathbf{r})^2\psi - c^2(1+h)[\partial^2\psi/\partial\mathbf{r}^2 - (m^2c^2/\hbar^2)\psi] = 0 \ . \tag{54}$$

It can be viewed as describing the quantal motion of a particle under the action of a weak force $-(mc^2/2)\partial h(\mathbf{r})/\partial\mathbf{r}$, as seen by an observer moving with the small velocity $-c\mathbf{g}(t)$. It can be derived directly from (52) by the coordinate transformations (20), in the limit h, $\mathbf{g} \rightarrow 0$.[12] It is worth noting, however, that there is still a slight inaccuracy in deriving this equation, arising from the fact that the operator $(1+h)(p^2 + m^2c^2)$ is not hermitean. It reflects the indefineteness in writing $(1+h)(p^2+m^2c^2)$ or $(p^2+m^2c^2)(1+h)$ when passing from (53) to (54). This indicates the ambiguities in quantizing the relativistic motion, and they are remedied by the theory of the quantal fields, as it is shown below.

The above equation can be written more conveniently as

$$(i\hbar\partial/\partial t - c\mathbf{gp})^2\psi - c^2(1+h)(p^2 + m^2c^2)\psi = 0 \ , \tag{55}$$

where $\mathbf{p} = -i\hbar\partial/\partial\mathbf{r}$ and $i\hbar\partial/\partial t$ stands for the energy E.

We introduce the operator

$$H^2 = c^2(1+h)(p^2 + m^2c^2) = c^2(p^2 + m^2c^2) + c^2h(p^2 + m^2c^2) \ , \tag{56}$$

which is time-independent, and treat the h-term as a small perturbation. It is easy to see, in the first-order of the perturbation theory, that the wavefunctions are labelled by momentum $\mathbf{p}$, and

[11] Einstein's (1905) quantization of energy and de Broglie's (1923) quantization of momentum follow immediately by this assumption, which gives a meaning to the Bohr-Sommerfeld quantization rules (Bohr, 1913, Sommerfeld, 1915). The quantal operators was first seen as matrices by Heisenberg, Born, Jordan, Pauli (1925-1926).

[12] It has to be compared with the Klein-Gordon equation $(i\hbar\partial/\partial t - e\varphi)^2\psi - c^2[(i\hbar\partial/\partial\mathbf{r} + e\mathbf{A}/c)^2 + m^2c^2] = 0$ for a particle with charge e in the electromagnetic field $(\varphi, \mathbf{A})$, which, hystorically, was first considered for the Hydrogen atom (Schrodinger, Klein, Gordon, 1926). There, the forces come by the electromagnetic gauge field.

are plane waves with a weak admixture of plane waves of the order of h; we denote them by $\varphi(\mathbf{p})$. Similarly, in the first-order of the perturbation theory, the eigenvalues of H^2 can be written as $E^2(p) = c^2(1+\overline{h})(p^2+m^2c^2)$, where $\overline{h} = (1/V)\int d\mathbf{r}\cdot h$, V being the volume of normalization. We have, therefore, $H^2\varphi(\mathbf{p}) = E^2(p)\varphi(\mathbf{p})$. Now, we look for a time-dependent solution of equation (55) $(i\hbar\partial/\partial t - c\mathbf{gp})^2\psi = H^2\psi$, which can also be written as $(i\hbar\partial/\partial t - c\mathbf{gp})\psi = H\psi$, where ψ is a superposition of eigenfunctions

$$\psi = \sum_{\mathbf{p}} c_{\mathbf{p}}(t)e^{-iE(p)t/\hbar}\varphi(\mathbf{p}) \ . \tag{57}$$

We get

$$\dot{c}_{\mathbf{p}'} = -(i/\hbar)\sum_{\mathbf{p}} c_{\mathbf{p}}e^{-i[E(p)-E(p')]/\hbar}c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}} \ , \tag{58}$$

where $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ is the matrix element of the momentum $\mathbf{p}$ betwen the states $\varphi(\mathbf{p}')$ and $\varphi(\mathbf{p})$. We assume $c_{\mathbf{p}} = c^0_{\mathbf{p}} + c^1_{\mathbf{p}}$, such as $c^0_{\mathbf{p}'} = 0$ for all $\mathbf{p}' \neq \mathbf{p}$ and $c^0_{\mathbf{p}} = 1$, and get

$$\dot{c}^1_{\mathbf{p}'} = -(i/\hbar)e^{-i[E(p)-E(p')]/\hbar}c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}} \ , \tag{59}$$

which can be integrated straightforwardly. The square $\left|c^1_{\mathbf{p}'}\right|^2$ gives the transition probability from state $\varphi(\mathbf{p})$ in state $\varphi(\mathbf{p}')$.[13]

