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Other few remarks on the motion. Vibrations of macroscopic ensembles of particles (Lecture twelve of the Course of Theoretical Physics)

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Abstract

The emergent dynamics of the density motion is formulated for macroscopic ensembles of non-relativistic, interacting particles in terms of a displacement field. The formulation is valid both for classical and quantum-mechanical ensembles at equilibrium, as well as for ensembles at statistical equilibrium. The dynamics of the displacement field is expressed in terms of coupled harmonic oscillators, which exhibit vibrations and wave-like motion; if needed, the motion of the displacement field can be quantized. Examples are given for short-range interaction (gases, liquids, solids, or atomic nucleus) and (long-range) Coulomb interaction (plasmas). Ensembles with short-range interaction may exhibit a self-energy (oscillation eigenfrequency), which may lead to resonance. The density motion in ensembles of particles interacting through short-range forces is equivalent with the motion of the deformation of homogeneous, isotropic solids with only one Lame coefficients (dilatational motion).

Newton's law. A particle with mass m and position vector $\mathbf{r}$ moves according to Newton's law

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} \ , \tag{1}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ is the velocity and $\mathbf{F}$ denotes the force. It is sufficiently general to assume that the force is given by a potential Φ, through

$$\mathbf{F} = -grad\Phi \ . \tag{2}$$

If the potential Φ is independent of time, then, multiplying equation (1) by $\mathbf{v}$, we get the law of energy conservation

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \Phi\right) = 0 \ , \tag{3}$$

where $mv^2/2$ is the kinetic energy and Φ is the potential energy. A more general form of the Newton's law may be derived from the extremum of the mechanical action

$$S = \int dt\mathcal{L}(\mathbf{r}, \mathbf{v}) = \int dt(mv^2/2 - \Phi) \ , \tag{4}$$

where $\mathcal{L}$, defined by equation (4), is the lagrangian; the resulting Lagrange equation of motion are

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \frac{\partial\mathcal{L}}{\partial\mathbf{r}} = 0 \ ; \tag{5}$$

multiplying equation (5) by $\mathbf{v}$ we get the law of conservation of the hamiltonian (energy)

$$\frac{d}{dt}\left(\mathbf{v}\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\right)-\frac{\partial\mathcal{L}}{\partial\mathbf{v}}\frac{\partial\mathbf{v}}{\partial t}-\mathbf{v}\frac{\partial\mathcal{L}}{\partial\mathbf{r}}=\frac{d}{dt}\left(\mathbf{v}\frac{\partial\mathcal{L}}{\partial\mathbf{v}}-\mathcal{L}\right)=0\ , \tag{6}$$

where $H=\mathbf{v}\partial\mathcal{L}/\partial\mathbf{v}-\mathcal{L}$ is the hamiltonian (energy); introducing momentum $\mathbf{p}=\partial\mathcal{L}/\partial\mathbf{v}$ and using equation of motion (5) (which now reads $d\mathbf{p}/dt=\partial\mathcal{L}/\partial\mathbf{r}=\mathbf{F}$), we get the Hamilton equations of motion

$$\frac{d\mathbf{p}}{dt}=-\frac{\partial H}{\partial\mathbf{r}}\ ,\ \frac{d\mathbf{r}}{dt}=\frac{\partial H}{\partial\mathbf{p}}\ . \tag{7}$$

Moreover, from equation (4) and Hamilton equations we get $\partial S/\partial\mathbf{r}=\mathbf{p}$ and $\partial S/\partial t=dS/dt-\mathbf{v}\partial S/\partial\mathbf{r}=\mathcal{L}-\mathbf{vp}=-H$, which are the Hamilton-Jacobi equations.

The Newton's law is obtained from the general Lagrange, Hamilton, Jacobi formalism by $\mathbf{p}=m\mathbf{v}$ (and $\mathbf{F}=\partial\mathcal{L}/\partial\mathbf{r}=-\partial H/\partial\mathbf{r}$), which amounts to the kinetic term $mv^2/2$ in lagrangian (hamiltonian).

We note that energy is conserved due to the absence of the time in the expression of the force (lagrangian, hamiltonian); the energy conservation is due to the uniformity of time. Similarly, from Newton's law $d\mathbf{p}/dt=\mathbf{F}$ the momentum is conserved if the force is absent, which means the uniformity of space (Galilei's principle of inertia). By an infinitesimal rotation $\delta\boldsymbol{\varphi}$ the vectors change like $\delta\mathbf{r}=\delta\boldsymbol{\varphi}\times\mathbf{r}$; the change of the lagrangian is

$$\begin{aligned}\delta\mathcal{L}=\tfrac{\partial\mathcal{L}}{\partial\mathbf{r}}(\delta\boldsymbol{\varphi}\times\mathbf{r})+\tfrac{\partial\mathcal{L}}{\partial\mathbf{v}}(\delta\boldsymbol{\varphi}\times\mathbf{v})=\delta\boldsymbol{\varphi}\left(\mathbf{r}\times\tfrac{\partial\mathcal{L}}{\partial\mathbf{r}}+\mathbf{v}\times\tfrac{\partial\mathcal{L}}{\partial\mathbf{v}}\right)=\\ =\delta\boldsymbol{\varphi}\left(\mathbf{r}\times\dot{\mathbf{p}}+\mathbf{v}\times\mathbf{p}\right)=\delta\boldsymbol{\varphi}\tfrac{d}{dt}(\mathbf{r}\times\mathbf{p})\ ;\end{aligned} \tag{8}$$

hence, if the space is uniform under rotations the angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ is conserved; its equation of motion is

$$\frac{d\mathbf{L}}{dt}=\mathbf{r}\times\mathbf{F}\ , \tag{9}$$

where $\mathbf{K}=\mathbf{r}\times\mathbf{F}$ is the torque of the force. We can indeed check that $\partial\mathcal{L}/\partial\boldsymbol{\varphi}=\partial\mathcal{L}/\partial\mathbf{r}(\partial\mathbf{r}/\partial\boldsymbol{\varphi})=\mathbf{r}\times\mathbf{F}$. There are not other symmetries of the clasical motion of a material point, or a rigid solid, to lead to other conservation laws (constants of motion).

We may imagine that a source of motion is not only a gradient of a scalar function, but also a vortex. Force may also be given by a *curl*, $\mathbf{F}=curl\mathbf{A}$ (since any vector derive from a *grad* and a *curl*); it is natural to admit that $\mathbf{A}$ depends only on the magnitude r of the position vector. Then, we get from Newton's law $d\mathbf{p}/dt=curl\mathbf{A}$ the equation $d^2\mathbf{p}/dt^2=\mathbf{p}\times\mathbf{B}$, where $\mathbf{B}=\mathbf{A}^{'}(r)/mr$. We may assume that the direction of the vector $\mathbf{B}$ is fixed, at least locally; then $p_x=ae^{-\omega t}\cos\omega t$, $p_y=be^{-\omega t}\sin\omega t$, where $\omega=|B|^{1/2}/\sqrt{2}$; we can see that, although with an interesting trajectory, the motion ceases after a while.[1]

Many particles. Let us write Newton's law for many interacting particles as

$$m_i\frac{d\dot{x}_\alpha(i)}{dt}=-\sum_j{}^{'}\frac{\partial\Phi(ij)}{\partial x_\alpha(i)}\ , \tag{10}$$

where i, j label the particles, α is cartesian label and the prime over summation means $i\neq j$.; the notation $\Phi(ij)$ means that the potential Φ depends on the coordinates $\mathbf{x}(i)$ and $\mathbf{x}(j)$, where $i\neq j$.

[1]For vorticial motion see M. Apostol, "Fluids, fluid vortices and the theory of Electricity nad Magnetism", J. Theor. Phys. **135** (2006).

With sufficient generality we may assume for $\Phi(ij)$ a dependence on $\mid \mathbf{x}(i)-\mathbf{x}(j)\mid$. We multiply equation (10) by $\dot{x}_\alpha(i)$, sum over α and use the identity

$$\frac{d\Phi(ij)}{dt}=\frac{\partial\Phi(ij)}{\partial x_\alpha(i)}\dot{x}_\alpha(i)+\frac{\partial\Phi(ij)}{\partial x_\alpha(j)}\dot{x}_\alpha(j) \tag{11}$$

(for $i\neq j$). We get

$$\frac{d}{dt}\left(\frac{1}{2}m_i\dot{x}_\alpha^2(i)+\sum_j{}'\Phi(ij)\right)=\sum_j{}'\frac{\partial\Phi(ij)}{\partial x_\alpha(j)}\dot{x}_\alpha(j)\ , \tag{12}$$

where the summation over α is implicit. On the left in this equation we have the energy

$$E_i=\frac{1}{2}m_i\dot{x}_\alpha^2(i)+\sum_j{}'\Phi(ij) \tag{13}$$

of the i-th particle; equation (12) can also be written as

$$dE_i=\sum_j{}'\frac{\partial\Phi(ij)}{\partial x_\alpha(j)}dx_\alpha(j)\ ; \tag{14}$$

hence $\partial E_i/\partial x_\alpha(j)=\partial\Phi(ij)/\partial x_\alpha(j)$ and $\partial E_i/\partial x_\alpha(i)=0$; these two equalities are mutually contradictory, since $\partial^2E_i/\partial x_\beta(i)\partial x_\alpha(j)$ is both zero and nonzero ($\partial^2\Phi(ij)/\partial x_\beta(i)\partial x_\alpha(j)$); in fact, E_i does not exist. Hence we conclude that the motion of a collection of at least three interacting classical particles cannot be determined (in terms of trajectories of individual particles); it is not integrable (the separation of the center of mass makes one particle independent of the motion of the rest).[2] This basic limitation of the mechanical theory was first indicated by Poincare, in relation to the motion of the Earth, the Sun and the Moon;[3] subsequently it motivated the so-called "chaos theory" and the "theory of dynamical systems".[4]

Displacement field. Let us consider the lagrangian

$$\mathcal{L}=\sum_i\frac{1}{2}m_i\dot{\mathbf{x}}^2(i)-\sum_i\Phi_1(i)-\frac{1}{2}\sum_{ij}{}'\Phi_2(ij)\ , \tag{15}$$

of many interacting particles, where $\Phi_1(i)=\Phi_1(\mathbf{x}(i))$ is an external field (one-particle function) and $\Phi_2(ij)=\Phi_2(\mathbf{x}(i),\mathbf{x}(j))$ is an interaction (two-particle function; usually, we take $\Phi_2(ij)=\Phi_2(|\mathbf{x}(i)-\mathbf{x}(j)|)$; the prime over the summation sign means $i\neq j$.

