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On the Landau damping and the penetration depth in a plasma model (Lecture thirteenth of the Course of Theoretical Physics)

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The behaviour of a model of classical plasma, relevant for (quite fully) ionized gases and, with (important) modifications, for elementary quasi-electron excitations in metals and charge carriers in semiconductors, is analyzed with respect to its response to an external electric field. The model, derived from the ideal classical gas, is characterized by a universal mean freepath, which depends only on the temperature. The Debye length, mean freepath, collision frequency and the plasma frequency are introduced. The motion of the plasma under the action of an external uniform oscillating electric field is described by the equation of motion and the dielectric function. The "collisionless" regime is characterized. The propagation of an electromagnetic plasma is reviewed and the extinction theorem is discussed.

The Boltzmann equation is derived by taking into account the effect of the statistical motion in plasma on the equation of motion of a macroscopic displacement field. A comparison is made with solids and fluids, where the statistical motion is neglected. As it is well known, the motion in solids and fluids is ascribed to macroscopic, but small, regions of matter, a circumstance which is not available in plasma, where such regions are electrically neutral.

First, the response of the plasma to an initial perturbation of the distribution function is computed. In this context the notion of emergent dynamics is emphasized, which consists in the occurrence of a damped response of the macroscopic plasma as a consequence of the causal dynamics of the individual particles, subject to the thermal motion. This is the well-known Landau damping of the collective plasma eigenmode. Second, the response of a semi-infinite plasma is derived. It is shown that both the Landau damping and the penetration depth (attenuation coefficient, extinction length) have the same origin, which resides in the thermal transport effect. The (non-uniform) surface electric field generated by plasma as a reaction to the external electric field is computed. The role of the initial and boundary conditions is emphasized in solving the Boltzmann equation in plasma.

Introduction. We consider a gaseous plasma, which consists of point-like charges $-q$ (*e.g.*, electrons) with mass m, moving (quasi-) freely in a (quasi-) rigid uniform background of massive charges q (*e.g.*, ions). We assume that the gas is governed by the equation of state of the ideal classical gas $p = nT$, where p is the pressure, n is the density (concentration) and T is the temperature. In normal conditions ($p = 1bar = 10^5 N/m^2$, $T = 300K$) the density is $n = 2.5\times10^{19}cm^{-3}$, which means a mean inter-particle separation $a \simeq 34$Å. The gas is indeed classical, since $\hbar^2/ma^2 \ll T$, $\hbar$ being the Planck's constant; or $\hbar/m\sqrt{T/m} \ll a$, where $v_{th} = \sqrt{T/m}$ is the thermal velocity.

Actually, the neutralizing background consists of point ions with charge q, placed at positions $\mathbf{r}_i$; the Coulomb potential in plasma is

$$\Phi(r) = \sum_i \frac{q}{\mid \mathbf{r} - \mathbf{r}_i \mid} - \int d\mathbf{r} \frac{qn(\mathbf{r}')}{\mid \mathbf{r} - \mathbf{r}' \mid} \ ,$$

where i labels the ions; we can see that for a quasi-uniform distribution of ions, with the same density as the electrons, the potential is vanishing, which justifies the asssumption of (quasi-) free motion for the electrons.

The conditions of normal temperature and densities are highly unrealistic conditions. Usually, a gas is ionized to a very small extent (due to weak ionizing agents, where the electron-ion recombination plays an important role), such that the electron density in usual plasmas is of the order $10^{10}cm^{-3}$ and the electron temperature is much higher than the room temperature (for instance, 10^4K); the pressure is then much lower, and the mean inter-particle separation is much larger. The electron temperature may be much higher than the ion temperature (then plasma is said to be "non-thermal"). Since the electron temperature is high, the electron gas may be in equlibrium; since the atoms (including ions) density is high, the atoms (with the ions) are in equilibrium; since the electron density is low in comparison with atoms density, and the electrons are much lighter than the atoms, the electrons are not, usually, in equilibrium with the atoms. We may view the electrons decoupled from the atoms and ions, except for the electron-ion electromagnetic interaction; however, since the ion mass is much larger than the electron mass, the effects of this interaction are weak. Consequently, the assumption of a rigid uniform background may be viewed as justified. However, a "free" classical plasma is considered to be unstable; when ionized, one may expect the gas to get a small cohesion energy, its density to increase and its temperature to decrease; in fact, the mobile charges restore the equilibrium of the neutral gas, and the plasma, it is argued, does not exist, except for a continuously acting external energetic ionizing agent (high electric fields, high temperature, etc).[1]-[4] Nevertheless, we consider this state of transient, quasi-equlibrium of the gaseous plasma, with a rigid uniform background and, for the sake of the numerical illustrations, we consider normal conditions of temperature and pressure.

There exist a few scales of energy of this plasma. First, there is the internal Coulomb energy $q^2/a \simeq 7 \times 10^{-13}erg \simeq 5 \times 10^3K$ per particle (electron charge $q = 4.8 \times 10^{-10}esu$, $1K = 1.38 \times 10^{-16}erg$); second, there is the temperature $T = 300K = 4 \times 10^{-14}erg$; third, suppose we apply an external electric field $E_0 = 10^3V/m \simeq 3 \times 10^{-2}esu$ (which is a high field among the common fields achieved in the laboratory); it gives an electric energy $\mathcal{E}_{el} = (1/8\pi)E_0^2a^3 \simeq 10^{-24}erg$ ($\simeq 10^{-8}K$) per particle; we have $\mathcal{E}_{el} \ll T \ll q^2/a$. In normal conditions the mobile charges have statistical positions $\mathbf{r}$ and velocities $\mathbf{v}$, governed by the Maxwell distribution; the thermal velocity is of the order $v_{th} = \sqrt{T/m} \simeq 6 \times 10^6cm/s$ (electron mass $m \simeq 10^{-27}g$), which is sufficiently low to enable us to neglect the magnetic field and magnetic energy; in addition, the contribution of the magnetic moments is small, such that we may view the plasma as unmagnetized.

Such a model of classical plasma is relevant for (quite fully) ionized gases. However, it may give valuable insights into solid-state plasmas of electrons in metals and the charge carriers in semiconductors, with some (important) modifications. Electrons in metals are quantum-mechanical; they are governed by the Fermi distribution; only the quasi-electron excitations are active in transport, which, being rare, have some features in common with a classical gas; the Landau theory of the normal Fermi liquid is the most convenient tool for treating the electrons in metals.[5] The charge carriers in semiconductors have another important particularities: their density is low and variable, their effective mass is low and they are governed by Maxwell distribution.