It follows that an observer in a non-uniform translation might see quantal transitions between the states of a relativistic particle, providing the frequencies in the Fourier expansion of $\mathbf{g}(t)$ match the difference in the energy levels. In the zeroth-order of the perturbation theory the eigenfunctions $\varphi(\mathbf{p})$ are plane waves, and the matrix elements $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ of the momentum vanish, so there are no such transitions to this order. In general, if the total momentum is conserved, as for free or interacting particles, these transitions do not occur. In the first order of the perturbation theory for the external force represented by h the matrix elements of the momentum do not vanish, in general, and we may have transitions, as an effect of a non-uniform translation. Within this order of the perturbation theory the matrix elements of the momentum are of the order of h, and the transition amplitudes given by (59) are of the order of gh. We can see that the time-dependent term of the order of gh neglected in deriving equation (55) produces corrections to the transition amplitides of the order of gh^2, so its neglect is justified.

In general, the solution of the second-order differential equation (54) can be approached by using the Fourier transform. Then, it reduces to a homogeneous matricial equation, where labels are the frequency and the wavevector $(\omega, \mathbf{k})$, conveniently ordered. The condition of a non-trivial solution is the vanishing of the determinant of such an equation. This gives a set of conditions for the ordered points $(\omega, \mathbf{k})$ in the $(\omega, \mathbf{k})$-space, but these conditions do not provide anymore an algebraic connection between the frequency ω and the wavevector $\mathbf{k}$.[14] This amounts to saying that for a given ω the wavevectors are not determined, and, conversely, for a given wavevector $\mathbf{k}$ the frequencies are not determined, *i.e.* the quantal states do not exist in fact, anymore. The particle exhibits quantal transitions, which make its quantal state undetermined. The same conclusion can also be seen by introducing a non-uniform translation in the phase of a plane wave, expanding the plane wave with respect to this translation, under certain restrictions, and then using the time Fourier expansion of the translation. The frequency of the original plane wave

[13] The same interaction gives an irrelevant phase factor for the original state $\varphi(\mathbf{p})$.

[14] This may be related to the well-known problem of the Ising models. Though it is true that for the statistical sum we need the eigenvalues, it is also true that we need also the weight of these eigenvalues, which depend on the symmetries. In general, we do not have these weights for the label-ordered eigenvalues, except for two dimensions, where the hamiltonian can be diagonalized in its own original labels, in fact.

changes correspondingly, which indicates indeed that there are quantal transitions. One may say that for a curved space as the one represented by the metric given here, the quantization question has no meaning anymore, or it has the meaning given here.

In the non-relativistic limit, the above Klein-Gordon equation becomes

$$i\hbar\partial\psi/\partial t = H\psi = (mc^2 + p^2/2m + \varphi)\psi + c\mathbf{gp}\psi \ , \tag{60}$$

which is Schrodinger's equation up to the rest energy mc^2, and one can see more directly the perturbation $c\mathbf{gp} = -\mathbf{Vp}$. It is worth noting that the derivation of Schrodinger's equation holds irrespectively of the ambiguities related to the quantization of the Hamilton-Jacobi equation. It follows, that under the conditions mentioned above, *i.e.* in the presence of a (non-trivial) external field φ, an observer in a non-uniform translation may observe quantal transitions in the non-relativistic limit, due to the non-inertial motion.[15] Obviously, the frequency of this motion must match the quantal energy gaps, for such transitions to be observed.

Dirac equation. Fields. As it is well-known, the Klein-Gordon equation (52) has a serious drawback. It conserves the density $\psi^*(\partial\psi/\partial t) - (\partial\psi^*/\partial t)\psi$, which, because of negative frequencies (energies), is not always positive; therefore, we cannot take it as a density of probability, as we can for Schrodinger's equation, where the energies are positive only. It has been thought for a while that this difficulty is circumvented by Dirac equation[16]

$$i\hbar\partial\psi/\partial t = H\psi = (\alpha c\mathbf{p} + \beta mc^2)\psi \ , \tag{61}$$

where α, β are the well-known Dirac matrices and ψ is a four-components column. Indeed, applying twice the operator in the *rhs* of equation (56), we get the Klein-Gordon equation $E^2\psi = (c^2p^2 + m^2c^4)\psi$, since $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$, and the conserved density of probability is now $\overline{\psi}\psi$, which is positive, $\overline{\psi}$ being the complex-conjugate four-components line .