We consider a macroscopic collection of classical, non-relativistic particles at rest at some points $\mathbf{x}(i)$; we assume a dense, uniform distribution of such points, such that we may pass from summations over i to integration over $\mathbf{x}$; in integrals, the position vector $\mathbf{x}$ will be denoted by $\mathbf{r}$. We assume the existence of a uniform density n of particles (concentration), which does not depend on position, nor on the time. Both the number N of particles and the volume V they occupy are macroscopic, *i.e.* $N\gg1$ and $V\gg v$, where v is the mean volume attributed to one particle

[2] M. Apostol, "The Many-Body Theory: its logic along the years", J. Theor. Phys. **152** (2007).

[3] See, for instance, M. Apostol, "Moon's problem", J. Theor. Phys. **117** (2005).

[4] The point related to the non-integrability of the equations of the mechanical motion lies in fact at the core of some arguments which claim the instability of Coulomb interacting many particles (plasma), either classical, or quantum-mechanical. Indeed, the repulsive interaction of an electron with the rest of $N-1$ electrons in an ensemble of N electrons is compensated by the attractive interaction of the same electron with N ions; it would follow that it remains one uncompensated electron-ion interaction, which would make the plasma unstable.

($v = V/N$, $nv = 1$). We introduce the external field Φ_1 and the interaction Φ_2 and are interested in the change produced by these agencies in the state of the collection of particles, especially its motion. The external field Φ_1 may depend on the time, but it is assumed that the interaction Φ_2 does not.

A similar problem can be formulated for an ensemble of interacting particles at equilibrium; although the motion is not integrable, at the initial moment of time the particles have positions; we are interested in the change produced in these positions by an external field and the interaction, especially, in the motion of the ensemble, if it exists. Specifically, we are interested in what kinds of motion the ensemble may support, if any. The interaction takes part in realizing the equilibrium, but an external field brings the interaction again in action. At equilibrium the density may, in general, be non-uniform, but these non-uniformities are either limited to small regions, or they are very small, such that we may assume, in a first approximation, a uniform density. A non-uniform density can be included in the considerations which follow.

Also, we can consider the problem described above for an ensemble of particles at thermal equilibrium. In this case, the positions and the momenta of the particles are not determined; they are subject to a statistical distribution $f(\mathbf{x}, \mathbf{p})$, where $\mathbf{x}$ is the ensemble of positions and $\mathbf{p}$ is the ensemble of momenta. The motion is described usually by the change in this distribution f, governed by kinetic equations (in particular Boltzmann equation). An external field perturbs the ensemble of particles, but, since the change in entropy is a second-order change at equibrium, there exists equilibrium, "kinetic" motion of f limited to first-order changes. However, an external perturbation brings the interaction into play, which implies second-order changes and, possibly, mechanical ("kinetic") motion; the entropy of the ensemble may increase on this occasion, in order to compensate for the decrease brought about by the ordered mechanical motion sustained by the ensemble. At a given moment, when the perturbation is introduced, the particles are at some, unknown, points $\mathbf{x}(i)$, statistically distributed.

A set of quantum-mechanical particles is described by a multi-particle wavefunction obeying the Schodinger equation. At the moment of introducing the external perturbation there exist positions where the probabilistic existence of quantum-mechanical particles hold. The approach described below is valid for quantum-mechanical particles.

Let us assume that the perturbation generates a displacement

$$\mathbf{x}(i) \to \mathbf{x}(i) + \mathbf{y}(i) \ , \tag{16}$$

where we consider small values of the changes $\mathbf{y}(i)$ in positions. We assume that the displacement field $\mathbf{y}(i)$ has two components,

$$\mathbf{y}(i) = \mathbf{u}(i) + \boldsymbol{\xi}(i) \ , \tag{17}$$

where the component $\mathbf{u}(i)$ has a slow spatial and temporal dependence, while the component $\boldsymbol{\xi}(i)$ has a fast spatial and temporal dependence; the component $\mathbf{u}$ corresponds to a "macroscopic" motion, while the component $\boldsymbol{\xi}$ corresponds to a "microscopic" motion. We are interested in the motion of the coordinates $\mathbf{u}$, the rapid $\boldsymbol{\xi}$-motion being the undetermined mechanical motion, or the statistical motion, or the quantum-mechanical motion. We shall average the microscopic motion, both spatially and temporally. The average with respect to time gives $\overline{\boldsymbol{\xi}}(i) = 0$, $\overline{\dot{\boldsymbol{\xi}}}(i) = 0$ and the vanishing of all the odd powers of $\boldsymbol{\xi}(i)$ and $\dot{\boldsymbol{\xi}}(i)$. Similar results gives the spatial average. The spatial avearge is performed by means of a coarse-graining procedure, in the sense that summations over i are understood as

$$\sum_i = \sum_{\mathbf{r},i} \ , \tag{18}$$

where $\mathbf{r}$ are some arbitray points and i in the second summation in equation (18) denotes the particles in the neighbourhood of each point $\mathbf{r}$. It is assumed that the results of the coarse-graining procedure does not depend on the choice of the points $\mathbf{r}$ and the choice of the neighborhoods of these $\mathbf{r}$-points. The $\mathbf{r}$-points are distributed with a uniform density n. According to this coarse-graining procedure and the rest of our assumptions, the macroscopic part $\mathbf{u}(i)$ of the displacement field depends only on the position $\mathbf{r}$ which defines the neighbourhood the i-th particle belongs to. We keep the notation $\mathbf{u}(i)$ but we understand everywhere the splitting summation procedure given in equation (18). It is this two-scale assumptions, both in space and time, which makes possible the separation of the macroscopic motion from the microscopic one.

Also, a two-scale decompositon is used for the external field Φ_1 and the interaction Φ_2,

$$\Phi_1 = \Phi_{1s} + \Phi_{1f} \ , \ \Phi_2 = \Phi_{2s} + \Phi_{2f} \ , \tag{19}$$

where the suffixes s and f denotes the slow and, respectively, fast components of these functions, both in space and time. The slow components correspond to macroscopic fields and interactions.

Density. The particles density can be represented as

$$\rho(\mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{x}(i)) = \frac{1}{V} \sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{x}} \sum_i e^{-i\mathbf{q}\mathbf{x}(i)} \ ; \tag{20}$$

we can see both the fast spatial dependence in passing from one δ-function to another and the slow dependence associated with the envelope of the δ-functions; we are interested in the slow dependence, which means that we may restrict to small wavevectors $\mathbf{q}$ in the Fourier decompostion given by equation (20).

There exists a useful relation between the change $\delta\rho$ in density ρ and the displacement $\mathbf{u}$, as expected, which justifies the denomination of density motion (density-fluctuations motion) for the macroscopic $\mathbf{u}$-motion; we note that there exists an important difference between the non-uniform density ρ and the uniform density n (concentration), corresponding to the Fourier component $\mathbf{q} = 0$ of ρ; in contrast with n, the non-uniform density ρ includes the density fluctuations, corresponding to the components $\mathbf{q} \neq 0$. Let us see how the density ρ changes under a displacement field $\mathbf{y}(i)$; we have

$$\begin{gathered} \widetilde{\rho}(\mathbf{x}) = \tfrac{1}{V} \textstyle\sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{x}} \sum_i e^{-i\mathbf{q}(\mathbf{x}(i)+\mathbf{u}(i)+\boldsymbol{\xi}(i))} = \\ = \tfrac{1}{V} \textstyle\sum_{\mathbf{q}} e^{i\mathbf{q}\mathbf{x}} \sum_i e^{-i\mathbf{q}\mathbf{x}(i)} \left[1 - i\mathbf{q}\mathbf{u}(i) - i\mathbf{q}\boldsymbol{\xi}(i) - \tfrac{1}{2}(\mathbf{q}\mathbf{u}(i) + \mathbf{q}\boldsymbol{\xi}(i))^2 + ...\right] \ ; \end{gathered} \tag{21}$$

we apply the time and space average in this equation; consequently, the $\boldsymbol{\xi}$-term and the mixed term which includes $\mathbf{u}(i)\boldsymbol{\xi}(i)$ disappear, while the average of the $\boldsymbol{\xi}^2$-term gives a constant; we express its contribution in terms of a constant $\boldsymbol{\xi}$; it is easy to see that we are left with

$$\widetilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) - \partial_\alpha \rho(\mathbf{x}) \cdot u_\alpha(\mathbf{x}) + \frac{1}{2} \partial_\alpha \partial_\beta \rho(\mathbf{x}) \cdot [u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) + \xi_\alpha \xi_\beta] + ... \ . \tag{22}$$