The mobile charges move in a small self-consistent potential φ arising from the mutual Coulomb repulsion and the background Coulomb attraction; the Maxwell distribution $\sim e^{q\varphi/T}$ brings a small contribution $nq^2\varphi/T$ to the charge imbalance, which determines the Debye screening length $\lambda_D = \sqrt{T/4\pi nq^2} = a\sqrt{Ta/4\pi q^2}$ in the Poisson equation of the potential; at room temperature it is of the order $\lambda_D \simeq a/14 \simeq 2.5$Å. We may view the mobile charges as "neutral" particles with a radius of the order λ_D. For high temperatures and low densities the Debye length may be large, and many particles may be included in the region defined by this length; however, the velocity of these particles is high, such that they spends litle time in the Debye region; we can see that the collision time is the important parameter, which should be compared with the time scale of the motion in plasma. On the other hand, the displacement caused by external fields is, usually, very small, such that neither the plasma equilibrium is affected by this displacement, nor the displacement is affected by collisions.

The background of the statistical motion, as expressed by the thermal velocities $\mathbf{v}$, brings a characteristic feature to the behaviour of this plasma, related to the particle collisions, their mean freepath, relaxation time and collision frequency. We can see easily that the mean freepath of the classical gas of mobile charges is of the order $\Lambda = a(a/d)^2$, where $d \simeq 2\lambda_D$ is the geometrical diameter of the particles; for $d \simeq 5$Å and $a \simeq 34$Å we get $\Lambda \simeq 1.6 \times 10^3$Å $= 1.6 \times 10^{-5}$cm; it corresponds to a collision frequency $v_{th}/\Lambda \simeq 4 \times 10^{11}s^{-1}$. Similar numerical estimations can be made for metals and semiconductors; in typical metals at room temperature $\Lambda \simeq 10^3 - 10^4$Å, in typical semiconductors at room temperature $\Lambda \simeq 10^2$Å.[6] It is worth noting that using the Debye length $\lambda_D = a\sqrt{Ta/4\pi q^2}$ as the radius of classical particles, we get a mean freepath $\Lambda = \pi q^2/T$, which is independent of the nature of the plasma. The plasma model introduced here exhibits a universal meanfree path, which depends only on the temperature (and the particle charge). Interaction is always present in plasma, in order to ensure the equilibrium. If the plasma density is low, and the Debye length is larger than the mean-inter-particle separation a (which is the usual case for most of the plasmas) the mean freepath is of the order of a and collision frequency is $v_{th}/a \simeq 2 \times 10^{13}s^{-1}$. We shall see below that these collision frequencies are not relevant for plasma, due to the long-range interaction: this interaction produces small displacements of the charges, extended over large distances (collective motion), which makes plasma to be usually in the collisionless regime. The long-range character of the Coulomb forces in plasma makes the "collision frequency" be zero (practically, the particles in plasma do not collide, though the equilibrium is achieved by the long-range interaction).

Another special feature of the behavior of the plasma is its eigenfrequency (plasma frequency). Let us assume that the mobile charges move entirely, as a compact body, in a rectangular box, by a small distance u along a certain direction perpendicular to the surface; the surface charge $-q/a^2$ suffers a change $(q/a^3)u$ (along one direction) and generates an electric field $2\pi(q/a^3)u$ and a force $-2\pi(q^2/a^3)u$ acting upon each particle; the two surfaces generate a force $-4\pi(q^2/a^3)u$; the equation of motion $m\ddot{u} = -4\pi(q^2/a^3)u$ leads to an eigenfrequency $\omega_0 = \sqrt{4\pi nq^2/m}$, which is the plasma frequency; for $n = 2.5 \times 10^{19}cm^{-3}$, $q = 4.8 \times 10^{-10}esu$ and $m = 10^{-27}g$ (electrons) it is of the order $\omega_0 = 2.6 \times 10^{14}s^{-1}$ ($\nu \simeq 4 \times 10^{13}s^{-1}$); this frequency is in the infrared range (for metals this frequency is higher, for semiconductors it is lower). The plasma frequency, generated by the internal field, is associated with any disturbance occurring in plasma. It is due to the long-range character of the Coulomb force (and the related point-like character of the mobile charges). It is worth noting that the surfaces bounding the plasma may be extended to infinity, and the plasma frequency thus estimated here is a bulk property.

Uniform electric field. Let us assume a uniform, oscillating electric field $\mathbf{E}_0(t) = \mathbf{E}_0 \cos\omega t$, acting in plasma, where ω denotes the field frequency. The field may arise from a transverse electromagnetic field with the wavelength c/ω much larger than the size d of the plasma sample

($\omega d/c \ll 1$), where c is the speed of light in vacuum; or, it may be a longitudinal field generated inside a capacitor. The electric field $\mathbf{E}_0(t)$ produces a small displacement $\mathbf{u}(t)$ of the mobile charges, superposed upon the statistical motion; this displacement generates a total change $(-2q/a^3)\mathbf{u}$ in the surface charge density, an internal (reaction) electric field $\mathbf{E} = (-4\pi q/a^3)\mathbf{u}$ and an internal reaction force $q\mathbf{E}$; it is more convenient here to work with a charge q assigned to each particle (electron); the equation of motion reads

$$m\ddot{\mathbf{u}} = q\mathbf{E}_0(t) + q\mathbf{E} = q\mathbf{E}_0(t) - 4\pi nq^2\mathbf{u} \ . \tag{1}$$

This is the equation of motion of a harmonic oscillator with frequency ω_0, acted by a periodical force; it may exhibit the resonance phenomenon, when $\omega \simeq \omega_0$; then, the displacement may be affected by collisions; on the other hand, if the frequency is very small the motion has the character of a quasi-uniform drift motion, which implies $\ddot{\mathbf{u}} \simeq \dot{\mathbf{u}}/\tau$, where τ is a relaxation time and $\gamma = 1/\tau$ is the collision frequency; a current density $\mathbf{j} = nq\dot{\mathbf{u}}$ occurs then, where $\mathbf{j} = (nq^2/m\gamma)\mathbf{E}_t$, where $\mathbf{E}_t = \mathbf{E}_0(t) + \mathbf{E}$ is the total field; it follows that $\sigma = nq^2/m\gamma$ is the drift electric conductivity (static conductivity). Therefore, a damping (friction) term should be included in equation (1), which reads now