It is impossible, in general, to get a similar Dirac equation for equation (61), because the operators $(1 - h/2)(E - c\mathbf{gp})$ and $\alpha c\mathbf{p} + \beta mc^2$, which represent the square roots of the two sides of equation (53) do not commute anymore. Nevertheless, if we limit ourselves to the first order of the perturbation theory, we can see that the operator H^2 defined in the previous section reduces to $c^2(p^2 + m^2)$ providing we redefine the energy levels such as to include the factor $1 + \bar{h}$. Within this approximation, we get the Dirac equation

$$(i\hbar\partial/\partial t - c\mathbf{gp})\psi = (\alpha c\mathbf{p} + \beta mc^2)\psi \ , \tag{62}$$

where ψ contains now a weak admixture of plane waves, of the order of h. It is worth noting that this equation is the Dirac equation given by (61), subjected to the translation $\mathbf{r} = \mathbf{r}' + \mathbf{R}$, and $t = t'$. The non-uniform translation in the *lhs* of equation (62) gives now quantal transitions.

Though conserving a probability, the Dirac equation has still negative energies, which raise difficulties in giving them a meaning. On the other hand, it turned out that the Dirac equation corresponds to a particle of spin 1/2, so the original Klein-Gordon equation (52), which has no spin at all, still remains to be given a meaning, with respect to its lack of a well-defined probability. The solution to all these problems was given by the notion of quantal fields. The ψ in

[15] M. Apostol, J. Theor. Phys. **140**, **142** (2006). A suitable unitary transformation of the wavefunction (for instanceexp$(-i\mathbf{Rp}/\hbar)$) can produce such an interaction in the time-dependent *lhs* of the Schrodinger equation, but, at the same time, it produces an equivalent interaction in the hamiltonian, such that the Schrodinger equation is left unchanged. Such unitary transformations are related to symmetries (Wigner's theorem, 1931) and they are different from a change of coordinates.

[16] Dirac (1928).

these equations are not wavefunctions anymore, but fields. They are expanded in series of wavefunctions which satisfy differential equations, but the coefficients of the expansion are operators of creating and annihilating particles, with definite Bose-Einstein or Fermi-Dirac commutation relations. This is called the "second quantization".[17] The probability which had no meaning for the Klein-Gordon equation is now the charge, which may be negative too. It appeared the necessity of introducing antiparticles in the quantal field theory. The fields move in their own right, and lagrangians have to be established for this motion, which involves essentially the motion of the number of particles.[18] This way, the whole problem of the quantal relativistic motion receives a different meaning.

A scalar field in a curved space. Let[19]

$$S = \int dx^0 d\mathbf{r}\sqrt{-g}\cdot\left[(\partial_i\psi)(\partial^i\psi) + (m^2c^2/\hbar^2)\psi^2\right] \tag{63}$$

be the lagrangian for the (real) scalar field ψ, where $g = -\Delta^2 = -(1+h+g^2)$ is the determinant of the metric given by (4). It is easy to see that the principle of least action for ψ in a flat space leads to the Klein-Gordon equation (53). For the metric given by (4), and neglecting g^2-terms, we get a generalized Klein-Gordon equation

$$\begin{gathered}(\partial/\partial t + c\mathbf{g}\partial/\partial\mathbf{r})\tfrac{1}{\sqrt{1+h}}(\partial/\partial t + c\mathbf{g}\partial/\partial\mathbf{r})\psi - \\ -c^2(\partial/\partial\mathbf{r})\sqrt{1+h}(\partial/\partial\mathbf{r})\psi + \sqrt{1+h}(m^2c^4/\hbar^2)\psi = 0\ .\end{gathered} \tag{64}$$

We can apply the same perturbation approach to this equation as we did for equation (53). Doing so, we get equation (55) and an additional term $i(c^2\hbar/2)(\partial h/\partial\mathbf{r})\mathbf{p}$, which yields no difficulties in the perturbation approach. The resulting equation reads

$$(i\hbar\partial/\partial t - c\mathbf{g}\mathbf{p})^2\psi - c^2(1+h)(p^2+m^2c^2)\psi + (ic^2\hbar/2)(\partial h/\partial\mathbf{r})\mathbf{p}\psi = 0\ . \tag{65}$$