This is an expansion in powers of u_α and ξ_α, which can be obtained directly from equation (20):

$$\begin{gathered} \widetilde{\rho}(\mathbf{x}) = \textstyle\sum_i \delta(\mathbf{x} - \mathbf{x}(i) - \mathbf{u}(i) - \boldsymbol{\xi}(i)) = \rho(\mathbf{x}) - \sum_i u_\alpha(i) \partial_\alpha \delta(\mathbf{x} - \mathbf{x}(i) + \\ + \tfrac{1}{2} \textstyle\sum_i \left[u_\alpha(i) u_\beta(i) + \xi_\alpha \xi_\beta\right] \partial_\alpha \partial_\beta \delta(\mathbf{x} - \mathbf{x}(i) + ... \ , \end{gathered} \tag{23}$$

where the averages were taken; equation (23) can also be written as

$$\begin{gathered} \widetilde{\rho}(\mathbf{x}) = \rho(\mathbf{x}) - \textstyle\sum_i u_\alpha(\mathbf{x}) \partial_\alpha \delta(\mathbf{x} - \mathbf{x}(i)) + \\ + \tfrac{1}{2} \textstyle\sum_i \left[u_\alpha(\mathbf{x}) u_\beta(\mathbf{x}) + \xi_\alpha \xi_\beta\right] \partial_\alpha \partial_\beta \delta(\mathbf{x} - \mathbf{x}(i) + ... \ , \end{gathered} \tag{24}$$

which is equation (22). The change in density given by equation (24) is due to the change in position; we are interested in the change in density brought about by transport, or particles flow; this latter variation of density is given by transferring the derivatives upon $\mathbf{u}$ ($\boldsymbol{\xi}$ is a constant, so its contribution disappears); in addition, we replace $\mathbf{x}$ by $\mathbf{r}$; we get

$$\widetilde{\rho}(\mathbf{r}) = \rho(\mathbf{r}) \left[1 + div\mathbf{u} + \frac{1}{2}\partial_\alpha \partial_\beta \left(u_\alpha u_\beta \right) + ... \right] \ . \tag{25}$$

It is worth noting that the change in density is given by

$$\delta\rho(\mathbf{r}) = \rho(\mathbf{r}) - \widetilde{\rho}(\mathbf{r}) = -\rho(\mathbf{r}) \left[div\mathbf{u} + \frac{1}{2}\partial_\alpha \partial_\beta \left(u_\alpha u_\beta \right) \right] + ... \tag{26}$$

(since the displacement field decreases the density). We can see that a displacement field induces a change

$$\delta n = -ndiv\mathbf{u} - \frac{1}{2} n \partial_\alpha \partial_\beta \left(u_\alpha u_\beta \right) + ... \tag{27}$$

in the uniform density n. The first term in equation (27) can also be obtained by the following well-known procedure: let $\mathbf{r} = (x, y, z)$ and $\mathbf{r} + \Delta\mathbf{r} = (x + \Delta x, y, z)$ be two arbitrary points; for a displacement field u_x the change in particle density is

$$\delta n = \frac{n u_x(\mathbf{x}) - n u_x(\mathbf{x} + \Delta\mathbf{x})}{\Delta x} = -n \partial_x u_x \ ; \tag{28}$$

obviously, for a general displacement we have $\delta n = -ndiv\mathbf{u}$.

It is worth noting that the change in the total number of particles is given by a surface term; for a constant number of particles this term should be cancelled out by an additional condition imposed upon $\mathbf{u}$; a similar additional condition should be imposed, if we use the conservation of the mean number of particles.

Lagrangian. Let us see how the lagrangian given by equation (15) is changed if a displacement field is introduced. Since $\dot{\mathbf{x}}(i) \to \dot{\mathbf{y}}(i)$, the kinetic term in equation (15) becomes

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{y}}(i)^2 \ ; \tag{29}$$

similarly, the external field contribution to the lagrangian and the interaction lagrangian become

$$\mathcal{L}_1 = -\sum_i \Phi_1(\mathbf{x}(i) + \mathbf{y}(i)) = -\sum_i \left[\Phi_1(i) + \partial^i_\alpha \Phi_1(i) \cdot y_\alpha(i) + \frac{1}{2} \partial^i_\alpha \partial^i_\beta \Phi_1(i) \cdot y_\alpha(i) y_\beta(i) + ... \right] \tag{30}$$

and, respectively,

$$\begin{gathered} \mathcal{L}_2 = -\tfrac{1}{2} \sum\nolimits'_{ij} \Phi_2(|\mathbf{x}(i) - \mathbf{x}(j) + \mathbf{y}(i) - \mathbf{y}(j)|) = \\ = -\tfrac{1}{2} \sum\nolimits'_{ij} \{ \Phi_2(ij) + \partial^i_\alpha \Phi_2(ij) \cdot (y_\alpha(i) - y_\alpha(j)) + \\ + \tfrac{1}{2} \partial^i_\alpha \partial^i_\beta \Phi_2(ij) \cdot (y_\alpha(i) - y_\alpha(j))(y_\beta(i) - y_\beta(j)) + ... \} \ ; \end{gathered} \tag{31}$$

due to the symmetry of the function $\Phi_2(ij)$ the interaction term can also be written as

$$\begin{gathered} \mathcal{L}_2 = -\tfrac{1}{2} \sum\nolimits'_{ij} \{ \Phi_2(ij) + 2\partial^i_\alpha \Phi_2(ij) \cdot y_\alpha(i) + \\ + \partial^i_\alpha \partial^i_\beta \Phi_2(ij) \cdot (y_\alpha(i) y_\beta(i) - y_\alpha(i) y_\beta(j)) + ... \} \ , \end{gathered} \tag{32}$$

or

$$\mathcal{L}_2 = -\tfrac{1}{2}\sum_{ij}^{'}\Phi_2(ij) - \sum_i \partial_\alpha^i\left(\sum_j^{'}\Phi_2(ij)\right)\cdot y_\alpha(i) - \\ -\tfrac{1}{2}\sum_i \partial_\alpha^i\partial_\beta^i\left(\sum_j^{'}\Phi_2(ij)\right)y_\alpha(i)y_\beta(i) + \tfrac{1}{2}\sum_{ij}^{'}\partial_\alpha^i\partial_\beta^i\Phi_2(ij)\cdot y_\alpha(i)y_\beta(j) + \dots \ . \tag{33}$$

The equations of motion for $\mathbf{y}(i)$ are

$$m_i\ddot{y}_\alpha(i) + \partial_\alpha^i\partial_\beta^i\left[\Phi_1(i) + \sum_j^{'}\Phi_2(ij)\right]\cdot y_\beta(i) - \\ -\sum_j^{'}\partial_\alpha^i\partial_\beta^i\Phi_2(ij)\cdot y_\beta(j) = -\partial_\alpha^i\left[\Phi_1(i) + \sum_j^{'}\Phi_2(ij)\right] \ . \tag{34}$$

We can see that equations (34) are equations of motion of an ensemble of coupled harmonic oscillators subject to an external field. As it is well known, the system of equations (34) can be solved; the solution implies the diagonalization of the spatial coupling, which can be realized by a Fourier transform; the ensemble exhibits wave-like normal modes, which are known as phonons (or vibrations of the positions denoted by i). However, the motion remains non-integrable, since the positions i are arbitrary.

Emergent motion. We introduce the notations

$$E_\alpha(i) = -\partial_\alpha^i\left[\Phi_1(i) + \sum_j^{'}\Phi_2(ij)\right] \ , \ E_{\alpha\beta}(i) = -\partial_\alpha E_\beta(i) = \partial_\alpha^i\partial_\beta^i\left[\Phi_1(i) + \sum_j^{'}\Phi_2(ij)\right] \ , \\ I_{\alpha\beta}(ij) = \partial_\alpha^i\partial_\beta^i\Phi_2(ij) \ ; \tag{35}$$

the equations of motion (34) can be written as

$$m_i\ddot{y}_\alpha(i) + E_{\alpha\beta}(i)y_\beta(i) - \sum_j^{'}I_{\alpha\beta}(ij)y_\beta(j) = E_\alpha(i) \ ; \tag{36}$$

in this equation we replace $\mathbf{y}(i)$ by $\mathbf{u}(i)+\boldsymbol{\xi}(i)$ and use the decomposition in slow and fast components of the fields (equations (19)):

$$m_i[\ddot{u}_\alpha(i) + \ddot{\xi}_\alpha(i)] + [E_{\alpha\beta s}(i)u_\beta(i) + E_{\alpha\beta s}(i)\xi_\beta(i) + E_{\alpha\beta f}(i)u_\beta(i) + E_{\alpha\beta f}(i)\xi_\beta(i)] - \\ -\sum_j^{'}[I_{\alpha\beta s}(ij)u_\beta(j) + I_{\alpha\beta s}(ij)\xi_\beta(j) + I_{\alpha\beta f}(ij)u_\beta(j) + I_{\alpha\beta f}(ij)\xi_\beta(j)] = E_{\alpha s}(i) + E_{\alpha f}(i) \ ; \tag{37}$$

by using the average procedure the mixed terms of the type (s,ξ) and (f,u) are vanishing; at the same time, we drop the term $I_{\alpha\beta f}(ij)\xi_\beta(j)$ because we assume that the fast microscopic motions of the i-th and j-th particles are uncorrelated. The term $E_{\alpha\beta f}(i)\xi_\beta(i)$ is small, such that the fast motion is determined by the equation

$$m_i\ddot{\xi}_\alpha(i) = E_{\alpha f}(i) \ ; \tag{38}$$

the solution is of the type $\xi_\alpha(i) = -E_{\alpha f}(i)/m_i\omega^2$, where ω is a characteristic frequency of the fast motion; the term $E_{\alpha\beta f}(i)\xi_\beta(i)$ becomes

$$E_{\alpha\beta f}(i)\xi_\beta(i) = -\partial_\alpha E_{\beta f}(i)\frac{E_{\beta f}(i)}{m_i\omega^2} = -\partial_\alpha\frac{E_{\beta f}^2(i)}{2m_i\omega^2} \ ; \tag{39}$$

taking the time average we get

$$\frac{\overline{E_{\beta f}^2}(i)}{2m_i\omega^2} = \frac{1}{2}m_i\overline{\dot{\boldsymbol{\xi}}^2}(i) \ , \tag{40}$$

i.e. the time-averaged kinetic term

$$T_f(i) = \frac{1}{2} m_i \overline{\dot{\boldsymbol{\xi}}^2}(i) \ ; \tag{41}$$

it acts like an external field $\partial_\alpha T_f$ in the macroscopic motion. Equally well, we may use $\left[\overline{E^2_{\beta f}}(i)\right]^{1/2}$ for this field.[5] Usually, the spatial average of T_f does not depend on the spatial coordinates, such that the corresponding force is vanishing. It may happen that the fast component of the external field is absent; then the fast motion is determined by equation $m_i \ddot{\xi}_\alpha + E_{\alpha\beta f}(i)\xi_\beta(i) = 0$, with corresponding average procedure; in this case the fast motion does not affect in any way the **u**-motion.