$$\ddot{\mathbf{u}} + \omega_0^2\mathbf{u} + \gamma\dot{\mathbf{u}} = \frac{q}{m}\mathbf{E}_0(t) \ , \tag{2}$$

where $\gamma \ll \omega_0$ ($\gamma > 0$). The damping coefficient should be small, otherwise the motion is lost, being dominated by collisions. It is convenient to write $\mathbf{E}_0(t) = \mathbf{E}_0 e^{-i\omega t}$and take the real part. Equation (2) has a transient part of the solution (governed by γ), arising from the solution of the homogeneous equation (which serves also to the fulfilment of the initial conditions), and a particular (stationary) solution. The particular solution is

$$\mathbf{u} = -\frac{q\mathbf{E}_0}{m}\frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma} \ ; \tag{3}$$

the internal field is

$$\mathbf{E} = \frac{\omega_0^2}{\omega^2 - \omega_0^2 + i\omega\gamma}\mathbf{E}_0 \tag{4}$$

and the total field is

$$\mathbf{E}_t = \mathbf{E}_0 + \mathbf{E} = \frac{\omega^2 + i\omega\gamma}{\omega^2 - \omega_0^2 + i\omega\gamma}\mathbf{E}_0 \ . \tag{5}$$

From equation (2) we get the rate of energy dissipation $dQ/dt = mn\gamma\dot{\mathbf{u}}^2$ per unit volume; its time average is

$$\overline{\frac{dQ}{dt}} = \gamma\frac{E_0^2}{8\pi}\frac{\omega_0^2\omega^2}{(\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2} \ . \tag{6}$$

A displacement u (along a direction) may change the density $n = 1/a^3$ into $1/a^2(a+u) \simeq -u/a^4 = -n(u/a)$; the change can be viewed as $\delta n = -n div\mathbf{u}$; the Gauss equation becomes $div\mathbf{E} = -4\pi nq div\mathbf{u} = -4\pi div\mathbf{P}$, where $\mathbf{P} = nq\mathbf{u} = -\mathbf{E}/4\pi$ is the polarization; and $\mathbf{E}_0+\mathbf{E}+4\pi\mathbf{P} = \mathbf{E}_t + 4\pi\mathbf{P} = \mathbf{D} = \mathbf{E}_0$ is the electric displacement (induction); χ in $\mathbf{P} = \chi\mathbf{E}_t$ is the electric susceptibility and $\varepsilon = 1 + 4\pi\chi$ in $\mathbf{D} = \varepsilon\mathbf{E}_t$ is the dielectric function. We get from the above equations

$$\varepsilon = 1 - \frac{\omega_0^2}{\omega^2 + i\omega\gamma} \ . \tag{7}$$

The Faraday equation $curl\mathbf{E}_0 = -(1/c)\partial\mathbf{H}_0/\partial t$ is satisfied approximately, according to $\omega d/c \ll 1$, and its remaining part $curl\mathbf{E} = 0$ is satisfied by the uniform field $\mathbf{E}$. Similarly, the Maxwell-Ampere equation $curl\mathbf{H}_0 = (1/c)\partial\mathbf{E}_0/\partial t$ is satisfied approximately, while the remaining part is

$(1/c)\partial\mathbf{E}/\partial t + (4\pi/c)\mathbf{j} = 0$; it follows

$$\mathbf{j} = \frac{i\omega}{4\pi}\mathbf{E} = \frac{i\omega_0^2}{4\pi}\frac{1}{\omega + i\gamma}\mathbf{E}_t \tag{8}$$

and the electric conductivity

$$\sigma = \frac{i\omega_0^2}{4\pi}\frac{1}{\omega + i\gamma} \ ; \tag{9}$$

we may check the static conductivity $\omega_0^2/4\pi\gamma$ (for $\omega \to 0$); in addition, we have

$$\varepsilon = 1 + \frac{4\pi i\sigma}{\omega} \ . \tag{10}$$

In general, from Maxwell equations $\varepsilon div\mathbf{E} = 0$, $div\mathbf{H} = 0$, $curl\mathbf{E} = (i\omega/c)\mathbf{H}$ and $curl\mathbf{H} = -(i\omega\varepsilon/c)\mathbf{E}$, we get

$$\Delta\mathbf{E} + \tfrac{\varepsilon\omega^2}{c^2}\mathbf{E} = 0 \ , \ \Delta\mathbf{H} + \tfrac{\varepsilon\omega^2}{c^2}\mathbf{H} = 0 \tag{11}$$

(in the absence of the magnetization), which show that the velocity of the electromagnetic waves is modified in the plasma (in matter) to $c \to c/\sqrt{\varepsilon}$; similarly, the wavelength is modified to $\lambda = c/\omega \to \lambda/\sqrt{\varepsilon}$. Making use of the dielectric function derived above, we can see that the electromagnetic waves do not penetrate the plasma (do not propagate) for frequencies $\omega < \omega_0$; for higher frequencies near ω_0 the wavelength is large, and the condition of uniform field is satisfied for large samples. Making use of equation (10), we can see that the damping of the electromagnetic fields in plasma (matter) are governed by the electric conductivity (skin effect). We note that equations (11) are practically identically zero for uniform fields.

The collisions dominate the motion for low frequencies, or at resonance. The damping factor should be small, otherwise the motion is quickly dissipated. However, a non-uniform motion is practically disentangled from collisions, provided $u \ll \Lambda$. This is the so-called condition of a "collisionless" plasma; it is a condition of self-consistency. The greatest displacement, of the order $u \simeq qE_0/m\omega_0\gamma$, is achieved for $\omega \simeq \omega_0$; making use of the formula $\Lambda = \pi q^2/T$, we get $\gamma \gg v_{th}E_0/\pi q\omega_0$, or, with the numerical data used here, $\gamma \gg 15E_0$; this is a reasonable condition for usual electric fields. We can see that the mechanical motion is practically independent of the thermal motion. First-order variations leave the entropy unchanged at equilibrium, because the entropy at equilibrium is maximal. We may call this motion "kinetic" motion, since it is accomodated by the thermal motion and it accomodates the thermal motion. The damping coefficient is small, but not vanishing.

It is worth noting that the above formulae are valid for an infinite plasma (with bounding surfaces at infinity). The zeros of the dielectric functions, which make the displacement infinite (at resonance) give the frequencies of the plasmonic eigenmodes. For a finite plasma the eigenfrequency ω_0 is modified, and damped surface plasmonic eigenmodes may appear.

