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Notes on Quantum Electrodynamics and Quantum Field Theory
(Lecture fourteenth of the Course of Theoretical Physics)

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Abstract

Quantum-mechanical coordinates are undetermined; therefore, they cannot be subject to relativist transformations. Quantized free fields are undetermined; therefore, they cannot be subject to relativist transformations. Quantum-mechanical interaction cannot be subject to relativist transformations. The description of scattering experiments in terms of relativist quantized free fields leads to inconsistencies. The renormalization techniques are improper and do not work for the whole series of perturbations. Dirac equation has not a classical limit. Quantum Electrodynamics and Quantum Field Theory are constructed on the irreducible incrongruency between Quantum Mechanics and Relativity.

1 Introduction

These Notes include an attempt of a critical analysis of Quantum Electrodynamics and Quantum Field Theory. Details concerning notation, calculation and derivation of particular mathematical relations can be found in Landau and Lifshitz (Berestetskii, Lifshitz, Pitaevskii) *Quantum Electrodynamics*, Jauch and Rohrlich *Photons and Electrons*, Weinberg *Quantum Theory of Fields*, Dyson *Advanced Quantum Mechanics*. We deal here only with questions of principle. Although there is a consensus that, in spite of their computational success, Quantum Electrodynamics and Quantum Field Theory exhibit serious difficulties at the conceptual level, there is little preoccupation of tracing back the origin of these dificulties. We attempt here to show that these difficulties originate in the profound incongruency between Quantum Mechanics and Relativity. Quantum Mechanics assumes instantaneous positions, which may be completely undetermined, while the Theory of Relativity assumes determined positions which can be related by signals propagating with veocities lower than, or equal with, the speed of light in vacuum. The Theory of Relativity deals with a space-time dynamics, while Quantum Mechanics envisages only a time evolution, for a global set of positions. These difficulties do not appear in the original Dirac's theory of radiation (Fermi), which is consistently quantum-mechanical and not relativist, while they are exhibited by the current descriptions of these theories, because the results of the quantum-fields interaction are interpreted in terms of relativist free fields. This leads to a meaningless effect of the interaction.

Quantum Mechanics deals with small amounts of mechanical action, which means small amounts of energy (energy quanta) in reasonably long times and small amounts of momenta (momentum quanta) over reasonably long distances. On the other hand, the Theory of Relativity deals with high velocities, which imply high motion energies and momenta. Therefore, a Quantum Electrodynamics, or a Quantum Field Theory, would make sense only during very short durations of time and over very short distances. Such theories would explore the internal structure of the small particles, as in high-energy scattering experiments. Indeed, these theories were developed by the advent of the particle accelerators.

But, is there an internal structure of the small, elementary particles? The high-energy scattering experiments revealed many elementary particles, such that the two theories became a theory of elementary particles and their interactions. The theory of the elementary particles is based on internal symmetries and local gauge symmetries, the latter associated with differential operators, reminiscent of quantum-mechanical operators, all relativist invariant. To what extent the Quantum Mechanics and the Theory of Relativity are present, survive and keep their meaning, in the theory of elementary particles?

Quantum Mechanics views the time and position coordinates as partially, or totally, undetermined, according to the uncertainty relations, while the Theory of Relativity consider the time-space coordinates exactly determined, and subject to Lorentz transformations. It seems that a union of these two theories would be impossible in space and time, or, if still is, to what a price? The quantization in Quantum Electrodynamics and Field Theories pertains to fields, which exhibit quanta which may be created or destroyed; then, the relativist transformations do not apply to such undetermined quantum-mechanical objects. Both Relativity and Quantum Mechanics have a positive content and limitations. Moreover, the time-space structure is irrelevant in scattering experiments, what matters in these experiments is what happens during the short time and over the small distances of the interaction. However, the coordinates are there and, even not relevant, we must deal with them. These Notes attempt to analyze and answer to some extent such questions.

Quantum Electrodynamics has begun with Einstein's notion of photon. It was realized that the photon is a quantum of electromagnetic field, it is each field component which is quantized, while the relativist nature of the field vector is not affected. However, when quantized the fields are not determined, such that they cannot be subject to relativist transformations. At this level, it seems that Quantum Mechanics and Relativity are not compatible with one another. The photon is also represented by a plane wave, which is a solution of the Maxwell equations for radiation, and the plane waves have a well-determined frequency and a well-determined wavevector, which makes the time and the position perfectly (completely) undetermined. This is contrary to the spirit of the Theory of Relativity, which requires well-determined coordinates. Moreover, the quanta of energy and momentum of the photon are obtained from the plane wave by applying the well-known quantum-mechanical operators, as if the plane wave would be a wavefunction, though there is no Schrodinger equation for the photon. This uncertainty in time-energy and momentum-position, which is specific to Quantum Mechanics, and which appears naturally in field equations, contradicts the Theory of Relativity. The escape from this conflictual situation is the realization that in Relativity there exist unphysical coordinates, namely those placed on space-like surfaces (outside the light cone). Therefore, the fields live on such space-like surfaces, where the first quantization (*i.e.* Quantum Mechanics in space and time) may act, but without any meaning, other than a formal one. Therefore, the fields are relativist, their quantization exists in their space, where the coordinates are the fields, while the time and the position, which would generate a conflictual situation between Relativity and Quantum Mechanics, are relegated to the status of undetermined, unphysical, unmeasurable parameters, having only a formal role. The quantization of the electromagnetic radiation implies only the time, while a Lorentz transformation requires coordinates too. This is a basically conflictual situation in an attempt to reconcile the two theories.