After applying the average procedure and decoupling the microscopic motion we are left with the "macroscopic" equations of motion

$$m_i \ddot{u}_\alpha(i) + E_{\alpha\beta s}(i) u_\beta(i) - \sum_j{}' I_{\alpha\beta s}(ij) u_\beta(j) = E_{\alpha s}(i) + \partial_\alpha T_f(i) \ ; \tag{42}$$

we may replace $\mathbf{x}(i)$ in this equation by $\mathbf{r}$, give up the suffix s (understanding that we are left only with macroscopic fields) and replace the summation over j by an integral (we note that the restriction $i \neq j$ is not effective anymore); also, the mass m_i may be replaced by its spatial average m; we get

$$m\ddot{u}_\alpha(\mathbf{r}) + E_{\alpha\beta}(\mathbf{r}) u_\beta(\mathbf{r}) - n \int d\mathbf{r}' I_{\alpha\beta}(\left|\mathbf{r} - \mathbf{r}'\right|) u_\beta(\mathbf{r}') = E_\alpha(\mathbf{r}) + \partial_\alpha T_f(\mathbf{r}) \ , \tag{43}$$

where

$$\begin{aligned} E_\alpha(\mathbf{r}) &= -\partial_\alpha \left[\Phi_1(\mathbf{r}) + n \textstyle\int d\mathbf{r}' \Phi_2(\left|\mathbf{r} - \mathbf{r}'\right|)\right] \ , \\ E_{\alpha\beta}(\mathbf{r}) &= \partial_\alpha \partial_\beta \left[\Phi_1(\mathbf{r}) + n \textstyle\int d\mathbf{r}' \Phi_2(\left|\mathbf{r} - \mathbf{r}'\right|)\right] \ , \\ I_{\alpha\beta}(\left|\mathbf{r} - \mathbf{r}'\right|) &= \partial_\alpha \partial_\beta \Phi_2(\left|\mathbf{r} - \mathbf{r}'\right|) \ . \end{aligned} \tag{44}$$

Since the coordinates $\mathbf{x}(i)$ are eliminated through integration, the equations of motion (43) are integrable. They are equations of motion for a field, the displacement field **u**. These equations define an emergent dynamics,[6] associated with the motion of the density fluctuations. Since the field **u** depends on both the time and position, the total derivative $\dot{\mathbf{u}} = d\mathbf{u}/dt$ may be replaced by the partial derivative $\partial\mathbf{u}/\partial t$; we note that the field **u** is defined at the position $\mathbf{x}(i)$, which, according to our hypotheses, is fixed.

Effective lagrangian. Let us apply the average procedure to the lagrangian given by equations (29), (30) and (33). The kinetic energy (equation (29)) reads

$$T = \sum_i \frac{1}{2} m_i (\dot{\mathbf{u}}^2(i) + 2\dot{\mathbf{u}}\dot{\boldsymbol{\xi}} + \dot{\boldsymbol{\xi}}^2(i)) \ ; \tag{45}$$

by the time average the mixed $\dot{\mathbf{u}}\dot{\boldsymbol{\xi}}$-term disapears.

In the remaining part $\mathcal{L}_1 + \mathcal{L}_2$ of the lagrangian we neglect the mixed terms arising from slow-fast combinations with an odd number of fast contributions, including the lack of correlations between

[5] See, for instance, M. Apostol and L. C. Cune, "Molecular dynamics in high electric fields", Chem. Phys. **472** 262-269 (2016).

[6] P. W. Anderson, "More is different", Science **177** 393-396 (1972).

the partners of the (ij)-pairs; we are left with

$$\mathcal{L}_1+\mathcal{L}_2=\sum_i E_{\alpha s}(i)u_\alpha(i)+\sum_i E_{\alpha f}(i)\xi_\alpha(i)-$$

$$-\tfrac{1}{2}\sum_i[E_{\alpha\beta s}(i)u_\alpha(i)u_\beta(i)+E_{\alpha\beta s}(i)\xi_\alpha(i)\xi_\beta(i)+E_{\alpha\beta f}(i)u_\alpha(i)\xi_\beta(i)+E_{\alpha\beta f}(i)u_\beta(i)\xi_\alpha(i)]+ \tag{46}$$

$$+\tfrac{1}{2}\sum_{ij}^{'}[I_{\alpha\beta s}(ij)u_\alpha(i)u_\beta(j)+I_{\alpha\beta f}(ij)\xi_\alpha(i)\xi_\beta(j)]\ ;$$

we take the variation of the lagrangian with respect to the coordinates $u_\alpha(i)$ and $\xi_\alpha(i)$, and consider the leading contributions to the equations of motion for the coordinates $\xi_\alpha(i)$; making use of equation (45) these equations of motion is equation (38),

$$m_i\ddot{\xi}_\alpha(i)=E_{\alpha f}(i)\ ; \tag{47}$$

we solve this equation and introduce the solution in the lagrangian; we can see that the terms quadratic in ξ_α are small in comparison with the rest of the terms, and may be neglected (in the equations of motion these terms disappear by averaging); in addition, the $\dot{\boldsymbol{\xi}}^2$-kinetic term and the term $\sum_i E_{\alpha f}(i)\xi_\alpha(i)$ give the lagrangian for the ξ-motion, and they can be left aside; the terms $E_{\alpha\beta f}(i)u_\alpha(i)\xi_\beta(i)$ and $E_{\alpha\beta f}(i)u_\beta(i)\xi_\alpha(i)$ give the slow contribution of the term T_f in equation (42); finally, we get

$$\mathcal{L}=\sum_i\tfrac{1}{2}m_i\dot{\mathbf{u}}^2(i)-\tfrac{1}{2}\sum_i E_{\alpha\beta s}(i)u_\alpha(i)u_\beta(i)+\tfrac{1}{2}\sum_{ij}^{'}I_{\alpha\beta s}(ij)u_\alpha(i)u_\beta(j)+$$
$$+\sum_i E_{\alpha s}(i)u_\alpha(i)+\sum_i\partial_\alpha T_f(i)u_\alpha(i)\ , \tag{48}$$

which leads to the equations of motion (42). By spatial averaging and passing to integration, the macroscopic lagrangian is

$$\mathcal{L}=n\int d\mathbf{r}\tfrac{1}{2}m\dot{\mathbf{u}}^2(\mathbf{r})-\tfrac{1}{2}n\int d\mathbf{r}E_{\alpha\beta}(\mathbf{r})u_\alpha(\mathbf{r})u_\beta(\mathbf{r})+\tfrac{1}{2}n^2\int d\mathbf{r}d\mathbf{r}^{'}I_{\alpha\beta}(\left|\mathbf{r}-\mathbf{r}^{'}\right|)u_\alpha(\mathbf{r})u_\beta(\mathbf{r}^{'})+$$
$$+n\int d\mathbf{r}E_\alpha(\mathbf{r})u_\alpha(\mathbf{r})+n\int d\mathbf{r}\partial_\alpha T_f(\mathbf{r})u_\alpha(\mathbf{r})\ , \tag{49}$$

which leads to the equations of motion (43).

We note that the decoupling of the macroscopic motion from the microscopic one is accomplished as a consequence of the recognition of the two distinct space-time scales. If this distinction is not sharp, then mixed terms are present in dynamics; such a typical term is the mixed kinetic term $\sum_i m_i\dot{\mathbf{u}}\dot{\boldsymbol{\xi}}$, which leads to an uncertainty in energy of the order $\sqrt{\overline{\varepsilon}_u\overline{\varepsilon}_\xi}$, where $\overline{\varepsilon}_{u,\xi}$ are the mean energies per particle, corresponding to the u- and, respectively the ξ-motion. This uncertainty in energy leads to dissipation of energy and relaxation.[7]

Also, it is worth noting that we have assumed that the displacement u is much smaller than the wavelengths; if a particle has a long excursion $u(i)$, its place is taken by another identical particle, and we may attribute the displacement to this process, such that the displacement is indeed small; similarly, small displacements can be be viewed as long ones, by transferring each of them to neighbouring, identical particles. Non-inertial motion may give rise to a "large" displacement field, leading to electromagnetic emission and gyroelectric and gyromagnetic effects.[8] According

[7]See, for instance, M. Apostol, "Dynamics of collective density modes in multi-component molecular mixtures", Phys. Chem. Liquids **47** 35-44 (2009).

[8]M. Apostol, "Non-inertial electromagnetic effects in matter. Gyromagnetic efffect", Solid State Commun. **152** 1567-1571 (2012).

to this picture we may say that the formalism of the **u**-motion described here corresponds to the so-called Euler equations of motion for fluids.