The equation of motion of the displacement of the mobile charges in plasma is the basic element of the Drude-Lorentz model of matter polarization.[7]-[9]

If we include the motion of the ions, then we have $-nqdiv\mathbf{u}$ for the charge variation of the electrons (with mass m, charge q and displacement $\mathbf{u}$) and $nqdiv\mathbf{v}$ for the charge variation of the ions (mass M, charge $-q$ and displacement $\mathbf{v}$); the internal field is $\mathbf{E} = -4\pi nq(\mathbf{u} - \mathbf{v})$ and the equations of motion read $\ddot{\mathbf{u}} + \omega_e^2(\mathbf{u} - \mathbf{v}) = q\mathbf{E}_0/m$ and $\ddot{\mathbf{v}} - \omega_i^2(\mathbf{u} - \mathbf{v}) = -q\mathbf{E}_0/M$, where $\omega_e^2 = 4\pi nq^2/m$ and $\omega_i^2 = 4\pi nq^2/M$; we can see that the relative motion is governed by $\omega_\mu^2 = 4\pi nq^2/\mu$, where μ given by $1/\mu = 1/m + 1/M$ is the reduced mass, as expected; since $m \ll M$ we may neglect the ions motion ($\mu \simeq m$).

Non-uniform transverse field (radiation field). Let $\mathbf{E}_0(\mathbf{r},t)$ be a transverse electric field, with the magnetic field $\mathbf{H}_0(\mathbf{r},t)$ (radiation field); these fields are proportional to $e^{-i\omega t+i\mathbf{kr}}$, where $\omega = ck$ is the frequency and $\mathbf{k}$ is the wavevector ($\mathbf{kE}_0 = \mathbf{kH}_0 = 0$). These fields satisfy the Maxwell equations $div\mathbf{E}_0 = 0$, $div\mathbf{H}_0 = 0$, $curl\mathbf{E}_0 = -(1/c)\partial\mathbf{H}_0/\partial t$, $curl\mathbf{H}_0 = (1/c)\partial\mathbf{E}_0/\partial t$. In plasma, the electric field generates a transverse displacement $\mathbf{u}$, $div\mathbf{u} = 0$ (the displacement determined by the magnetic field is too small); consequently, the polarization charge is zero, but there exists a polarization current $\mathbf{j} = nq\dot{\mathbf{u}}$, $div\mathbf{j} = 0$; an internal magnetic field $\mathbf{H}$ and an internal electric field $\mathbf{E}$ appear, obeying the Maxwell equations

$$\begin{gathered} div\mathbf{E} = 0 \ , \ \ div\mathbf{H} = 0 \ , \\ curl\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \ , \ \ curl\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}nq\dot{\mathbf{u}} \ ; \end{gathered} \tag{12}$$

the equation of motion is

$$\ddot{\mathbf{u}} + \gamma\dot{\mathbf{u}} = \frac{q}{m}(\mathbf{E}_0 + \mathbf{E}) \ . \tag{13}$$

From these equations we get immediately

$$\begin{gathered} \mathbf{u} = -\frac{q\mathbf{E}_0}{m}\frac{\omega^2 - c^2k^2}{\omega^2(\omega^2-\omega_0^2-c^2k^2)+i\omega\gamma(\omega^2-c^2k^2)} \ , \\ \mathbf{E} = \frac{\omega_0^2\omega^2}{\omega^2(\omega^2-\omega_0^2-c^2k^2)+i\omega\gamma(\omega^2-c^2k^2)}\mathbf{E}_0 \ , \\ \mathbf{E}_t = \mathbf{E} + \mathbf{E} = \frac{(\omega^2+i\omega\gamma)(\omega^2-c^2k^2)}{\omega^2(\omega^2-\omega_0^2-c^2k^2)+i\omega\gamma(\omega^2-c^2k^2)}\mathbf{E}_0 \ ; \end{gathered} \tag{14}$$

we can see that $\mathbf{u} = 0$, $\mathbf{E} = -\mathbf{E}_0$and $\mathbf{E}_t = 0$, *i.e.* the plasma (matter) reacts to the electromagnetic wave such that it annihilates it; in an infinite plasma (in infinite matter) the electromagnetic fields cannot be propagated. This is the Ewald-Oseen theorem of extinction.[10]-[12] For a finite sample of plasma the eigenmodes of these equations can be propagated, with frequencies given by $\Omega^2 = \omega_0^2 + c^2k^2$; these modes are the polaritonic modes; they are responsible for the refraction of the electromagnetic waves at surface.

Boltzmann equation. The displacement in the position of a particle produced by an external field depends on the initial conditions; the initial conditions for a particle in statistical motion are the position $\mathbf{r}$ and velocity $\mathbf{v}$. It follows that we should consider a displacement $\mathbf{u}$ which is a function of position $\mathbf{r}$ and velocity $\mathbf{v}$: $\mathbf{u}(\mathbf{r},\mathbf{v})$; and, of course, a function of time $\mathbf{u}(\mathbf{r},\mathbf{v},t)$. The Drude-Lorentz model, where $\mathbf{u}$ is a function of time only, is a simplification (another simplification is the displacement $\mathbf{u}(\mathbf{r},t)$ of a fluid or a solid, where the statistical motion is neglected). In this case, the equation of motion reads

$$\frac{\partial\mathbf{w}}{dt} + (\mathbf{v}grad)\mathbf{w} + \gamma\mathbf{w} = \frac{1}{m}\mathbf{K} \ , \tag{15}$$

where $\mathbf{w} = d\mathbf{u}/dt$, $\mathbf{K}$ is the force and $w \ll v$; it is worth noting the occurrence of the transport term in equation (15) (which in fluids becomes $(\mathbf{w}grad)\mathbf{w}$ and in solids it is absent; in fluids and solids the statistical motion is neglected, since the motion is considered for macroscopical, but small, regions; in solids the velocity $\mathbf{w}$ is small and may be neglected).