It is worth noting that in the limit $\mathbf{g}\rightarrow 0$ this is an exact equation. The qualitative conclusions derived above for equation (55), as regards the quantal transitions produced by the non-uniform translation, remain valid, though, we have now a language of fields. It follows that a quantal particle, either relativistic or non-relativistic, in a curved space of the form analyzed herein becomes a wave packet from a plane wave (or even forms a bound state), as a consequence of the forces, and, at the same time, suffers quantal transitions, due to the time-dependent metric (as if in a non-inertial translation for instance). This gives no meaning to the problem of the quantization in curved spaces, or it gives the meaning discussed here.

The density L of lagrangian in the action $S = \int dt d\mathbf{r}\cdot L$ given by (63) gives the momentum $\Pi = \partial L/\partial(\partial\psi/\partial t)$ and the hamiltonian density $\Pi\partial\psi/\partial t - L$. The quantized field reads

$$\psi = \sum_{\mathbf{p}}(c\hbar/2\sqrt{\varepsilon})(a_{\mathbf{p}}e^{-i\varepsilon t/\hbar + i\mathbf{p}\mathbf{r}/\hbar} + a^+_{\mathbf{p}}e^{i\varepsilon t/\hbar - i\mathbf{p}\mathbf{r}/\hbar})\ , \tag{66}$$

[17]Fock (1933), Furry and Oppenheimer (1934) for electrons; Pauli and Weisskopf (1934) for scalar particles. After previous work by Born, Heisenberg and Jordan (1926), Dirac (1927), Jordan (1927), Jordan and Wigner (1928), Fermi (1930, 1932).

[18]Heisenberg and Pauli (1929).

[19]In general, the action for fields must be written by replacing the flat metric η_{ij} by the curved metric g_{ij} (including $\sqrt{-g}$ in the elementary volume of integration) and replacing the derivatives ∂_i by covariant derivatives D_i. The latter requirement can produce technical difficulties, in general. However, for a scalar field or for the electromagnetic field the D_i has the same effect as ∂_i, so the former are superfluos.

and

$$\Pi = -i\sum_{\mathbf{p}}(\sqrt{\varepsilon}/c)(a_{\mathbf{p}}e^{-i\varepsilon t/\hbar+i\mathbf{pr}/\hbar} - a^{+}_{\mathbf{p}}e^{i\varepsilon t/\hbar-i\mathbf{pr}/\hbar}) \ , \tag{67}$$

where $\varepsilon = c\sqrt{m^2c^2+p^2}$ and $[\psi(t,\mathbf{r}),\Pi(t,\mathbf{r}')] = i\hbar\delta(\mathbf{r}-\mathbf{r}')$ with usual commutation relations for the bosonic operators $a_{\mathbf{p}}$, $a^{+}_{\mathbf{p}}$ and a normalization of one $\mathbf{p}$-state in a unit volume. The hamiltonian is obtained by integrating its density given above over the whole space. It can be written as $H = H_0 + H_{1h} + H_{1g}$, where

$$H_0 = \int d\mathbf{r}\cdot\left[c^2\Pi^2/4 + (\partial\psi/\partial\mathbf{r})^2 + (m^2c^2/\hbar^2)\psi^2\right] = \sum_{\mathbf{p}}(\varepsilon/2)(a_{\mathbf{p}}a^{+}_{\mathbf{p}} + a^{+}_{\mathbf{p}}a_{\mathbf{p}}) \tag{68}$$

is the free hamiltonian,

$$H_{1h} = \int d\mathbf{r}\cdot(\sqrt{1+h}-1)\left[c^2\Pi^2/4 + (\partial\psi/\partial\mathbf{r})^2 + (m^2c^2/\hbar^2)\psi^2\right] \tag{69}$$

is the interacting part due to the external field h and

$$H_{1g} = -(c/2)\int d\mathbf{r}\cdot[\Pi(\mathbf{g}\partial\psi/\partial\mathbf{r}) + (\mathbf{g}\partial\psi/\partial\mathbf{r})\Pi] = -(c/2)\sum_{\mathbf{p}}(\mathbf{gp})(a_{\mathbf{p}}a^{+}_{\mathbf{p}} + a^{+}_{\mathbf{p}}a_{\mathbf{p}}) \tag{70}$$

is the time-dependent interaction. Perturbation theory can now be applied systematically in the first-order of $\mathbf{g}$ and all the orders of h, with the same results as those described above: the quanta will scatter both their wavevectors and their energy. Similar field theories can be set up for charged particles, or for particles with spin 1/2 and for photons, moving in a curved space given by the metric (4).