Equilibrium. It may happen that the switching on of the external field and the interaction induced by the ensuing change in positions destroy the equilibrium of the ensemble of particles; it may happen that the ensemble is driven to a new state of equilibrium, or, simply, the motion collapses. For instance, the second term on the right in E_α in equations (44) generates a field which may destroy the equilibrium. We can look for a new state of equilibrium by introducing $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1$ and solve for $\mathbf{u}_0$ in the new equilibrium state. Internal fields generated by interaction disappear on performing such a redefinition of the equilibrium state. Consequently, we disregard such terms in the equations of motion, which become

$$\begin{aligned} m\ddot{u}_\alpha(\mathbf{r}) + n\partial_\alpha\partial_\beta \int d\mathbf{r}'\Phi_2(\left|\mathbf{r}-\mathbf{r}'\right|)\cdot u_\beta(\mathbf{r}) - n\partial_\alpha\partial_\beta \int d\mathbf{r}'\Phi_2(\left|\mathbf{r}-\mathbf{r}'\right|)u_\beta(\mathbf{r}') = \\ = -\partial_\alpha\Phi_1(\mathbf{r}) - \partial_\alpha\partial_\beta\Phi_1(\mathbf{r})\cdot u_\beta(\mathbf{r}) \ , \end{aligned} \tag{50}$$

where we omit the T_f-term. The second term on the left in this equation is a self-energy (self-interaction) term, while the third term on the left is an interaction term. The first term on the right in equation (50) is a typical external-field interaction term, while the second term on the right can be viewed as a ponderomotive force due to the (slow) spatial variation of the external field.

Equations (50) may satisfy some necessary conditions of equilibrium, but they may not necessarily satisfy sufficient conditions of equilibrium. The general criterion of equilibrium (stability) is a minimum of the energy of the ensemble. For instance, for short-range interaction it is necessary to have a positive self-energy, which implies a repulsive short-range interaction; however, the equilibrium requires also an attractive overall force, which should be provided by external forces, *e.g.*, acting as boundary conditions. For (long-range) Coulomb interaction the repulsive self-energy of the ensemble is infinite; it is easy to see that a neutralizing, rigid background of electric charges provides exactly a self-energy term with opposite sign, such that the self-energy term should be omitted in equations (50); formally, such a term is of the same form as the second term on the right in equations (50) (*i.e.*, an "external" field provided by the internal background). Consequently, for Colomb interacting charges the equations of motion read

$$m\ddot{u}_\alpha(\mathbf{r}) - n\partial_\alpha\partial_\beta \int d\mathbf{r}'\Phi_2(\left|\mathbf{r}-\mathbf{r}'\right|)u_\beta(\mathbf{r}') = -\partial_\alpha\Phi_1(\mathbf{r}) - \partial_\alpha\partial_\beta\Phi_1(\mathbf{r})\cdot u_\beta(\mathbf{r}) \ . \tag{51}$$

Also, we note that the equations of motion (50) and (51) remain valid for a non-uniform density n, which should be introduced under the integral sign in the interaction terms.

Energy conservation. If we multiply the equations of motion (50) by $\dot{u}_\alpha$, sum over α and make use of the symmetry of the tensors involved in these equations, we get the energy conservation

$$\begin{aligned} \tfrac{d}{dt}\left[\tfrac{1}{2}m\dot{\mathbf{u}}^2 + \tfrac{1}{2}n\partial_\alpha\partial_\beta\left(\int d\mathbf{r}'\Phi_2\right)\cdot u_\alpha u_\beta\right] - n\partial_\alpha\partial_\beta\int d\mathbf{r}'\Phi_2 u'_\beta\cdot\dot{u}_\alpha = \\ = -\partial_\alpha\Phi_1\cdot\dot{u}_\alpha - \tfrac{1}{2}\partial_\alpha\partial_\beta\Phi_1\tfrac{d}{dt}\left(u_\alpha u_\beta\right) \ , \end{aligned} \tag{52}$$

where in square brackets we have the energy density and in the term on the right we have the density of mechanical work done by external forces per unit time; the second term on the left is a mechanical work done by interaction (the interaction is assumed independent of time). We note that the mechanical work done by the ponderomotive forces generated by the external field is an energy density. Integrating over $\mathbf{r}$ we get the conservation of the total energy.

Boundary conditions. We can see from equations (50) that the internal force which acts upon the u-field is

$$F_\alpha(\mathbf{r}) = -n\partial_\alpha\partial_\beta\left(\int d\mathbf{r}^{'}\Phi_2\right)\cdot u_\beta + n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2 u_\beta^{'} \ , \tag{53}$$

where $u_\beta^{'}$ stands for $u_\beta(\mathbf{r}^{'})$. For a short-range interaction $\Phi_2(\mid \mathbf{r}-\mathbf{r}^{'} \mid) = v\Phi_2\delta(\mathbf{r}-\mathbf{r}^{'})$, where Φ_2 is a constant and v is the volume associated with one particle ($nv = 1$), the self-energy term in equation (53) disappears, and we are left with the equation of motion

$$m\ddot{\mathbf{u}} - \Phi_2 grad\cdot div\mathbf{u} = \mathbf{E} \tag{54}$$

(where we neglect the ponderomotive force) and the internal force

$$F_\alpha(\mathbf{r}) = \Phi_2\partial_\alpha\partial_\beta u_\beta \ . \tag{55}$$

The total force acting upon the volume V is

$$\int_V d\mathbf{r}F_\alpha = \Phi_2\oint_S dSs_\beta(\partial_\alpha u_\beta)\mid_S \ , \tag{56}$$

where S is the surface which bounds the volume V and s_β are the components of the normal $\mathbf{s}$ to the surface S. It is convenient to introduce the curvilinear cordinates s along the normal to the surface S (denoted also by z or x_3, for instance) and t (for instance, x, y or $x_1\,x_2$), parallel to the surface S (lying in the plane paralel with the plane tangent to S in any point). Then, we can see that the contributions arising from $\alpha = t$ in equation (56) are zero, and we are left with

$$\int_V d\mathbf{r}F_t = 0 \ , \ \int_V d\mathbf{r}F_s = \Phi_2\oint_S dS(\partial_s u_s)\mid_S \ ; \tag{57}$$

it follows that

$$n\Phi_2 s_\beta(\partial_s u_\beta)\mid_S = n\Phi_2(\partial_s u_s)\mid_S = -P_s \tag{58}$$

is the force acting (normal, inwards) upon the unit area of the surface; equation (58) serves as boundary conditions for the u-motion with short-range forces. We can note that equations (54) with the boundary conditions given by equation (58) define the deformation motion of a homogeneous, isotropic solid with only one Lame coefficient (dilatational coefficient λ with the notations of Elasticity). The motion is governed by a potential Φ, such that $\mathbf{u} = grad\Phi$. The surface force P_s in equation (58) is the pressure.

For (long-range) Coulomb interaction we consider equations of motion (51) and point charges q interacting through the Coulomb potential $\Phi_2 = q^2/\mid \mathbf{r}-\mathbf{r}^{'} \mid$; we get

$$n^2 s_\beta\left(\partial_s\int d\mathbf{r}^{'}\Phi_2 u_\beta^{'}\right)\mid_S = n^2\left(\partial_s\int d\mathbf{r}^{'}\Phi_2 u_s^{'}\right)\mid_S = -P_s \tag{59}$$

for the surface force; if the displacement is given by a potential Φ, $\mathbf{u} = grad\Phi$, effecting the surface integral in the total force, we get

$$-4\pi n^2 q^2\Phi\mid_s = -P_s \tag{60}$$

for the boundary conditions. For a uniform displacement the surface force is vanishing for a short-range interaction (equation (58), the surface is free) and non-vanishing for a Coulomb interaction (in both cases the number of particles is conserved).

It is worth noting that a non-uniform density changes the boundary conditions given above.

Short-range interaction. Let us consider an ensemble of particles at equilibrium, interacting by short-range forces and subject to an external field; the equations of motion for the displacement field are

$$m\ddot{u}_\alpha + n\partial_\alpha\partial_\beta \int d\mathbf{r}^{'}\Phi_2 \cdot u_\beta - n\partial_\alpha\partial_\beta \int d\mathbf{r}^{'}\Phi_2 u_\beta^{'} = -\partial_\alpha\Phi_1 \ , \tag{61}$$

where we omit the ponderomotive force $-\partial_\alpha\partial_\beta\Phi_1 \cdot u_\beta$ arising from the external field (equations (50)). For a short-range interaction $\Phi_2(\mid \mathbf{r} - \mathbf{r}^{'} \mid) = v\Phi_2\delta(\mathbf{r} - \mathbf{r}^{'})$, where Φ_2 is a constant, the equations of motion (61) become

$$m\ddot{u}_\alpha - \Phi_2\partial_\alpha div\mathbf{u} = -\partial_\alpha\Phi_1 \tag{62}$$

($vn = 1$). We can see that the self-energy term disappears (it does not for a non-uniform density). Taking the div in equation (62) we get the wave equation

$$m\frac{\partial^2}{\partial t^2}\delta n - \Phi_2\Delta\delta n = n\Delta\Phi_1 = -ndiv\mathbf{E} \ , \tag{63}$$

for the density, with $c = \sqrt{\Phi_2/m}$ the wave velocity ($\Phi_2 > 0$). It describes the "kinetic" sound (or "anomalous" sound, distinct from the hydrodinamic, equilibrium sound) in interacting condensed matter.[9]