There exists a fundamental difference in the way Quantum Mechanics and Relativity view the plane wave and, consequently, all functions made of plane waves. In Relativity the plane wave is viewed as an oscillating function with well determined values at any point. Quantum Mechanics views the plane wave as defined only at those points where the plane wave have the same values, *i.e.* at those points where the phases differ by an integral multiple of 2π. Consequently, Quantum Mechanics attributes an uncertainty to the phase of the plane wave and a corresponding uncertainty in coordinates, frequency and wavevector (energy and momentum). The solution offered by the space-like surface to this conflictual situation raises serious difficulties, because it restricts the space Δx and the time Δt to $\Delta x \geq c\Delta t$; we cannot go inside the region $c\Delta t$ during the time Δt; and, of course, $\Delta x \geq c\hbar/\Delta E$. This means that even with a large amount of energy we cannot probe as small a region of space as we would like, *e.g.* the structure of an elementary particle; we can only probe larger regions of space, which, nonetheless, are unphysical. Essentially the same objections have been raised by Landau and Peierls to Quantum Electrodynamics and Quantum Field Theory. Moreover, for plane waves $\Delta x \longrightarrow \infty$, so we can have information about scattering processes which occur in small space-time regions only from asymptotic incoming and outgoing plane waves, whose scattering time is infinite.

Quantum Electrodynamics continued with the electron. Since its non-relativist motion is quantized, it is natural to extend the (first) quantization to the relativist electron; we get thereby the Klein-Gordon equation. Here it appears immediately a difficulty, related to the rest energy. It is necessary to view this energy as a motion energy, in order to have a consistent extension of the (first) quantization. Such an assumption has far-reaching implications; it amounts to view the electron as an energy and momentum quantum (like the photon), which can be created or destroyed. Therefore, it obeys anticommutation rules and, moreover, since the charge is conserved, there should exist positrons (antiparticles). The existence of the spin requires matrices, which must be at least 4×4 matrices; these account both for spin and antiparticles and bring an important simplification to the quadratic Klein-Gordon equation, which becomes equivalent with the linear Dirac equation.

This whole scheme of second quantization, which views fundamental particles like photon and electron as quanta, *i.e.* assign a quantum-mechanical nature to their very existence and not to their motion, as in the original quantum-mechanical scheme, leads to profound difficulties with their interaction. We emphasize that the second quantization is possible only by assuming that the time-position coordinates are unphysical space-like parameters which are integrated out as for global (delocalized) particles. But such an integration does not warrant meaningful results; we may only accept such a procedure as long as it may look a reasonable one; unfortunately, in the interaction problem it shows its artificial character, because we integrate the interaction indiscriminately over space- and time-like coordinates. In Quantum Electrodynamics and Quantum Field Theory the very existence of the particles is quantized (second quantization), while the fields are defined in relativist unphysical regions (as required by the first quantization and allowed by Relativity). In addition, quantized fields cannot be subject to relativist transformations, since they are undetermined.

If there exist such inconsistencies at the fundamental level of the relativist quantum-mechanical dynamics, how can be understood the success of Quantum Electrodynamics and Quantum Field Theory? The answer resides in the fact that these theories make little use of either relativist or quantum-mechanical dynamics. Their results are based on the symmetries required by the Lorentz group and the internal symmetry groups and the ensuing conservation laws. The whole physics of elementary particles is a classification of particles (quanta) according to symmetry principles, including the unitary symmetries. However, the understanding of these theories in the relativist and quantum-mechanical framework necessitates clarification. The computational success of these theories is circumstantial.

The Theory of Relativity is a classical theory (*i.e.*, a non-quantum-mechanical one); it views functions well-defined in space and time, at exact time-positions coordinates, which are related by Lorentz tranformations (space and time reversal included). The meaning of this theoretical framework consists in the existence of experiments which show the existence of such objects by measurement; this is the "reality content" of this theory (Einstein), as for any physical theory. Quantum Mechanics views functions defined in space and time at coordinates which are uncertain, undetermined, according to its uncertainty principle. Again, the meaning of this quantum-mechanical theoretical framework is the existence of experiments which produce such objects by measurement. Therefore, a union between Relativity and Quantum Mechanics is impossible, in the sense that the first class of relativist experiments is different from the second class of quantum-mechanical experiments. Indeed, this is the case, and this was the original formulation of radiation theory.