Electromagnetic field in curved spaces. Photons. The action for the electromagnetic field is

$$S = -(1/16\pi c)\int dx^0 d\mathbf{r}\cdot\sqrt{-g}F_{ij}F^{ij} \ , \tag{71}$$

where the electromagnetic fields F_{ij} are given by the potentials A_i through $F_{ij} = \partial_i A_j - \partial_j A_i$. This leads immediately to the first pair of Maxwell equations (the free equations) $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ and the principle of least action gives the second pair of Maxwell equations

$$\partial_j(\sqrt{-g}F^{ij}) = 0 \ . \tag{72}$$

In the presence of charges and currents the *rhs* of equation (72) contains the current, conveniently defined. The antisymmetric tensor F_{ij} consists of a vector and a three-tensor in spatial components, the latter being representable by another vector, its dual. Let these vectors be denoted by $\mathbf{E}$ and $\mathbf{B}$. Similarly, by raising or lowering the suffixes we can define other two vectors, related to the former pair of vectors, and denoted by $\mathbf{D}$ and $\mathbf{H}$. Then, the Maxwell equations obtained above take the usual form of Maxwell equations in matter, namely $curl\mathbf{E} = -(1/c\sqrt{\gamma})\partial(\sqrt{\gamma}\mathbf{B})/\partial t$, $div\mathbf{B} = 0$ (the free equations) and $div\mathbf{D} = 4\pi\rho$, $curl\mathbf{H} = (1/c\sqrt{\gamma})\partial(\sqrt{\gamma}\mathbf{D})/\partial t + (4\pi/c)(\rho\mathbf{v})$, where ρ is the density of charge divided by $\sqrt{\gamma}$ and $\gamma_{\alpha\beta} = -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00}$ is the spatial metric (*div* and *curl* are conveniently defined in the curved space). For our metric, and neglecting g^2, the matrix γ reduces to the euclidean metric of the space ($\gamma = 1$).

We use $A_0 = 0$, $F_{0\alpha} = \partial_0 A_\alpha$ and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We define an electric field $\mathbf{E} = grad\mathbf{A}$ and a magnetization field $\mathbf{B} = -curl\mathbf{A}$. Then, neglecting g^2, equation (72) can be written as

$$div[(\mathbf{E} + g\times\mathbf{B})/\Delta] = 0 \tag{73}$$

and

$$\frac{\partial}{c\partial t}[(\mathbf{E}+\mathbf{g}\times\mathbf{B})/\Delta] = curl[\Delta\mathbf{B}+\mathbf{g}\times\mathbf{E}/\Delta] \ , \tag{74}$$

where $\Delta=\sqrt{1+h}$. One can see that we may have a displacement field $\mathbf{D}=(\mathbf{E}+\mathbf{g}\times\mathbf{B})/\Delta$ and a magnetic field $\mathbf{H}=\Delta\mathbf{B}+\mathbf{g}\times\mathbf{E}/\Delta$, and the Maxwell equations $div\mathbf{D}=0$, $\partial\mathbf{D}/c\partial t=curl\mathbf{H}$ without charges.

Equations (73) and (74) can be solved by the perturbation theory, for small values of h and $\mathbf{g}$, starting with free electromagnetic waves as the unperturbed solution. Doing so, we arrive immediately at the result that the solution must be a wave packet, and the frequencies are not determined anymore, in the sense that either for a given wavevector we have many frequencies or for a given frequency we have many wavevectors. This can be most conveniently expressed in terms of photons which suffer quantal transitions.