This result is valid as long as the mean inter-particle separation a goes to zero ($a \to 0$); for a small, but finite a, there exist additional contributions to the equations of motion; for instance, the second term in equation (61) is of the order $na\Phi_2 u_\alpha$. Let us estimate these contributions for an ensemble of particles confined to a sphere of radius R. With the origin in the centre of the sphere we get

$$J = \int d\mathbf{r}^{'}\Phi_2 = 2\pi\int_0^R dr^{'}r^{'2}\int_{-1}^1 du\Phi_2(\sqrt{r^2 + r^{'2} - 2rr^{'}u}) \ ; \tag{64}$$

making use of the substitution $\rho = \sqrt{r^2 + r^{'2} - 2rr^{'}u}$ this integral becomes

$$J = \frac{2\pi}{r}\int_0^R dr^{'}r^{'}\int_{|r-r^{'}|}^{r+r^{'}} d\rho\cdot\rho\Phi_2(\rho) = \frac{2\pi}{r}\int_{-a}^a dx(r+x)\int_{|x|}^{2r+x} d\rho\cdot\rho\Phi_2(\rho) \ , \tag{65}$$

where $r^{'} = r + x$; the leading contribution to this integral is

$$J = 4\pi aF(2r) \ , \tag{66}$$

where F is the primitive function of $\rho\Phi_2(\rho)$; the second term in equation (61) becomes

$$n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2 \cdot u_\beta = n\partial_\alpha\partial_\beta J \cdot u_\beta \simeq 16\pi na\Phi_2(2r)u_\alpha \ . \tag{67}$$

We can see that this term yields an eigenfrequency for the u-oscillators located in the core of the ensemble; for $r < a/2$ it reduces to a particle. For a short-range (repulsive) interaction this eigenfrequency, denoted by ω_c, is given approximately by

$$\omega_c^2 \simeq \frac{16\pi na\Phi_2}{m} \ . \tag{68}$$

[9]Called "densiton" waves in M. Apostol, "Dynamics of collective density modes in multi-component molecular mixtures", Phys. Chem. Liquids **47** 35-44 (2009); they prevail in solids, as phonons, are hindered in gases by the hydrodynamic sound and are inter-mixed with the hydrodynamic sound in liquids. It is worth noting that the displacement in the hydrodynamic sound is small (much smaller than the mean inter-particle separation a), while the displacement in the u-motion described here is larger than a (but smaller than the wavelegths). For small displacement the interaction is replaced by the compressibility coefficient.

It is worth noting that the estimation of the integral J in equation (66) holds for any space region the ensemble is confined to, not only for a sphere.

The third term in equations (61) can be written as

$$-n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2 u_\beta^{'}=n\partial_\alpha\int d\mathbf{r}^{'}\partial_\beta^{'}\Phi_2 u_\beta^{'}=n\partial_\alpha\oint_S d\mathbf{S}^{'}(\Phi_2\mathbf{u}^{'})-n\partial_\alpha\int d\mathbf{r}^{'}\Phi_2 div^{'}\mathbf{u}\ ; \tag{69}$$

we can check that for a short-range interaction the surface term is vanishing (for any $\mathbf{r}$ inside the sphere), while the second term on the right in equation (69) is $-\Phi_2\partial_\alpha div\mathbf{u}$, which appears in equation (62). On the other hand, it is easy to see that the surface term is of the form $\partial_\alpha f(r)$, where $f(r)$ is a function of r resulted from the surface integration; its magnitude is of the order $a^2\Phi_2 u_R$, where u_R is the mean radial displacement on the surface; we get

$$n\partial_\alpha\oint_S d\mathbf{S}^{'}(\Phi_2\mathbf{u}^{'})\simeq na\Phi_2 u_R\frac{X_\alpha}{R} \tag{70}$$

for the surface term, which may be neglected in comparison with the external force.

Taking into account all these results we get the equations of motion

$$m\ddot{u}_\alpha+m\omega_c^2 u_\alpha-\Phi_2\partial_\alpha div\mathbf{u}=-\partial_\alpha\Phi_1\ \ , \tag{71}$$

or

$$\ddot{\mathbf{u}}+\omega_c^2\mathbf{u}-c^2 grad\cdot div\mathbf{u}=\mathbf{E}\ ; \tag{72}$$

the ω_c-term is valid only for the core $r<a/2$; for the rest of the ensemble $\omega_c=0$; we may take an average eigenfrequency ω_c extended to the whole ensemble.

If the external field $\mathbf{E}=\mathbf{E}_0e^{-i\omega t}$ does not depend on position (for instance, an electric field in the dipole approximation) we get $\mathbf{u}=-\mathbf{E}_0e^{-i\omega t}/(\omega^2-\omega_c^2)$ (a similar solution is obtained if a spatial dependence would be present). We can see that there exists a resonance in the response of the ensemble as a whole, which may be called a "giant" resonance. We note that for a uniform displacement the total internal force in equation (72) is $\int d\mathbf{r}m\omega_c^2\mathbf{u}\sim a^4$, which may be neglected (it is taken over by the rest of the ensemble and compensated by boundary forces).

The above results can be applied to the atomic nucleus.[10] For the mean separation distance we take $a=10^{-13}cm$ $(1fm)$, the nucleon mass is $m\simeq 10^{-24}g$ and for the interaction energy Φ_2 we take the mean cohesion energy per nucleon, $\Phi_2\simeq 10MeV$; we get $\hbar\omega_c\simeq 10MeV$ (where $\hbar$ is Planck's constant).

Coulomb interaction. For ensembles of electric charges at equlibrium, interacting through (long-range) Coulomb forces, the self-energy is absent in the equations of motion, which read (equations (51))

$$m\ddot{u}_\alpha-n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2 u_\beta^{'}=-\partial_\alpha\Phi_1(\mathbf{r}) \tag{73}$$

(we leave aside the ponderomotive term). However, it is worth noting that a short-range interaction may be present, as, for instance, the molecular electric field acting upon electrons in the Drude model of polarizable matter,[11] which may generate a self-energy.

The interaction term

$$E_\alpha^{mf}(\mathbf{r})=n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2(\left|\mathbf{r}-\mathbf{r}^{'}\right|)u_\beta(\mathbf{r}^{'}) \tag{74}$$

[10]M. Apostol and M. Ganciu, "Coupling of (ultra-) relativistic atomic nuclei with photons", AIP Advances **3** 112133 (2013).

[11]M.Apostol, *Essays in Electromagnetism and Matter*, Lambert (2013).

is a mean field; it may be determined self-consistently from the equations of motion. For Coulomb forces the mean field is determined from the equations of the electric field. Indeed, for an infinite ensemble, where we may leave aside the surface term, we can write

$$E_\alpha^{mf}(\mathbf{r}) = n\partial_\alpha \int d\mathbf{r}^{'}\partial_\beta \Phi_2(\left|\mathbf{r}-\mathbf{r}^{'}\right|)u_\beta(\mathbf{r}^{'}) = -n\partial_\alpha \int d\mathbf{r}^{'}\partial'_\beta \Phi_2(\left|\mathbf{r}-\mathbf{r}^{'}\right|)u_\beta(\mathbf{r}^{'}) = \\ = -\partial_\alpha \int d\mathbf{r}^{'}\Phi_2(\left|\mathbf{r}-\mathbf{r}^{'}\right|)\delta n(\mathbf{r}^{'}) \ ; \tag{75}$$

for an ensemble of point charges q interacting by a Coulomb potential (plasma) we get easily

$$div\mathbf{E}^{mf}(\mathbf{r}) = -\int d\mathbf{r}^{'}\Delta_{\mathbf{r}}\Phi_2(\left|\mathbf{r}-\mathbf{r}^{'}\right|)\delta n(\mathbf{r}^{'}) = 4\pi q^2\delta n(\mathbf{r}) = -4\pi n q^2 div\mathbf{u}(\mathbf{r}) \ , \tag{76}$$

whence

$$\mathbf{E}^{mf}(\mathbf{r}) = -4\pi n q^2\mathbf{u}(\mathbf{r}) \ ; \tag{77}$$

we can see that the interaction is "solved" for such an ensemble, and the equations of motion (73) become

$$m\ddot{\mathbf{u}} + 4\pi n q^2\mathbf{u} = \mathbf{E} \ , \tag{78}$$

where $\omega_p = (4\pi nq^2/m)^{1/2}$ is the plasma frequency (as it is well known, the resonance with the frequency ω_p is called plasmon).

For a finite-size ensemble the surface term of the interaction should be included. For a uniform external field it is sufficient to expand the displacement $\mathbf{u}$ and the Coulomb interaction $q^2/\left|\mathbf{r}-\mathbf{r}^{'}\right|$ in series of Legendre polynomials $P_l(\cos\theta)$, where θ is the angle between $\mathbf{r}$ and $\mathbf{r}^{'}$; apart from normal modes, we can see that only the component $l=0$ (u) is driven by the external field, with $E^{mf} = -(4\pi nq^2/3)u$;[12] it follows that the plasma eigenfrequency for a sphere is $\omega_p = (4\pi nq^2/3)^{1/2}$ ("spherical" plasmon).[13] Since such modes occur for finite-size ensembles they are also called sometime "surface" plasmons, though damped modes are properly called surface plasmons. We note that the surface term disappears also for a uniform displacement of a plasma confined to an infinite slab, in which case the plasma frequency is $\omega_p = (4\pi nq^2/m)^{1/2}$; since the displacement is uniform, we can view it as being restricted to the surfaces of the slab; then, the mean field can be viewed as a (de-) polarizing field; in the static case, it compensates the external field (as in conductors).