Since a free classical field, which exhibits well-determined coordinates and is relativist, can be decomposed in a superposition of plane waves, we may extend its relativist character to these plane waves; the plane waves may be used as free quantum-mechanical fields. This extension is not harmful and is legitimate, as long as there is no experiment to determine the quantum-mechanical character of these free fields. Indeed, this is the case, because any experiment which would determine the quantum-mechanical character of these fields would imply a detection, *i.e.* an interaction, which would transform the free fields into interacting fields. Now, suppose that we make an experiment to detect the quantum-mechanical character of the fields, which would mean to transform them into interacting fields; in other words, suppose that we make an experiment where the free quantum-mechanical fields are set in interaction, *i.e.* they become interacting fields. Since such an experiment is a quantum-mechanical one, it implies necessarily an extended space (cavity) and an extended time interval, as in scattering experiments, such that the coordinates are not determined anymore; in such an experiment the interacting fields behave as a global entity. If we wish to determine momenta with high accuracy, the extension of the spatial cavity should be large; if we wish to determine energies with high accuracy, the extension of the time interval should be large. For such a global experimental set-up the (proper) Lorentz transformations are not applicable anymore; any attempt to apply them would only displace the set-up as a whole in space and time. Consequently, in such a quantum-mechanical experiment the preservation of the relativist character of the original free fields, which are employed to construct the interacting fields, cannot be checked. There is no relativist experiment to detect the relativist character of the interacting fields, such that the requirement of checking their relativist character is meaningless. It would be imposible to change locally the reference frame, such that the relativist invariance to be in danger to be lost, though we cannot, in fact, prove its persistance, because any change of the reference frame is global. As long as there is no experiment to prove or disprove a property, any such property is not determined. In particular, the quantum-mechanical evolution equation of the interaction needs not be relativist. However, the field interaction is interpreted in terms of relativist free fields, which leads to meaningless results. There exist divergencies which can be removed in every finite order of the perturbation theory by renormalization in such an approach, but they cannot be removed from the whole perturbation series. The profound reason for this difficulty resides in the incongruency betweem Quantum Mechanics and the Theory of Relativity.

In a summarizing sentence, if we view the fields as quantum-mechanical objects, then the coordinates have no sense, only time is relevant and a time evolution (not for stationary states). We cannot check the relativist character of such objects, because the Relativity applies to both time and coordinates; in addition, the quantized fields are undetermined and cannot be subject to relativist transformations. In Quantum Electrodynamics and Quantum Field Theories we insist to do both operations, though in fact we do only Quantum Mechanics; which explains to some extent the success of the calculations; the difficulties related to convergence and divergencies originate in the insistence of working with relativist free fields in higher orders of the perturbation theory, where the conflictual nature of the two approaches arises. The renormalization techniques are only a limited way of redefining the lowest order of the perturbation theory. If we include the whole perturbation series, there will not be anymore a finite order to become the lowest, and the renormalization will become impossible.

In Quantum Electrodynamics and Quantum Field Theory we are interested in scattering experiments. In quantum-mechanical Relativity do not exist bound states (at least as long as we work with disentangled particles); there may exist only limited relativist corrections to bound states. A scattering experiment should be described in terms of relativist and quantum-mechanical free fields (incoming and outgoing fields), while the interaction is quantum-mechanical but not relativist. We can see that the relativist regime is preserved in interaction from the perturbation series which is written with free fields. This basic contradiction undermines Quantum Electrodynamics and Quantum Field Theory. It is worth noting that the quantum-mechanical character of the fields is associated not only with the second quantization (*i.e.*, the existence or the non-existence of the particles), but also with the space-time motion of the quanta (first quantization). Indeed, we get the photon energy $\hbar\omega$ by applying the operator $i\hbar\partial/\partial t$ to a plane wave and we derive the field equation of the electron by using the operators $i\hbar\partial/\partial t$ and $-i\hbar grad$. The relativist transformations and unitary transformations may be related, which suggests compatibility, coexistence and correspondence between Relativity and Quantum Mechanics, but this relation, simply, does not apply to scattering experiments.

2 Photons

2.1 Electromagnetic field

The electromagnetic field in vacuum consists of two real vectors **E** and **H**, called electric field and magnetic field, respectively, which obey Maxwell's equations

$$\begin{gathered} div\mathbf{E} = 4\pi\rho \ , \ div\mathbf{H} = 0 \ , \\ curl\mathbf{E} = -\tfrac{1}{c}\tfrac{\partial\mathbf{H}}{\partial t} \ , \ curl\mathbf{H} = \tfrac{1}{c}\tfrac{\partial\mathbf{E}}{\partial t} + \tfrac{4\pi}{c}\mathbf{j} \ . \end{gathered} \tag{1}$$

in these equations ρ is the electrical charge density, $\mathbf{j}$ is the electrical current density and c is the speed of light in vacuum. From the above equations we see easily that the charge and the current satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + div\mathbf{j} = 0 \ , \tag{2}$$

which shows the conservation of the electrical charge. The two homogeneous Maxwell's equations (1) are immediately satisfied by using the scalar potential Φ and the vector potential $\mathbf{A}$, through

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - grad\Phi \ , \ \mathbf{H} = curl\mathbf{A} \ , \tag{3}$$

while the other two, inhomogeneous, equations become

$$\begin{gathered} \tfrac{1}{c^2}\tfrac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = 4\pi\rho + \tfrac{1}{c}\tfrac{\partial}{\partial t}\left(\tfrac{1}{c}\tfrac{\partial\Phi}{\partial t} + div\mathbf{A}\right) \ , \\ \tfrac{1}{c^2}\tfrac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \tfrac{4\pi}{c}\mathbf{j} - grad\left(\tfrac{1}{c}\tfrac{\partial\Phi}{\partial t} + div\mathbf{A}\right) \ . \end{gathered} \tag{4}$$

We can see that the potentials satisfy wave equations (with sources), providing we impose the condition

$$\frac{1}{c}\frac{\partial\Phi}{\partial t}+div\mathbf{A}=0 \tag{5}$$

(subsidiary condition); indeed, the continuity equation (charge conservation) reduces the number of solution (and source) parameters to three. Equation (5) is called the Lorenz gauge.