The quantization of the electromagnetic field in a curved space proceeds in the usual way. The action given by (71) can be written as

$$S=(1/8\pi)\int dtd\mathbf{r}\cdot\Delta(\mathbf{D}^2-\mathbf{B}^2)=(1/8\pi)\int dtd\mathbf{r}\cdot(1/\Delta)[\mathbf{E}^2+2\mathbf{E}(\mathbf{g}\times\mathbf{B})-\Delta^2\mathbf{B}^2] \tag{75}$$

which exhibits the well-known density of lagrangian in the limit h, $\mathbf{g}\rightarrow 0$. We change now to the covariant vector potential $\mathbf{A}\rightarrow-\mathbf{A}$, such that $\mathbf{E}=-\partial\mathbf{A}/c\partial t$ and $\mathbf{B}=curl\mathbf{A}$. Leaving aside the factor $1/8\pi$, the momentum is given by $\Pi=\partial L/\partial(\partial\mathbf{A}/\partial t)=(2/\Delta c^2)(\partial\mathbf{A}/\partial t-\mathbf{g}\times\mathbf{B})$. The vector potential is represented as

$$\mathbf{A}_\alpha=\sum_{\alpha\mathbf{p}}(c\hbar/2\sqrt{\varepsilon})[a_{\alpha\mathbf{p}}\mathbf{e}^\alpha e^{-i\varepsilon t/\hbar+i\mathbf{pr}/\hbar}+h.c.] \tag{76}$$

and the momentum by

$$\Pi_\alpha=-i\sum_{\alpha\mathbf{p}}(\sqrt{\varepsilon}/c)[a_{\alpha\mathbf{p}}\mathbf{e}^\alpha e^{-i\varepsilon t/\hbar+i\mathbf{pr}/\hbar}-h.c.] \ , \tag{77}$$

where $\mathbf{e}^\alpha$ is the polarization vector along the direction α, perpendicular to $\mathbf{p}=\hbar\mathbf{k}$ (we assume the transversality condition $div\mathbf{A}=0$), $\varepsilon=\hbar\omega=cp$, ω is the frequency and $\mathbf{k}$ is the wavevector. The commutation relations are the usual bosonic ones, and we get the hamiltonian $H=H_0+H_{1h}+H_{1g}$, given by

$$H_0=\int d\mathbf{r}\cdot(c^2\Pi^2/4+B^2)=\sum\nolimits_{\alpha\mathbf{p}}(\varepsilon/2)(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}}+a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}) \ ,$$

$$H_{1h}=\int d\mathbf{r}\cdot(\sqrt{1+h}-1)(c^2\Pi^2/4+\mathbf{B}^2) \ , \tag{78}$$

$$H_{1g}=-\sum\nolimits_{\alpha\mathbf{p}}(\mathbf{gp}/2)(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}}+a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}) \ .$$

Systematic calculations can now be performed within the perturbation theory, and we can see that quantal transitions between the photonic states may appear, starting with the hg-order of the perturbation theory. Therefore, an observer moving with a non-uniform velocity is able to see a "blue shift" in the frequency of the photons "acted" by a force like the gravitational one.[20] The shift occurs obviously at the expense of the energy of the observer's motion.[21]

[20] This is similar with the Unruh effect (1976).

[21] It is worth investigating the change in the equilibrium distribution of the black-body radiation as a consequence of the non-uniform translation in a gravitational field. The frequency shift amounts to a change of temperature, which increases, most likely, by $\Delta T/T\sim(g\bar{h})^2$, with temporal and spatial averages (for the quantization of the black-body radiation see Fermi, 1932). In this respect, the effect discussed here, though related to the Unruh effect, is different. The Unruh effect assumes rather that the external non-uniform translation, as a macroscopic motion, consists of a coherent vacuum, so equilibrium photons can be created; the related increase in temperature is rather the measurement made by the observer of its own motion.

Other fields. A similar approach can be used for other fields in a curved space. In particular, it can be applied to spin-1/2 Dirac fields, with similar conclusions, though, technically, it is more cumbersome to write down the action for spinors in curved spaces.[22] The question of quantizing the gravitational field can also be tackled in a similar manner. Indeed, weak perturbations of the flat metric can be represented as gravitational waves,[23] which can be quantized by using the gravitational action $\int dx^0 d\mathbf{r}\cdot\sqrt{-g}R$, where R is the curvature of the space.[24] Now, we may suppose that these gravitons move in a curved space with the metric g. We may use the same gravitational action as before, where g is now the metric of the space and R contains the graviton field. Or, alternately, we expand $g = g_0 + \delta g$, where $g_0$is the background part and δg is the graviton part. We get a field theory of gravitons interacting with the underlying curved space, and we get quantal transitions of the gravitons, which gives a meaning to the quantization of the gravity, in the sense that either it is not possible or the gravitons suffer quantal transitions. The space and time (the gravitons) are then scattered statistically by matter (which in turn suffers a similar process) or by the non-inertial motion.