A non-uniform density occurs in principle in Coulomb interacting ensembles through the screening length. In a classical plasma the density is Boltzmann distributed, $n \sim e^{-\beta q\varphi}$, where $\beta = 1/T$ is the inverse of the temperature T, q is particle charge and φ is the potential; there exists a change $\delta n = -n\beta q\varphi$ in density, such that the Poisson equation for the potential is $\Delta\varphi = -4\pi q\delta(r) + 4\pi n\beta q^2\varphi$; with solution $\varphi = (q/r)e^{-\kappa r}$, where $\kappa = \sqrt{4\pi n\beta q^2}$; κ^{-1} is the Debye screening length. In normal conditions, a classical plasma has a pressure $p = 10^6 dyn/cm^2$, a temperature $T = 300K$ and a mean inter-particle separation $a = (T/p)^{1/3} \simeq 35$Å; for electrons $q = 4.8\times 10^{-10} esu$; the Debye screening length is $\kappa^{-1} \simeq 10^{-1}a$ $(q^2/a = 7\times 10^{-13}erg,\ T = 4\times 10^{-14}erg)$; it is too small to affect appreciably the density. For a degenerate plasma (cold, quantum-mechanical, higly-correlated plasma), like electrons in solids, the screening length is of the order of mean inter-particle separation,[14] which again justifies a quasi-uniform density.

[12] M. Apostol and G. Vaman, "Plasmons and diffraction of an electromagnetic plane wave by a metallic sphere", Progr. Electrom. Res. PIER **98** 97-118 (2009).

[13] It is also worth noting that the self-energy $n\partial_\alpha\partial_\beta\int d\mathbf{r}^{'}\Phi_2\cdot u_\beta$ for the Coulomb interaction in a sphere is $-(4\pi nq^2/3)u_\alpha$.

[14] L. C. Cune and M. Apostol, "Ground-state energy and geometric magic numbers for homo-atomic metallic clusters", Phys. Lett. **A273** 117-124 (2000).

Granular matter. Granular matter consists of microscopic grains (micro-domains), physically assembled, which, usually, carry an electric charge (the ensemble being electrically neutral); this is known as the Maxwell-Wagner-Sillars effect.[15] Let us consider a uniform distribution of such identical domains, with total density n, each with mass M and charge $\pm Q$. Under the action of a external, uniform electric field $\mathbf{E}$ the equations of motion for the displacement $\mathbf{u}_{\pm}$ corresponding to the two species of domains (equations (43), (73)) are

$$\begin{aligned} M\ddot{u}_{+\alpha} - \tfrac{n}{2}\int d\mathbf{r}^{'} I_{\alpha\beta}(\left|\mathbf{r}-\mathbf{r}^{'}\right|)[u_{+\beta}(\mathbf{r}^{'}) - u_{-\beta}(\mathbf{r}^{'})] = E_{\alpha}\ , \\ M\ddot{u}_{-\alpha} - \tfrac{n}{2}\int d\mathbf{r}^{'} I_{\alpha\beta}(\left|\mathbf{r}-\mathbf{r}^{'}\right|)[u_{-\beta}(\mathbf{r}^{'}) - u_{+\beta}(\mathbf{r}^{'})] = -E_{\alpha}\ ; \end{aligned} \tag{79}$$

the relative motion of any pair is described by

$$M\Delta\ddot{u}_{\alpha} - n\int d\mathbf{r}^{'} I_{\alpha\beta}(\left|\mathbf{r}-\mathbf{r}^{'}\right|)\Delta u_{\beta}(\mathbf{r}^{'}) = 2E_{\alpha}\ , \tag{80}$$

where $\Delta\mathbf{u} = \mathbf{u}_{+} - \mathbf{u}_{-}$, while the center of mass is free; the interaction is Coulombian. The mean field defined by

$$E_{\alpha}^{mf} = n\int d\mathbf{r}^{'} I_{\alpha\beta}(\left|\mathbf{r}-\mathbf{r}^{'}\right|)\Delta u_{\beta}(\mathbf{r}^{'}) \tag{81}$$

is $\mathbf{E}^{mf} = -4\pi n Q^2\Delta\mathbf{u}$ (equation (77)); we can see that it generates a plasma eigenfrequency

$$\omega_p = \left(\frac{4\pi n Q^2}{M}\right)^{1/2}\ . \tag{82}$$

We give here an estimation of this frequency. The grain density is $n = 1/R^3$, where R is the linear dimension of the grains. The number of atoms in a grain is $N = (R/a)^3$, where a is the linear dimension of an atom; we take $10^5 m$ for atomic mass, where $m = 10^{-27}g$ is the electron mass, such that the mass of a grain is $M = 10^5(R/a)^3 m$. The charge Q is determined from the condition of electrostatic equilibrium $Qq/R = \varepsilon = q^2/a$, where q is the electron charge ($q = 4.8\times 10^{-10} esu$); ε is the cohesion energy of an electron, which is approximately of the order q^2/a; we get $Q = q(R/a)$ and $\omega_p = (10^{-5}q^2 a/m)^{1/2}R^{-2}$. For $a = 3$Å and $R = 1\mu m$ ($10^{-4}cm$) we get $\omega_p \simeq 10^{11}s^{-1}$. More probably, the grains are larger and the charge Q is smaller, such that their plasma freqency may lie in the radio range.

An additional short-range interaction exists between the grains, like, for instance, a dipolar interaction; other interaction may be induced by the external field; consequently, the equilibrium may be modified under the action of an external field, and the plasma frequency may be modified, temporarily; it is known that relaxation phenomena are important in such matter.

In this context it is also wortwhile estimating the dipolar electromagnetic emission of a sample of granular matter. It is well known that the dipolar intensity (energy per unit time) is $I = 2d^2\omega^4/3c^3$, where $d = Q\Delta u$ is the electric dipole; since $M\omega^2\Delta u = -QE$, where E is the external electric field, we get $I \simeq Q^4E^2/M^2c^3$; taking $\mathcal{E}/V$ for E^2, where $\mathcal{E}$ is the external energy, we get the total emitted intensity $I \simeq (Q^4 n/M^2c^3\omega)I_{ex}$, where I_{ex} is the external, input intensity. Making use of the numerical data given above ($Q \simeq 10^{-6} esu$, $M \simeq 10^{-11}g$, $n \simeq 10^{11}cm^{-3}$, $\omega = \omega_p$), we get $I \simeq 10^{-33}I_{ex}$, which is an extremely small output, as expected, because the external energy is ineffective in the large grains volume.

[15] See, for instance, M. Apostol, S. Ilie, A. Petrut, M. Savu and S. Toba, "Induced displacive trasition in heterogeneous materials", Eur. Phys. J. Appl. Phys. **59** 10401 (2012).

Surface terms. As it is well known, the derivatives in the lagrangian and the equations of motion mean that the equations of motion are valid inside the domain occupied by the ensemble; on the other hand, the boundary conditions imply the limit of the functions and their derivatives on the surface; hence, the functions should be continuous and differentiable. It follows that a distinction can be made between volume and surface terms. Indeed, by using the Gauss's theorem in equations (50) we get

$$\begin{aligned} m\ddot{u}_\alpha - n\partial_\alpha \oint_S dS\Phi_2(|\mathbf{r}-\mathbf{R}|)\cdot u_s(\mathbf{r}) + n\partial_\alpha \oint dS\Phi_2(|\mathbf{r}-\mathbf{R}|)u_s(\mathbf{R}) - \\ -n\partial_\alpha \int_V d\mathbf{r}'\Phi_2(\left|\mathbf{r}-\mathbf{r}'\right|)div\mathbf{u}(\mathbf{r}') = E_\alpha \ , \end{aligned} \tag{83}$$

where u_s is the component normal to the surface S, $\mathbf{R}$ is a position vector on the surface S and V is the volume of the ensemble. We can see that the self-energy term is in fact a surface term;[16] the term which includes $div\mathbf{u}$ is a volume term. For a uniform displacement the surface terms in equations (83) cancel out (for short-range interaction). For Coulomb interaction the self-energy is absent and for a uniform displacement we are left only with the surface term. For a sphere $u_s = u\cos\theta$ for a uniform displacement u and the evaluation of the surface term in equation (83) gives $(4\pi nq^2/3)u$, as expected. It is worth noting that we may view the restriction of the displacement $\mathbf{u}$ to the domain V as $u\theta(s_0 - s)$, where θ is the step function, s is the coordinate along the normal to the surface and s_0 is the value of this coordinate on the surface; then, we may extend the integral over the volume V in equations (83) to the whole space by introducing the factor $\theta(s_0 - s')$; this factor can be absorbed in $div\mathbf{u}$, by using the identity

$$div\mathbf{u}\cdot\theta(s_0 - s') = div[\mathbf{u}'\theta(s_0 - s')] + u_{s'}\delta(s' - s_0) \ ; \tag{84}$$

we can see that the surface interaction term disappears in equations (83), which now read

$$m\ddot{u}_\alpha - n\partial_\alpha \oint_S dS\Phi_2(|\mathbf{r}-\mathbf{R}|)\cdot u_s(\mathbf{r}) - n\partial_\alpha \int d\mathbf{r}'\Phi_2(\left|\mathbf{r}-\mathbf{r}'\right|)div[\mathbf{u}'\theta(s_0 - s')] = E_\alpha \ . \tag{85}$$

where the volume integral is extended to the whole space; a factor $\theta(s_0 - s)$ may multiply the rest of the terms in equation (85), such that we solve this equation in the whole space and the restriction of the solution to the domain V is the solution of our problem. It is also worth noting that in Gauss's equation $div\mathbf{E} = 4\pi\rho$, where ρ is a charge density, the restriction to a finite domain implies $div[\mathbf{E}\theta(s_0 - s)] = div\mathbf{E}\cdot\theta(s_0 - s) - E_s\delta(s - s_0)$, which indicates the presence of (half) a surface charge density $E_s/4\pi$ (corresponding volume charge $E_s\delta(s - s_0)/4\pi$).