Equation (15) can be written in a more convenient form by making use of the distribution function $F(\mathbf{v})$ at equilibrium; it gives the density of particles

$$n = \int d\mathbf{v}F \ ; \tag{16}$$

the equilibrium is preserved if this function suffers a smal change

$$f = \delta F = -\mathbf{w}\frac{\partial F}{\partial \mathbf{v}} = -\frac{d\mathbf{u}}{dt}\frac{\partial F}{\partial \mathbf{v}} \ ; \tag{17}$$

we can see that f is a function of $\mathbf{r}, \mathbf{v}$ and t. We multiply equation (15) by $\partial F/\partial \mathbf{v}$ and get

$$\frac{\partial f}{\partial t} + (\mathbf{v}grad)f + \frac{1}{m}\mathbf{K}\frac{\partial F}{\partial \mathbf{v}} = -\gamma f \ ; \tag{18}$$

this is the Botzmann kinetic equation;[13] the term $-\gamma f$ accounts for collisions; the collision frequence γ should be small (and its dependence on velocities and position may be neglected); in the collisionless regime this term may be neglected. The meaning of the Boltzmann equation consists in the vanishing of the total change $df/dt + \gamma f = 0$. We note that $df/dt = 0$ means the mechanical equilibrium as expressed by the mechanical motion, while the term γf represents the agent needed for establishing the thermal equilibrium. The Boltzmann equation describes the way of accomodating small external perturbations to the thermal equilibrium (the "kinetic" motion). In this respect, it differs from the transport equations, which deal with (global) non-equilibrium.

It is worth noting that the first-order change in F preserves the equilibrium; higher-order corrections to this change either are negligible, or destroy the equilibrium; in this respect, we note the condition $w \ll v$. In this context, there exists another special point which deserves attention. There exists an internal force due to the variations of the pressure (variations of the density), which may contribute to the force $\mathbf{K}$. For a variation $\delta n = -ndiv\mathbf{u}$ of the mobile-charge density (leaving aside the $\mathbf{v}$-dependence), a change $\delta p = (\partial p/\partial n)\delta n = -n(\partial p/\partial n)div\mathbf{u}$ in pressure appears, and a force per unit volume $-grad\delta p = n(\partial p/\partial n)grad \cdot div\mathbf{u}$. At constant temperature this force is $nTgrad \cdot div\mathbf{u}$. Compared with the inertial force $m\ddot{\mathbf{u}}$, this is a great force; the changes it brings about are rapid, such that we should take this variation at constant entropy.[14] If we denote $c^2 = (\partial p/\partial n)_s/m$, where s is the entropy, this force reads $mc^2 grad \cdot div\mathbf{u}$, where c is the sound velocity.[1] This force may appear in fluids and solids; this force does not preserve the equilibrium of the underlying statistical motion; consequently, its place is not in the kinetic equation; it is a purely mechanical motion. Macroscopic, but small, regions of plasma can be considered, as in fluids or solids, but such regions are electrically neutral, and plasma is then a common gas.

In the presence of an external electric field $\mathbf{E}_0$, the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + (\mathbf{v}grad)f + \frac{q}{m}(\mathbf{E}_0 + \mathbf{E})\frac{\partial F}{\partial \mathbf{v}} = 0 \ , \tag{19}$$

where we leave aside the collision term; the internal field is given by

$$div\mathbf{E} = 4\pi q \int d\mathbf{v} f \tag{20}$$

($div\mathbf{E}_0 = 0$) or

$$\frac{\partial \mathbf{E}}{\partial t} = -4\pi q \int d\mathbf{v} \cdot \mathbf{v} f \ ; \tag{21}$$

equation (20) arises from a polarization charge, while equation (21) originates in a polarization current. It is worth noting that this latter equation may generate longitudinal electric fields which

[1] The equation of the adiabatic transformation of an ideal gas is $pV^\gamma = const$, where γ is the ratio of the two specific heats, $\gamma = c_p/c_v$ (at constant pressure and, respectively, constant volume); for a monoatomic gas $\gamma = 5/3$; it follows that the wave velocity c is given by $c = \sqrt{\gamma p/mn}$; compared with the isothermal velocity $v_{th} = \sqrt{p/mn} = \sqrt{T/m}$ we see that it is of the same order of magnitude as the thermal velocity v_{th}.

vary in space, as expected in a medium with electric charges. In general, the contribution of the magnetic field to the force term cannot be neglected (especially when the electric field is a transverse radiaton field). The internal field was introduced in Boltzmann equation by Vlasov (equation (19) is known as Vlasov equation).[15]

Plasma oscillations. Eigenfrequencies. First, we assume that the external field $\mathbf{E}_0$ is absent; we are interested in the response of the plasma to an initial perturbation $f_i = f(t = 0, \mathbf{r}, \mathbf{v})$. Since the plasma is uniform, we may restrict ourselves to a spatial dependence $\sim e^{i\mathbf{kr}}$ (F does not depend on $\mathbf{r}$, nor t); we take the coordinate, denoted by x, along the direction of the wavevector $\mathbf{k}$; since the field is directed along the x-direction we may integrate over the transverse velocities and use for the Maxwell distribution $F = n(\beta m/\pi)^{1/2}e^{-\beta mv^2}$, where v is the velocity along the x-direction and $\beta = 1/T$ is the reciprocal temperature; equations (19) and (20) become

$$\frac{\partial f}{\partial t} + ivkf + \frac{qE}{m}\frac{\partial F}{\partial v} = 0 \tag{22}$$

and

$$ikE = 4\pi q \int dvf \ . \tag{23}$$

In order to account for the initial condition ($t = 0$) we multiply equation (22) by the step function $\theta(t)$ and restrict ourselves to $t > 0$; equation (22) becomes

$$\frac{\partial f}{\partial t} + ivkf + \frac{qE}{m}\frac{\partial F}{\partial v} = f_i\delta(t) \ ; \tag{24}$$

now we may perform the Fourier transform with respect to the time, and get

$$-i(\omega - vk)f + \frac{qE}{m}\frac{\partial F}{\partial v} = f_i \ , \tag{25}$$

or

$$f = i\frac{f_i - \frac{qE}{m}\frac{\partial F}{\partial v}}{\omega - vk} \ ; \tag{26}$$

and, from equation (23),

$$E = 4\pi q\frac{\int dv\frac{f_i}{\omega - vk}}{k + \frac{4\pi q^2}{m}\int dv\frac{\partial F/\partial v}{\omega - vk}} \ ; \tag{27}$$

the change in the distribution function is given by

$$f = \frac{i}{\omega - vk}\left(f_i - \frac{4\pi q^2}{m}\frac{\partial F}{\partial v}\frac{\int dv\frac{f_i}{\omega - vk}}{k + \frac{4\pi q^2}{m}\int dv\frac{\partial F/\partial v}{\omega - vk}}\right) \ . \tag{28}$$