Maxwell's equations have an important symmetry, called the gauge symmetry, related to the charge conservation. Let us introduce the notations $\square=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta$, $L=\frac{1}{c}\frac{\partial\Phi}{\partial t}+div\mathbf{A}$ and $C=\frac{\partial\rho}{\partial t}+div\mathbf{j}$; the equations for the potentials can be written as

$$\square\Phi=4\pi\rho+\frac{1}{c}\frac{\partial L}{\partial t}\ ,\ \square\mathbf{A}=\frac{4\pi}{c}\mathbf{j}-gradL\ ; \tag{6}$$

the Lorenz condition is $L=0$ and the charge conservation is $C=0$; we note that $C=0$ from equations (6) even if $L\neq 0$. The fields given by equations (3) and equations (6) for potentials are invariant under the transformation

$$\mathbf{A}=\mathbf{A}'+grad\chi\ ,\ \Phi=\Phi'-\frac{1}{c}\frac{\partial\chi}{\partial t}\ , \tag{7}$$

where χ is an arbitrary function, but the Lorenz formation becomes

$$L=L'-\square\chi \tag{8}$$

and the charge conservation is

$$C'=C+\frac{1}{4\pi c}\square\square\chi=\frac{1}{4\pi c}\square\square\chi=0\ ; \tag{9}$$

it follows that the charge conservation requires $\square\chi=0$ and the invariance of the Lorenz condition ($L=L'$). The transformation given by equations (7) with the condition $\square\chi=0$ is called the gauge transformation.

If $\rho=0$ (and $div\mathbf{j}=0$) then the Lorenz condition becomes $div\mathbf{A}=0$. This is called the Coulomb gauge. Since $\square\Phi=0$ in this case, we may use the gauge function $\chi=\Phi$, which leaves the potentials ($\Phi'=0$, $\mathbf{A}'=\mathbf{A}$), the fields and the equations, including the condition $div\mathbf{A}=0$, unchanged. In vacuum $\rho=0$ implies $\mathbf{j}=0$; then, we are in the presence of electromagnetic radiation.

From the two equations (1) in the second raw we get the law of energy conservation

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(E^2+H^2\right)+\frac{c}{4\pi}div\left(\mathbf{E}\times\mathbf{H}\right)=\mathbf{jE} \tag{10}$$

and the law of momentum conservation

$$\rho E_i+\frac{1}{c}\left(\mathbf{j}\times\mathbf{H}\right)_i+\frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H})_i=\partial_j\sigma_{ij}\ , \tag{11}$$

where

$$\sigma_{ij}=\frac{1}{4\pi}\left[E_iE_j+H_iH_j-\frac{1}{2}\delta_{ij}\left(E^2+H^2\right)\right] \tag{12}$$

is the tensor of the electromagnetic stress. In equation (10)

$$\mathcal{E}=\frac{1}{8\pi}\left(E^2+H^2\right) \tag{13}$$

is the energy density,

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E}\times\mathbf{H}\right) \tag{14}$$

is the energy flux density (energy flow, Poynting vector) and $\mathbf{jE}$ is the density of mechanical work per unit time. In equation (11) $\rho\mathbf{E}+\frac{1}{c}\mathbf{j}\times\mathbf{H}$ is the Lorentz force density and $\mathbf{g}=\frac{1}{4\pi c}(\mathbf{E}\times\mathbf{H})=\mathbf{S}/c^2$ is the momentum density. We note that the spatial variation of the electromagnetic stress generates the time change of both the electromagnetic momentum $\mathbf{g}$ and the momentum of the charges and currents (Lorentz force). The density of the interaction energy of the electromagnetic field with charges and currents can be obtained from $\mathbf{jE}$.

2.2 Relativist notation

We use the contravariant position vector $x^\mu = (ct,\, x^i)$, $i = 1, 2, 3$ $(x,\, y,\, z)$, $\mu = 0, 1, 2, 3$, and the covariant position vector $x_\mu = (ct,\, -x^i)$, with $x^0 = x_0 = ct$ (metrics $g_{\mu\nu} = g^{\mu\nu} = (+,-,-,-)$); similarly, the derivatives are denoted by $\partial^\mu = \partial/\partial x_\mu = (\frac{\partial}{c\partial t},\, -grad)$ and $\partial_\mu = \partial/\partial x^\mu = (\frac{\partial}{c\partial t},\, grad)$; similar notations are used for any vector or tensor. Also, we use the notation

$$A^{\mu\nu} = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & -b_3 & b_2 \\ -a_2 & b_3 & 0 & -b_1 \\ -a_3 & -b_2 & b_1 & 0 \end{pmatrix} = (\mathbf{a},\mathbf{b}) \tag{15}$$

and $A_{\mu\nu} = (-\mathbf{a},\mathbf{b})$ for antisymmetrical tensors; we can see that $\mathbf{a}$ is a polar vector, while $\mathbf{b}$ can be expressed as an axial vector.