Rotations. A rotation of angular frequency Ω about some axis is an orthogonal transformation of coordinates defined locally by

$$d\mathbf{r}' = d\mathbf{r} + (\Omega\times\mathbf{r})\,dt \ , \tag{79}$$

such that the velocity is $\mathbf{v}' = \mathbf{v} + \Omega\times\mathbf{r}$ and

$$\begin{aligned} d\mathbf{v}' &= d\mathbf{v} + \left(\dot{\Omega}\times\mathbf{r}\right)dt + (\Omega\times\mathbf{v})\,dt + [\Omega\times(\mathbf{v}+\Omega\times\mathbf{r})]\,dt = \\ &= d\mathbf{v} + \left(\dot{\Omega}\times\mathbf{r}\right)dt + 2\,(\Omega\times\mathbf{v})\,dt + [\Omega\times(\Omega\times\mathbf{r})]\,dt \ . \end{aligned} \tag{80}$$

It is easy to see that in Newton's law for a particle of mass m there appears a force related to the non-uniform rotation ($\dot{\Omega}$), the Coriolis force $\sim \Omega\times\mathbf{v}$ and the centrifugal force $\sim\Omega^2$. The lagrangian $L = mv'^2/2 - \varphi$, where φ is a potential, leads to the hamiltonian

$$\begin{aligned} H &= mv^2/2 - m(\Omega\times\mathbf{r})^2/2 + \varphi = p^2/2m - \Omega(\mathbf{r}\times\mathbf{p}) + \varphi = \\ &= p^2/2m - \Omega\mathbf{L} + \varphi \ , \end{aligned} \tag{81}$$

where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is the angular momentum. We can see that neither the Coriolis force nor the centrifugal potential appear anymore in the hamiltonian. Instead, it contains the angular momentum. It is this hamiltonian which is subjected to quantization, so we may have quantal transitions between the states of the particle, providing these states do not conserve the angular momentum. This requires a force, as the one given by the potential φ.

The local coordinate transformation (79) leads to a distance given by

$$ds^2 = [1 + h - (\Omega\times\mathbf{r})^2/c^2]dx^{02} - 2[(\Omega\times\mathbf{r})/c]d\mathbf{r}dx^0 - d\mathbf{r}^2 \ , \tag{82}$$

where a static potential $\sim h$ is introduced as before, related to the potential φ in (81). It can be checked, through more laborious calculations, that the free motion in the curved space given by (82) is equivalent with the non-relativistic equations of motion given by (80).

A difficulty appears however in the above metric, related to the unbounded increase with $\mathbf{r}$ of the $\Omega\times\mathbf{r}$. Therefore, we drop out the square of this term in the g_{00}-term above, and keep

[22] It implies essentially the vierbeins.

[23] Einstein (1918).

[24] Though, there are difficulties in establishing a relativistically-invariant quantal theory for particles with helicity 2, like the gravitons. Another related difficulty is the general non-localizability of the gravitational energy.

only the first-order contributions in $\Omega\times\mathbf{r}$ in the subsequent calculations. As one can see, this approximation does not affect the hamiltonian (81). With this approximation, the metric given by (82) is identical with the metric given by equation (4), with the identification

$$\mathbf{g}=-(\Omega\times\mathbf{r})/c\ . \tag{83}$$

The $\Omega\times\mathbf{r}$ is exactly the rotation velocity $\mathbf{V}$, so we can apply directly the formalism developed above for a non-uniform translation to a non-uniform rotation. The only difference is that the $\mathbf{g}$ for rotations depends on the spatial coordinates too, beside its time dependence. The $\mathbf{g}$-interaction gives rise to terms of the type $\Omega\mathbf{L}$, and the evaluation of the matrix elements in the interacting terms becomes more cumbersome. It is worth keeping in mind the condition $\Omega r\ll c$ in such evaluations.

Conclusion. The quantal motion implies, basically, delocalized waves, like the plane wave, both in space and time. The general theory of relativity, gravitation or curved space as the one discussed here, arising from weak static forces and non-inertial motion, imply localized field, both in space and time. Consequently, the quantization is destroyed in those situations involved by the latter case, in the sense that quanta are scattered both in energy and the wavevector, and we have to deal there with transition amplitudes and probabilities, *i.e.* with a statistical perspective. The basic equations for the classical motion in these cases become meaningful only with scattered quanta. This shows indeed that the quantization is both necessary and illusory. The basic aspect of the natural world is its statistical character in terms of quanta.

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