This equation offers several interesting points. First, since we analyze the perturbation in terms of waves, the motion of the individual particles acquires a wave character too; such that an individual particle with velocity v exhibits a frequency vk. Then, we may see from equation (28) that there exist singularities in the distribution function for $v = \omega/k$ and divergent integrals arising from these singularities. At this point we should resort to the natural boundary condition which requires causality, *i.e.* $f = 0$ for $t < 0$. Therefore, the integration over ω shoud be performed in the lower half-plane, which requires a pole in that half-plane, *i.e.* ω in equation (28) should be replaced by $\omega + i\gamma$, $\gamma \rightarrow 0^+$. Had we retained the collision term in the Boltzmann equation, the γ-contributions toω would have arisen. We perform first the integration over ω in such terms and thereafter take the limit $\gamma \rightarrow 0^+$. In integrals with respect to velocity v we may take the limit $\gamma \rightarrow 0^+$. Therefore, the above equations should be written as

$$E = 4\pi q \frac{\int dv \frac{f_i}{\omega - vk + i0^+}}{k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega - vk + i0^+}} \tag{29}$$

and

$$f = \frac{i}{\omega - vk + i\gamma} \left(f_i - \frac{4\pi q^2}{m} \frac{\partial F}{\partial v} \frac{\int dv \frac{f_i}{\omega - vk + i0^+}}{k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega - vk + i0^+}} \right) . \tag{30}$$

Terms like $1/(\omega - vk + i0^+)$ lead to

$$\frac{1}{\omega - vk + i0^+} = P \frac{1}{\omega - vk} - i\pi\delta(\omega - vk) \ ; \tag{31}$$

such a circumstance makes the integrals over v in f finite; for instance,

$$\int dv \frac{f_i}{\omega - vk + i0^+} = P \int dv \frac{f_i}{\omega - vk} - \frac{i\pi}{v} f_i(v = \omega/k) \ , \tag{32}$$

where the principal value in this equation can be approximated satisfactorily by

$$P \int dv \frac{f_i}{\omega - vk} \simeq \frac{1}{\omega} \int dv f_i \ , \tag{33}$$

or

$$P \int dv \frac{f_i}{\omega - vk} \simeq \frac{k}{\omega^2} \int dv \cdot v f_i \ , \tag{34}$$

(for reasonably-behaving functions f_i). The situation is similar for the denominator involving $\partial F/\partial v$ in equation (28). Its zero provides an equation which gives the eigenfrequency of the distribution function. Due to the replacement $\omega \rightarrow \omega + i0^+$ this is a complex equation; it follows that the root ω of this equation (the eigenfrequency) acquires a damping factor, *i.e.*, we should solve this equation for $\omega \rightarrow \omega - i\Gamma$, where $\Gamma > 0$ is the damping coefficient. As we can see, we are led to admit that the causality implies a damping response. The causality, which acts at the level of the individual particles, leads to a damped behaviour of the collective mode, *i.e.* of the macroscopic behaviour. This is a profound consequence of the equations governing the condensed matter, and an instance of an emergent dynamics.[16, 17] The damping of the collective mode is known as the Landau damping.[18] We emphasize that the egenfrequency discussed here is a collective mode since it appears from an integral over velocities of the individual particles. It is the plasmon mode.

According to the scheme delineated above, we write

$$k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega - vk + i0^+} = k + \frac{4\pi q^2}{m} P \int dv \frac{\partial F/\partial v}{\omega - vk} - i \frac{4\pi^2 q^2}{m} \int dv \frac{\partial F}{\partial v} \delta(\omega - vk) \ , \tag{35}$$

or

$$k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega - vk + i0^+} \simeq k - \frac{4\pi n q^2}{m\omega^2} k - i \frac{4\pi^2 q^2}{mk} \frac{\partial F}{\partial v} \mid_{v=\omega/k} \tag{36}$$

(it is worth noting that the result of integration in equation (36) holds for any distribution function). It follows that the perturbation of the distribution function (equation (28)) should be written as

$$f = \frac{i}{\omega - vk + i\gamma} \left[f_i - \frac{4\pi q^2}{m} \frac{\partial F}{\partial v} \frac{\int dv \frac{f_i}{\omega - vk + i0^+}}{k \left(1 - \frac{\omega_0^2}{\omega^2} - i \frac{4\pi^2 q^2}{mk^2} \frac{\partial F}{\partial v} \mid_{v=\omega/k} \right)} \right] . \tag{37}$$

The initial condition

$$\frac{1}{2\pi}\int d\omega f = f_i \tag{38}$$

leads to

$$\int d\omega \frac{E(\omega)}{\omega - vk + i\gamma} = 0 \ , \tag{39}$$

where $E(\omega)$ is given by equation (27). Making use of the poles of the field $E(\omega)$ (equation (37)), we can see that the integral which includes $E(\omega)$ in equation (39) is indeed zero.

If we view ω as $\omega - i\Gamma$ we get

$$k + \frac{4\pi q^2}{m}\int dv \frac{\partial F/\partial v}{\omega - vk + i0^+} \simeq k(1-\omega_0^2/\omega^2) - i\left(\frac{2\gamma k}{\omega_0} + \frac{4\pi^2 q^2}{mk}\frac{\partial F}{\partial v}\mid_{v=\omega_0/k}\right) \ , \tag{40}$$

an equation which gives the damping coefficient

$$\Gamma = -\frac{2\pi^2 q^2\omega_0}{mk^2}\frac{\partial F}{\partial v}\mid_{v=\omega_0/k} \ . \tag{41}$$

External field. Let us consider a longitudinal electric field $E_0 e^{-i\omega t}$; the field is uniform and the position variable does not appear in Boltzmann equation, which reads

$$-i\omega f + \frac{q}{m}(E_0 + E)\frac{\partial F}{\partial v} = 0. \tag{42}$$

The Poisson equation

$$div\mathbf{E} = 4\pi q\int d\mathbf{v} f \tag{43}$$

is identically zero (f is an odd function of v); however, if we take the time derivative and use the continuity equation, we get

$$div\frac{\partial \mathbf{E}}{\partial t} = 4\pi q\int d\mathbf{v}\frac{\partial f}{\partial t} = -4\pi q\int d\mathbf{v}\cdot div\mathbf{v}f \ , \tag{44}$$

where $q\mathbf{v}f$ is the current density; therefore, we use $\partial\mathbf{E}/\partial t = -4\pi q\int d\mathbf{v}\cdot\mathbf{v}f$, or

$$i\omega E = 4\pi q\int dv\cdot vf \ . \tag{45}$$

We get immediately the solution

$$f = -i\frac{qE_0}{m}\frac{\omega}{\omega^2-\omega_0^2}\frac{\partial F}{\partial v} \ , \ E = E_0\frac{\omega_0^2}{\omega^2-\omega_0^2} \ ,$$
$$E_t = E_0 + E = E_0\frac{\omega^2}{\omega^2-\omega_0^2} \ . \tag{46}$$