The charge current is $j^\mu = (c\rho,\, \mathbf{j})$ and the equation of charge conservation reads $\partial_\mu j^\mu = 0$; the electromagnetic potential is $A^\mu = (\Phi,\, \mathbf{A})$ and the fields are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (equations (3)), *i.e.* $F^{\mu\nu} = (-\mathbf{E},\mathbf{H})$; the homogeneous Maxwell equations follow from the identity $\partial_\rho F_{\mu\nu}+\partial_\mu F_{\nu\rho}+\partial_\nu F_{\rho\mu} = 0$, while the inhomogeneous Maxwell equations read $\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c}j^\mu$. The energy-momentum tensor (computed like the derivation of the hamiltonian from the mechanical action)

$$T^{\mu\nu} = \frac{1}{4\pi}\left(-F^{\mu\rho}F_\rho^\nu + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{16}$$

can be written as

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \quad ; \tag{17}$$

it satisfies the equation of motion $\partial_\nu T_\mu^\nu = -\frac{1}{c}F_{\mu\nu}j^\nu$.

2.3 Photon

An electromagnetic field independent of charges and currents may be described by the Coulomb gauge $div\mathbf{A} = 0$. On changing the reference frame (by a Lorentz transformation) this condition may change, but we can always make a gauge transformation to preserve it. We are in the presence of an electromagnetic radiation. Maxwell's equations and the wave equations for potentials are invariant (covariant) under a Lorentz transformation. In addition, the wave equation for the potential $\mathbf{A}$ is invariant under spatial and temporal inversions (and spatial rotations). The potential

$\mathbf{A}$ and the electric field $\mathbf{E}$ are polar vectors, while the magnetic field and the potential change sign under temporal inversion. The wave equation for the potential $\mathbf{A}$ of radiation is separable in spatial and temporal coordinates; we can use Fourier decompositions, with wavevectors $\mathbf{k}$ and frequency $\omega = ck$ ($\omega > 0$). The Coulomb gauge indicates that the potential $\mathbf{A}$ is transverse, *i.e.* it is perpendicular to the wavevector $\mathbf{k}$; therefore it has two components, labeled by $\alpha = 1, 2$; these are called polarizations. Taking into account the symmetries, the most general form of decomposition of the potential $\mathbf{A}$ is

$$\mathbf{A} = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \mathbf{u_k} \sin \omega t \sin \mathbf{kr} \quad , \tag{18}$$

where the coefficients $\mathbf{u_k}$ are real and perpendicular to $\mathbf{k}$ and the summation is limited to half a space; V denotes the volume occupied by radiation. Aditional phases can be included, accounting for the origin of the space and the time. This is the original form of the potential used in Dirac's theory of radiation (Fermi). It is worth noting that equation (18) is not relativist invariant; it is used in the theory of radiation for interacting fields. In quantum-mechanical motion the spatial coordinates are not defined, and, also, for stationary states the time is not defined; therefore, the relativist trnasformations and invariance do not apply.

Equation (18) can also be written as a superposition of plane waves $e^{\mp i\omega t \pm i\mathbf{kr}}$ extended over the whole $\mathbf{k}$-space, where $\mathbf{u}_{-\mathbf{k}} = -\mathbf{u_k}$.

Let us give up the requirement of symmetries and write the more general field

$$\mathbf{A} = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \beta \left(\mathbf{a_k} e^{-i\omega t + i\mathbf{kr}} + \mathbf{a}^*_{\mathbf{k}} e^{i\omega t - i\mathbf{kr}} \right) \quad , \tag{19}$$

where the vectors $\mathbf{a_k}$, $\mathbf{a}^*_{\mathbf{k}}$ are perpendicular to $\mathbf{k}$ and β is a coefficient to be determined; the summation is performed over the whole space. Since $\mathbf{A}$ is real, we must have $\mathbf{a}^*_{-\mathbf{k}} = \mathbf{a_k}$. It is worth noting that this generalization involves complex coefficients $\mathbf{a_k}$, $\mathbf{a}^*_{\mathbf{k}}$.

Equation (19) can be put in a more general form. The wave equation for the potential $\mathbf{A}$ has plane waves $\mathbf{A_k}$ as elementary solutions in the infinite space (for other geometries the discussion is similar). The radiation plane waves are identified by their wavevectors $\mathbf{k}$. The Coulomb gauge shows that the electromagnetic potential is perpendicular to the wavector $\mathbf{k}$ of the plane wave, therefore the vector $\mathbf{A_k}$ has two components, denoted by $\alpha = 1, 2$ and called polarizations; or, there exist two transverse vectors (perpendicular to $\mathbf{k}$) which describe the vector potential $\mathbf{A_k}$, called polarization vectors (the condition $div\mathbf{A} = 0$, which becomes $\mathbf{kA_k} = 0$, is called the transversality condition). Therefore, the electromagnetic plane waves are identified by their wavectors $\mathbf{k}$ and their polarizations α (normal modes of the wave equation). We can write the components A^μ of the potential for a plane wave as $A^\mu_{\mathbf{k}\alpha} \sim e^\mu_{\mathbf{k}\alpha}$, where we can always chose $A^0 = 0$, $e^0 = 0$, $e^\mu = (0, \mathbf{e})$, $\mathbf{e}_{\mathbf{k}\alpha}\mathbf{k} = 0$ ($ek = 0$), $\mathbf{e}_{\mathbf{k}\alpha}\mathbf{e}^*_{\mathbf{k}\beta} = \delta_{\alpha\beta}$ and $e_{\mu\mathbf{k}\alpha} e^{\mu *}_{\mathbf{k}\beta} = -\delta_{\alpha\beta}$; for simplicity we omit sometimes the labels $\mathbf{k}\alpha$ and write simply $A \sim e$ for the vector with four components. We note that there are two polarization vectors e (each with components e^μ) for each wavevector $\mathbf{k}$, labelled by the suffix $\alpha = 1, 2$.