For short-range interaction a non-uniform density may appear near the surface,[17] in which case a self-energy is present in the equations of motion, which has the appearrance of surface contributions.

Fluids. There exists an important generalization of the description given here to the density motion, which arises from contemplating fluids. Let us consider small amounts of matter, sufficiently large to be viewed as macroscopic matter but, at the same time, sufficiently small to be viewed as particles. Being macroscopic, such particles are free of fast motions, in particular brownian motion, statistical or quantum-mechanical motion. Their dynamics is governed by equations (50)

[16] In the derivation made above of the eigenfrequency ω_c it was assumed that the short-range interaction is different from zero only inside a small domain of linear dimension of the order a and the self-energy was estimated on the surface of this small domain.

[17] For Coulomb interaction a double layer appears at the surface, which ensures the stability; it implies a specific surface density contribution (See, for instance, L. C Cune and M. Apostol, "Metallic clusters deposited on surface", J. Theor. Phys. **77** (2002) and L. C Cune and M. Apostol, "Metallic clusters and nanostructures", J. Theor. Phys. **78** (2002)).

or (51) (they can be electrically charged), with some important modifications. First, the positions denoted by i in deriving the equations of motion are not fixed anymore, since these particles may flow now. Consequently, the transiton from the total time derivative to the partial one is not warranted anymore, and we have to return to the total time derivative in the equations of motion; it is convenient to use the velocity in this case, instead of the displacement. Second, the density of these particles is, in general, non-uniform, and it may change; the fluid may be compressible; it is convenient to use a non-uniform mass density $\rho = mn$ instead of the concentration n. Let us consider that these particels interact by short-range forces given by the interaction $\Phi_2 = v\Phi_2\delta(\mathbf{r}-\mathbf{r}^{'})$, where v and Φ_2 are constants Then, the equations of motion (50) become

$$m\left(\frac{\partial v_\alpha}{\partial t} + (\mathbf{v}grad)v_\alpha\right) + v\Phi_2(\partial_\alpha\partial_\beta n)u_\beta - v\Phi_2\partial_\alpha\partial_\beta(nu_\beta) = E_\alpha \ , \tag{86}$$

where $\mathbf{v}$ is velocity, or

$$m\left(\frac{\partial v_\alpha}{\partial t} + (\mathbf{v}grad)v_\alpha\right) - v\Phi_2\partial_\beta n\cdot\partial_\alpha u_\beta - v\Phi_2\partial_\alpha(ndiv\mathbf{u}) = E_\alpha \ ; \tag{87}$$

we may consider here a constant, non-uniform displacement $\mathbf{u}^0$, such that the total displacement is $\mathbf{u}+\mathbf{u}^0$; we get

$$\begin{aligned} m\left(\tfrac{\partial v_\alpha}{\partial t} + (\mathbf{v}grad)v_\alpha\right) - v\Phi_2\partial_\beta n\cdot\partial_\alpha u_\beta - v\Phi_2\partial_\beta n\cdot\partial_\alpha u^0_\beta - v\Phi_2\partial_\alpha(ndiv\mathbf{u}) = \\ = v\Phi_2\partial_\alpha(ndiv\mathbf{u}^0) + E_\alpha \ ; \end{aligned} \tag{88}$$

we note that $-\Phi_2(ndiv\mathbf{u}^0) = p^0$ is pressure; introducing the mass density we get

$$\begin{aligned} \rho\left(\tfrac{\partial v_\alpha}{\partial t} + (\mathbf{v}grad)v_\alpha\right) - \tfrac{nv\Phi_2}{m}\partial_\beta\rho\cdot\partial_\alpha u_\beta - \tfrac{nv\Phi_2}{m}\partial_\beta\rho\cdot\partial_\alpha u^0_\beta - \tfrac{nv\Phi_2}{m}\partial_\alpha(\rho div\mathbf{u}) = \\ = -nv\partial_\alpha p + nE_\alpha \ , \end{aligned} \tag{89}$$

where p may include the internal pressure (actually, the variations of the internal pressure) beside the external pressure p_0. The existence of the internal pressure is an important feature of the condensed matter at statistical equilibrium. Since the fluid is viewed as being continuous (both at macroscopic and microscopic scales), we may take $vn = 1$, m the molecular mass and Φ_2 the interaction at the microscopic scale. Equation (89) becomes

$$\begin{aligned} \tfrac{\partial v_\alpha}{\partial t} + (\mathbf{v}grad)v_\alpha - \tfrac{\Phi_2}{\rho m}\partial_\beta\rho\cdot\partial_\alpha u_\beta - \tfrac{\Phi_2}{\rho m}\partial_\beta\rho\cdot\partial_\alpha u^0_\beta - \tfrac{\Phi_2}{\rho m}\partial_\alpha(\rho div\mathbf{u}) = \\ = -\tfrac{1}{\rho}\partial_\alpha p + f_\alpha \ , \end{aligned} \tag{90}$$

where $\mathbf{f}$ is force per unit mass. The continuity equation

$$\frac{\partial\rho}{\partial t} + div\rho\mathbf{v} = 0 \tag{91}$$

should be added (mass conservation).

Equations (90) and (91) describe the density motion of a fluid, *i.e.* the motion of the displacement field $\mathbf{u}$ (where $\mathbf{v} = d\mathbf{u}/dt$); it is worth noting that we should give the initial displacement $\mathbf{u}_0$ if an external pressure is present. If the internal pressure is viewed as an unknown, then the necessary supplementary equation is provided by the adiabatic motion of the ideal fluid, which reads $ds/dt = 0$, *i.e.* $\partial s/\partial t + \mathbf{v}grads = 0$, where s is the entropy per unit mass (or $\partial(\rho s)/\partial t + div(\rho s\mathbf{v}) = 0$,

making use of the continuity equation); if there exist friction or thermoconduction, the entropy is not conserved (and the ensemble is not at equilibrium). The density motion is, in general, a non-equilibrium motion; however, the change it may produce in entropy is small, since, on one hand, it affects a small amount of degrees of freedom ($\mathbf{uq} \ll 1$) and, on the other hand, it is produced by small external forces. An important simplification is obtained for a uniform density in the equilibrium state (which entails a vanishing $\mathbf{u}^0$ for ideal fluids), where equations (90) and (91) become

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}grad)\mathbf{v} - \frac{\Phi_2}{m} grad \cdot div\mathbf{u} = -grad(p/\rho) + \mathbf{f} \tag{92}$$

and $div\mathbf{v} = 0$; for small velocities we may neglect the term $(\mathbf{v}grad)\mathbf{v}$. For small displacements the interaction may be neglected and we get the well-known Euler equations for the fluid flow

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}grad)\mathbf{v} = -\frac{1}{\rho} gradp + \mathbf{f} \ , \tag{93}$$

where p may include an external pressure. For the (adiabatic) change in pressure we have $\delta p = (\partial p/\partial V)_S \delta V$ and $\delta V/V = -\delta n/n = div\mathbf{u}$; we get

$$\ddot{\mathbf{u}} - [-V(\partial p/\partial V)_S/\rho] grad \cdot div\mathbf{u} = \mathbf{f} \ , \tag{94}$$

which describe the sound. The Euler equations are equilibrium equations.

We can use the identity $\mathbf{v} \times curl\mathbf{v} = grad(v^2/2) - (\mathbf{v}grad)\mathbf{v}$ in Euler's equation, which becomes

$$\frac{\partial \mathbf{v}}{\partial t} + grad(v^2/2) - \mathbf{v} \times curl\mathbf{v} = -\frac{1}{\rho} gradp + \mathbf{f} \tag{95}$$

and

$$\frac{\partial}{\partial t}(curl\mathbf{v}) - curl(\mathbf{v} \times curl\mathbf{v}) = curl\mathbf{f} \tag{96}$$

(for an incompressible fluid); the fluid exhibits both a potential (irrotational, gradient) and a vorticial (solenoidal) flow. Equation (96) shows that the vorticity $\oint \mathbf{v}d\mathbf{l}$ equals the circulation of the force $\mathbf{f}$; for $\mathbf{f} = -grad\Phi_1$ the vorticity is conserved, $curl\mathbf{v} = 0$ and we may put $v = grad\psi$, where ψ is a scalar (velocity) potential; in that case, the flow is potential. Also, for small velocity we may consider the flow potential; then, from equation (95) we get

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}v^2 + \frac{1}{\rho}p + \Phi_1 = 0 \ , \tag{97}$$

which is a first integral of Euler's equation (for incompressible, irrotational flow). Equations (96) and (97) are similar with the Helmholtz decompositon for the motion of the elastic deformations. Similarly, under the same conditions, from equation (92) we get

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}v^2 - \frac{\Phi_2}{m} div\mathbf{u} + \frac{1}{\rho}p + \Phi_1 = 0 \ , \tag{98}$$

where $c = \sqrt{\Phi_2/m}$ is the wave velocity.

Concluding remarks. The motion of the density is considered in this paper by means of the displacement field. The disentanglement of the macroscopic density motion from the motion at the microscopic level is achieved by identifying two scales of slow and, respectively, fast motion. The equations of motion and the effective lagrangian are derived for short-range and (long-range) Colomb interactions, corresponding to gases, liquids, solids, atomic nuclei and, respectively, plasmas. The relevant boundary conditions are derived for this motion. For short-range interaction the density motion is the motion of the deformation of elastic, homogeneous, isotropic solids with only one Lame coefficient (dilatational motion). The density motion may imply diplacements

larger than the mean inter-particle separation, in contrast with the displacement in the hydrodynamic sound, which is limited to displacements smaller than the mean inter-particle separation. The latter provides equlibrium motion, with constant entropy, while the former addresses also the non-equilibrium motion. Also, the density motion implies the particle interaction, while the sound implies the compressibility coefficient.

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