These are well-known relations for quasi-static electric field in plasma. We can see that the total field E_t is vanishing in the static limit $\omega \to 0$, due to the reaction of the equal and opposite internal field E. The same result can be obtained from equation (5) (without dissipation), which gives the displacement

$$u = -\frac{qE_0}{m}\frac{1}{\omega^2-\omega_0^2} \ ; \tag{47}$$

from $div\mathbf{E} = -4\pi div\mathbf{P}$ and $div\mathbf{E} = 4\pi q\delta n = -4\pi nq div\mathbf{u}$ we get the polarization

$$P = nqu = -\frac{nq^2 E_0}{m}\frac{1}{\omega^2 - \omega_0^2} = \chi E_t \ , \tag{48}$$

where $\chi = -nq^2/m\omega^2$ is the electric susceptibility and $\varepsilon = 1 + 4\pi\chi = 1 - \omega_0^2/\omega^2$ is the dielectric function.

Surface field. Let us consider a semi-infinite plasma which occupies the half-space $x > 0$, bounded by the plane surface $z = 0$. In order to deal conveniently with the boundary conditions, we multiply equation (19) by the step function $\theta(x)$ and restrict ourselves to $x > 0$; the equation acquires on the right an additional term $vf_s\delta(x)$, where $f_s = f(x = 0)$ plays the role of a constant of integration (in fact, $f_s = f(x)$, $x \to 0^+$). The integral of f_s with respect to the velocities gives the surface particle concentration n_s. The Boltzmann equation becomes

$$\begin{gathered} -i\omega f + v\tfrac{\partial f}{\partial x} + \tfrac{q}{m}(E_0 + E + E_1)\tfrac{\partial F}{\partial v} = vf_s\delta(x) \ , \\ i\omega(E + E_1) = 4\pi q \textstyle\int dv \cdot vf \ ; \end{gathered} \tag{49}$$

we can check the relation $f_s = f(x \to 0^+)$ by direct integration of the firts equation (49) along a small distance perpendicular to the surface. We seek the solution as the sum $f = f_0 + f_1$, where

$$\begin{gathered} -i\omega f_0 + \tfrac{q}{m}(E_0 + E)\tfrac{\partial F}{\partial v} = 0 \ , \\ i\omega E = 4\pi q \textstyle\int dv \cdot vf_0 \end{gathered} \tag{50}$$

and

$$\begin{gathered} -i\omega f_1 + v\tfrac{\partial f_1}{\partial x} + \tfrac{q}{m}E_1\tfrac{\partial F}{\partial v} = vf_s\delta(x) \ , \\ \tfrac{\partial E_1}{\partial x} = 4\pi q \textstyle\int dv f_1 \ . \end{gathered} \tag{51}$$

The function f_0 is the uniform solution given by equation (46), while f_1 is given by

$$f_1 = \frac{i}{\omega - vk + i\gamma}\left[vf_s - \frac{4\pi q^2}{m}\frac{\partial F}{\partial v}\frac{\int dv\frac{vf_s}{\omega - vk + i0^+}}{k + \frac{4\pi q^2}{m}\int dv\frac{\partial F/\partial v}{\omega - vk + i0^+}}\right] \tag{52}$$

and

$$E_1 = 4\pi q\frac{\int dv\frac{vf_s}{\omega - vk + i0^+}}{k + \frac{4\pi q^2}{m}\int dv\frac{\partial F/\partial v}{\omega - vk + i0^+}} \ . \tag{53}$$

Let us first compute the field $E_1(x)$; to this end we need the denominator

$$D = k + \frac{4\pi q^2}{m}\int dv\frac{\partial F/\partial v}{\omega - vk + i0^+} \tag{54}$$

in the vicinity of its zeros; making use of equation (36) we get

$$D \simeq k(1 - \omega_0^2/\omega^2) - i\frac{4\pi^2 q^2}{mk}\frac{\partial F}{\partial v}\mid_{v=\omega/k} \ . \tag{55}$$

At this point we use the Maxwell distribution $F = n\sqrt{\beta m/\pi}e^{-\beta mv^2}$, where $\beta = 1/2T$ is half the reciprocal temperature; it is convenient to introduce the variable $\xi = \sqrt{\beta m}\omega/k$ and use the ratio ω_0/ω. We can see easily that the zeroes of D are given by $\xi^3 e^{-\xi^2} = -i\alpha$, where $\alpha =\mid \varepsilon \mid /2\sqrt{\pi}(1-\varepsilon)$; we consider first the case $\omega < \omega(\varepsilon < 0)$. We get two roots of the equation $D = 0$, placed in the upper half-plane, given by $k_{1,2} \simeq \frac{1}{2\alpha^{1/3}}\sqrt{\beta m}\omega(\pm\sqrt{3}+i)$. For k near $k_{1,2}$ the field $E_1(k)$ can be written as form

$$E_1(k) \simeq \frac{4\pi^2\alpha^{2/3}qv_{th}^2}{3\mid\varepsilon\mid\omega}\frac{\eta_{1,2}f_s(2\sqrt{2}\alpha^{1/3}v_{th}/\eta_{1,2})}{k-k_{1,2}} \ , \tag{56}$$

where $\eta_{1,2} = \pm\sqrt{3}+i$ and $v_{th} = 1/\sqrt{\beta m}$ is the thermal velocity. The calculation leads to

$$\begin{array}{c} E_1(x) = i\frac{4\pi^2\alpha^{2/3}qv_{th}^2}{3|\varepsilon|\omega}\{\eta f_s(2\sqrt{2}\alpha^{1/3}v_{th}/\eta)e^{i(\sqrt{3}\omega/2\sqrt{2}\alpha^{1/3}v_{th})x}- \\ -\eta^* f_s^*(-2\sqrt{2}\alpha^{1/3}v_{th}/\eta)e^{-i(\sqrt{3}\omega/2\sqrt{2}\alpha^{1/3}v_{th})x}\}e^{-(\omega/2\sqrt{2}\alpha^{1/3}v_{th})x} \ . \end{array} \tag{57}$$