The wave equation for the potential A requires a time dependence of the form $e^{\pm i\omega t}$, where $\omega = ck = c \mid \mathbf{k} \mid$. We note that $\omega > 0$ and the $\pm$ sign comes from the fact that the wave equation is a second-order equation in the time variable t. We write $A \sim e^{-i\omega t + i\mathbf{kr}}$, $e^{i\omega t - i\mathbf{kr}}$, where $\omega t - \mathbf{kr} = k_\mu x^\mu$ and $k_\mu = (\omega/c, -\mathbf{k})$. We note that the second exponential is the complex-conjugate of the former, in agreement with the fact that A is a real vector. Therefore, we can write the potential as

$$A = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\alpha} \beta \left(e_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} e^{-ikx} + e^*_{\mathbf{k}\alpha} a^*_{\mathbf{k}\alpha} e^{ikx} \right) \quad , \tag{20}$$

where β is a coefficient to be determined, V is the volume of the radiation, $a_{\mathbf{k}\alpha}$ are coefficients to be determined and kx is a notation for $k_\mu x^\mu$. We note that the gauge transformation (equations (7)) is $A_\mu \longrightarrow A_\mu - \partial_\mu \chi$ and $e_\mu \longrightarrow e_\mu + ik_\mu \chi$, such that the normalization condition $ee^* = -1$ and the transversality condition $ek = 0$ are preserved under a gauge transformation, since $k^2 = k_\mu k^\mu = 0$. Also, it is worth noting that $\omega = ck > 0$ depends on k (it should be written $\omega(\mathbf{k}) = \omega(k)$). Also, we may absorb the factors $e^{-i\omega t}$ and $e^{i\omega t}$ in $a_{\mathbf{k}\alpha}$ and $a^*_{\mathbf{k}\alpha}$, respectively.

However, it is worth noting that A given by equation (20) is not a relativist vector (but its Fourier components are), up to the factor $\beta/\sqrt{V}$ ($a_{\mathbf{k}\alpha}$ may be viewed as scalars). This may raise doubts about the validity of the results, athough the final results, obtained by working with such expressions, look like being relativist invariant. The wave equation satisfied by A for radiation and the gauge invariance define this vector (and the fields) up to a scale factor, but, once accepted, this circumstance requires to work with Fourier components of the field. The Fourier components of the potential vector preserve their form under a Lorentz transformation (due to the phase kx).

Planck succeeded in deriving the thermodynamics of the radiation field (black body radiation, *i.e.* radiation at statistical equilibrium at finite temperature) by analyzing, first, the field in terms of wavevectors and polarizations (*i.e.* the Fourier components of equation (20)) and, second, by assuming that each mode $\mathbf{k}\alpha$ has an energy $n \cdot \hbar\omega$, where $n = 0, 1, 2, ...$ is any positive integer (viewed by Planck as a statistical variable); $\hbar \simeq 10^{-27} erg \cdot s$ was called quantum of mechanical action. Since the electromagnetic radiation has a momentum given by the Poynting vector and energy conservation (equations (10)-(14)), it follows immediately that there exists also a momentum $n \cdot \hbar\omega/c$ directed along the wavector $\mathbf{k}$, *i.e.* a momentum $n \cdot \hbar\mathbf{k}$. Hence, Einstein assumed that there exists a quantum of energy $\hbar\omega$ and of momentum $\hbar\mathbf{k}$, which is absorbed and emitted in atomic processes (in particular in the photoelectric effect), by the motion of the electron. This quantum of energy and momentum was called photon. This was in conjunction with the atomic energy levels assumed by Bohr, since it led to the idea that the interaction of matter with the radiation proceeds by exchange of photons and transitions between atomic levels. The existence of energy levels for matter is the essence of Quantum Mechanics; the interaction of the photons with matter was described by Dirac's theory of radiation.[1]

The existence of a quantum of energy $\varepsilon = \hbar\omega$ and a quantum of momentum $\mathbf{p} = \hbar\mathbf{k}$ for photon may lead to the idea that the photon would be a quantum-mechanical particle (wave), with the wavefunction $\psi \sim e^{-ikx}/\sqrt{V}$, where the phase may be written as $-ikx = \frac{i}{\hbar}(-\varepsilon t + \mathbf{pr})$. This would be in accordance with de Broglie's duality particle-wave and with the uncertainty Heisenberg's relations $\Delta p_x \Delta x > \hbar/2$, $\Delta\varepsilon\Delta t > \hbar/2$; energy and momentum operators $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar grad$ would then be used, respectively. It is worth noting that the wave e^{-ikx} has a (minimal) phase uncertainty π, which shows itself when using the microscope, a circumstance employed by Heisenberg in deriving the uncertainty relations; this uncertainty implies $\Delta k_x \Delta x > \pi$, $\Delta\omega\Delta t > \pi$, which are in agreement with Heisenberg's uncertainty relations concerning $\mathbf{p}$ and ε. Unfortunately, there is no hamiltonian for a Schrodinger equation in such a quantum-mechanical picture, although the relation $\omega = ck$ may be viewed as being equivalent with $\omega^2 = c^2k^2$, or $\varepsilon^2 = c^2p^2$, which may lead to the wave equation $\partial_\mu \partial^\mu \psi = 0$ for such a wavefunction; and even a probability current $j^\mu = \frac{i}{2}(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) = k^\mu = (\omega/c, \mathbf{k})$ may be defined, which is conserved trivially.