We can see that an additional, non-uniform, electric field appears as a result of the presence of the surface, which is attenuated with an attenuation length (penetration depth, extinction length) $\lambda_e \simeq (1/\pi)^{1/6}[\mid\varepsilon\mid/(1-\varepsilon)]^{1/3}v_{th}/\omega$. For the frequency $\omega \simeq 10^7 s^{-1}$ the penetration depth is $\lambda_e \simeq 0.5cm$. It is worth noting that the penetration depth and the wavelength of the spatial oscillations have the same order of magnitude. For $\omega > \omega_0$ $(0 < \varepsilon < 1)$ there exists only one root $k_1 \simeq \frac{i}{\alpha^{1/3}}\sqrt{\beta m/2}\omega$ of the equation $D = 0$ $(\xi^3 e^{-\xi^2} = i\alpha)$ placed in the upper half-plane; the field is given by

$$E_1(x) = -\frac{8\pi^2\alpha^{2/3}qv_{th}^2}{3\varepsilon\omega}f_s(-i\sqrt{2}\alpha^{1/3}v_{th})e^{-(\omega/\sqrt{2}\alpha^{1/3}v_{th})x} \tag{58}$$

and the penetration depth is $\lambda_e \simeq (2/\pi)^{1/6}[\mid\varepsilon\mid/(1-\varepsilon)]^{1/3}v_{th}/\omega$; we note the absence of oscillations. Similar calculations of the penetration depth can be made for a plasma confined between two plane-parallel walls, or other geometries; the result depends on the boundary conditions incorporated in parameters like f_s. It is given by $f_s = \frac{1}{2\pi}\int dk f_1(k,v)$ (in fact the limit $x \rightarrow 0^+$ in the Fourier transform), where $f_1(k,v)$ is given by equation (52); it is easy to see that the integration of the first term in equation (52) gives f_s, while, making use of equation (25), the integration with respect to k of the term which includes $E_1(k)$ is zero.

Concluding remarks. According to equation (46), the distribution function changes as

$$F \rightarrow F + f = F - \frac{qE_0}{m}\frac{\omega}{\omega^2-\omega_0^2}\frac{\partial F}{\partial v}\sin\omega t \tag{59}$$

under the action of a uniform longitudinal electric field; this equation reads

$$F(v) \rightarrow F(v+A) \ , \ A = -\frac{qE_0}{m}\frac{\omega}{\omega^2-\omega_0^2}\sin\omega t \ . \tag{60}$$

The external forces are slow, such that the thermal equilibrium is achieved over their period. The thermal equilibrium is given by the maximum of the entropy, *i.e.* the maximum of $\int(F\ln F - \mathcal{E}F/T)$, where $\mathcal{E}$ is the energy and the integration is performed over the relevant phase-space variables. The preservation of the equilibrium under the action of the external forces implies a change in temperature $\delta T/T = \overline{\delta\mathcal{E}}/\overline{\mathcal{E}}$; in our case, $\overline{\delta\mathcal{E}}/\overline{\mathcal{E}} = A^2/\overline{v^2}$ and $\delta T = mA^2$, *i.e.*

$$\delta T = \frac{q^2E_0^2}{m}\left(\frac{\omega}{\omega^2-\omega_0^2}\right)^2\sin^2\omega t \ ; \tag{61}$$

this is an extremely small change in temperature, as expected.

The general equation (17) $f = -\mathbf{w}\frac{\partial F}{\partial \mathbf{v}}$ by which the Boltzmann equation is introduced gives an indication regarding the surface parameter f_s; indeed, it is natural to assume $\mathbf{w} = 0$ at the surface, *i.e.* $f_0 + f_s = 0$; it follows that we may take $f_s = -f_0$, where f_0 is given by equation (46); the magnitude of the field E_1 is of the order $E/\mid\varepsilon\mid$, where E is the internal uniform field given by equation (46).

A model of gaseous classical plasma is described, derived from the ideal classical gas, which is relevant for ionized gases, elementary quasi-electron excitations in metals and charge carriers in semiconductors. The model exhibits a universal mean freepath, governed by the Debye length. The eigenmodes (vibrations) of this plasma are investigated, *i.e.* the plasma modes, by means of the Boltzmannn equation, and the Landau damping is discussed, as arising from the causal dynamics of the particles subject to the thermal motion. The bulk and the surface response of the plasma is computed to a uniform external electric field (a longitudinal field) in the collisionless regime. Beside the uniform (bulk) reaction field, a non-uniform, damped, surface field is highlighted, and the penetration depth (extinction length, attentuation coefficient) is computed for a semi-infinite plasma. The calculations are performed by using the technique of generalized functions, which, in this case, is reflected in using surface (or initial) terms in the Boltzmannn equation.

A procedure similar with that presented here can be used for treating two coupled plasma, *e.g.* a charged beam propagating in a thermal plasma, where various instabilities can be analyzed.

Appendix. Let us estimate the integral

$$I=\int d\omega\frac{g(\omega)}{\omega-\omega_0+i0^+}=P\int d\omega\frac{g(\omega)}{\omega-\omega_0}-i\pi g(\omega_0) \tag{62}$$

(for $\omega_0>0$); if $g(\omega)=const$ the principal value is zero, and we are left with $-i\pi g(\omega_0)$. This is precisely the result given by the contour integration. Indeed,

$$P\int d\omega\frac{1}{\omega-\omega_0}=\ln\mid-\varepsilon\mid-\ln\mid-\infty\mid+\ln\mid\infty\mid-\ln\mid\varepsilon\mid\,,\;\;\varepsilon\rightarrow 0^+\;. \tag{63}$$

Otherwise (for integrable $g(\omega)$) the principal value is

$$P=P\int_0^\infty d\omega\left[\frac{g(\omega)}{\omega-\omega_0}-\frac{g(-\omega)}{\omega+\omega_0}\right]\;. \tag{64}$$

For even $g(\omega)$ we get

$$P=P\int_0^\infty d\omega\frac{2\omega_0 g(\omega)}{\omega^2-\omega_0^2}=-\frac{2}{\omega_0}\int_0^\infty d\omega g(\omega)\left(1+\frac{\omega^2}{\omega_0^2}+...\right)\;\;, \tag{65}$$

while for odd $g(\omega)$ we get

$$P=P\int_0^\infty d\omega\frac{2\omega g(\omega)}{\omega^2-\omega_0^2}=-\frac{2}{\omega_0^2}\int_0^\infty d\omega\cdot\omega g(\omega)\left(1+\frac{\omega^2}{\omega_0^2}+...\right)\;. \tag{66}$$

These are asymptotic series; for $g(\omega)$ rapidly vanishing at infinity, they converge rapidly.

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