Such a quantum-mechanical picture is inappropriate. First, we note that the photon has only one energy level, so it is not amenable to be described in standard quantum-mechanical terms, which require several energy levels. For photon there is no energy to be quantized (which reflects the non-existence of a classical hamiltonian). This would mean that Quantum Mechanics is not possible for photon, or it is trivial (empty of any content).

[1] M. Apostol, *Quantum Mechanics*, Cambridge International Science Publishing, Cambrdge (2018).

As shown above, in describing the photons we need plane waves, according to the relativist theory of radiation. Although there is no space-time wavefunction for photons, and a space-time quantum-mechanical description is not possible for them, we can see that there exists an uncertainty in the phase of the plane waves of the electromagnetic field, which shows that the plane waves are more general than a quantum-mechanical wavefunction. This uncertainty in the phase of the electromagnetic field has a profound significance, because it shows that we cannot measure exactly the coordinates x^μ for a plane wave (with well-defined, exact k_μ), which is a basic tenet of Relativity. Therefore, the Theory of Relativity is not valid for quantum-mechanical plane waves; consequently it is not valid for the photon, at least in its standard formulation. In working with plane waves of electromagnetic field we must admit that the coordinates x^μ are only unphysical parameters. Of course, the final, measurable results should not depend on x.

In classical Physics we have always amounts of mechanical action much larger than $\hbar$, such that we may omit $\hbar$ and asign the particles well-defined coordinates x^μ. Most of the classical waves are associated with a material medium, which basically, at each point, has a quantum-mechanical motion which implies large amounts of action, such that we may neglect again $\hbar$ in the uncertainty relations, such that we may have well-defined coordinates for these waves. But the electromagnetic field is an exception. It is not associated with a material body (ether), such that, even if we have a classical electromagnetic field, which implies a large amount of mechanical action, this motion is not associated with material particles. Both for quantum particles or quantum waves, with, more or less, a fixed momentum, or wavevector, we cannot define coordinates x^μ, such that, in their relativistic regime, we need to view the coordinates as unphysical parameters. This is also true for quantum electromagnetic plane waves. Since the Relativity requires to view the coordinates as unphysical parameters in all these cases, it follows that we cannot do a space-time quantum-mechanical description. In particular, we cannot give a meaning to the operators $i\hbar\partial^\mu$ in these cases, because x^μ are meaningless. We can see that the so-called union of the Quantum Mechanics with the Theory of Relativity is in fact impossible in space and time. Also, a quantum-mechanical description in any other terms than space-time is equally well impossible in the context of Relativity, because the Quantum Mechanics requires any set of canonical conjugate variables to be measurable with inaccuracy, while the Relativity may require these variables to be exactly measurable and obey the Lorentz transformations.

It follows that for photons viewed as particles-waves in space and time, neither Quantum Mechanics, nor the (standard) Theory of Relativity is possible. Then, in what sense there exists the photon and how does it interact with matter?

2.4 Quantization of radiation

We turn now to equation (19), which gives the electric and magnetic fields

$$\begin{aligned} \mathbf{E} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \frac{i\beta\omega}{c}\left(\mathbf{a}_{\mathbf{k}} e^{-i\mathbf{kr}} - \mathbf{a}^*_{\mathbf{k}} e^{i\mathbf{kr}}\right) \ , \\ \mathbf{H} &= \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} i\beta\left(\mathbf{k}\times\mathbf{a}_{\mathbf{k}} e^{-i\mathbf{kr}} - \mathbf{k}\times\mathbf{a}^*_{\mathbf{k}} e^{i\mathbf{kr}}\right) \ , \end{aligned} \tag{21}$$

where the time dependence is included in the coefficients $\mathbf{a}_{\mathbf{k}}$, $\mathbf{a}^*_{\mathbf{k}}$. We get

$$\begin{aligned} \int d\mathbf{r}E^2 &= \sum_{\mathbf{k}} \frac{\beta^2\omega^2}{c^2}\left(-\mathbf{a}_{\mathbf{k}}\mathbf{a}_{-\mathbf{k}} + \mathbf{a}_{\mathbf{k}}\mathbf{a}^*_{\mathbf{k}} - \mathbf{a}^*_{\mathbf{k}}\mathbf{a}^*_{-\mathbf{k}} + \mathbf{a}^*_{\mathbf{k}}\mathbf{a}_{\mathbf{k}}\right) \ , \\ \int d\mathbf{r}H^2 &= \sum_{\mathbf{k}} \frac{\beta^2\omega^2}{c^2}\left(\mathbf{a}_{\mathbf{k}}\mathbf{a}_{-\mathbf{k}} + \mathbf{a}_{\mathbf{k}}\mathbf{a}^*_{\mathbf{k}} + \mathbf{a}^*_{\mathbf{k}}\mathbf{a}^*_{-\mathbf{k}} + \mathbf{a}^*_{\mathbf{k}}\mathbf{a}_{\mathbf{k}}\right) \end{aligned} \tag{22}$$

and, from equation (13), the total energy

$$W = \frac{1}{8\pi}\int d\mathbf{r}\,(E^2+H^2) = \sum_{\mathbf{k}}\frac{\beta^2\omega^2}{4\pi c^2}\left(\mathbf{a_k}\mathbf{a_k^*}+\mathbf{a_k^*}\mathbf{a_k}\right) \quad . \tag{23}$$

In a simlar way we can compute the momentum

$$\begin{gathered}\mathbf{G} = \frac{1}{4\pi c}\int d\mathbf{r}\,(\mathbf{E}\times\mathbf{H}) = \\ = \sum_{\mathbf{k}}\frac{\beta^2\omega}{4\pi c^2}\mathbf{k}[(\mathbf{a_k}\mathbf{a_{-k}},-\mathbf{a_{-k}}\mathbf{a_k})+\mathbf{a_k}\mathbf{a_k^*}+ \\ +\mathbf{a_k^*}\mathbf{a_k}+\left(\mathbf{a_k^*}\mathbf{a_{-k}^*},-\mathbf{a_{-k}^*}\mathbf{a_k^*}\right)] \quad ,\end{gathered} \tag{24}$$

where the terms in brackets show the ambiguities arising in calculating $\mathbf{G}$. These ambiguities arise from contributions like $\mathbf{a_k}\times(\mathbf{k}'\times\mathbf{a_{k'}})\delta_{\mathbf{k},-\mathbf{k}'}$, which may be written either as $-\mathbf{k}\mathbf{a_k}\mathbf{a_{-k}}$ or as $\mathbf{k}'\mathbf{a_{-k'}}\mathbf{a_{k'}}$. Such ambiguities occur because of the generalization given by equation (19), which includes summation over the whole space with complex coefficients (*i.e.*, from the departure of equation (19) from the classical equation (18)). We agree to keep the order of the coefficients in equation (24) and write the result as

$$\begin{gathered}\mathbf{G} = \sum_{\mathbf{k}}\frac{\beta^2\omega}{4\pi c^2}\mathbf{k}\{\tfrac{1}{2}[\mathbf{a_k},\mathbf{a_{-k}},]+ \\ +\mathbf{a_k}\mathbf{a_k^*}+\mathbf{a_k^*}\mathbf{a_k}+\tfrac{1}{2}[\mathbf{a_k^*},\mathbf{a_{-k}^*}]\} \quad ,\end{gathered} \tag{25}$$

where $[\mathbf{a_k},\mathbf{a_{-k}}]=\mathbf{a_k}\mathbf{a_{-k}}-\mathbf{a_{-k}}\mathbf{a_k}$. We note that this is a convention without any motivation.

2.5 Second quantization

In equation (23) the coefficients $\mathbf{a_k}$, $\mathbf{a_k^*}$ do not depend on time. W can be viewed as a hamiltonian, with $\mathbf{a_k},\mathbf{a_k^*}$ canonical variables. Their quantization requires to view them as operators, with the commutators

$$[\mathbf{a}_{\mathbf{k},\alpha},\mathbf{a}^+_{\mathbf{k}'\alpha'}]=\delta_{\mathbf{kk}'}\delta_{\alpha\alpha'} \;\; , \;\; [\mathbf{a}_{\mathbf{k},\alpha},\mathbf{a}_{\mathbf{k}'\alpha'}]=0 \quad , \tag{26}$$

which lead to

$$W=\sum_{\mathbf{k}}\frac{\beta^2\omega^2}{2\pi c^2}\left(\mathbf{a_k^+}\mathbf{a_k}+1\right)=\sum_{\mathbf{k}\alpha}\frac{\beta^2\omega^2}{2\pi c^2}\left(a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+1/2\right) \tag{27}$$

and

$$\mathbf{G}=\sum_{\mathbf{k}}\frac{\beta^2\omega}{2\pi c^2}\mathbf{k}\left(\mathbf{a_k^+}\mathbf{a_k}+1\right)=\sum_{\mathbf{k}\alpha}\frac{\beta^2\omega}{2\pi c^2}\mathbf{k}\left(a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+1/2\right) \quad . \tag{28}$$

The eigenstates of W are defined by non-negative integers $n_{\mathbf{k}\alpha}\geq 0$, such that the action of the occupation number $n_{\mathbf{k}\alpha}=a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}$ is

$$n_{\mathbf{k}\alpha}\mid n_{\mathbf{k}\alpha}>=a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}\mid n_{\mathbf{k}\alpha}>=n_{\mathbf{k}\alpha}\mid n_{\mathbf{k}\alpha}> \;; \tag{29}$$

it follows immediately

$$\begin{gathered}a_{\mathbf{k}\alpha}\mid n_{\mathbf{k}\alpha}>=\sqrt{n_{\mathbf{k}\alpha}}\mid n_{\mathbf{k}\alpha}-1> \;, \\ a^+_{\mathbf{k}\alpha}\mid n_{\mathbf{k}\alpha}>=\sqrt{n_{\mathbf{k}\alpha}+1}\mid n_{\mathbf{k}\alpha}+1>\end{gathered} \tag{30}$$

and the existence of a vacuum state $\mid 0_{\mathbf{k}\alpha}>$, such that $a_{\mathbf{k}\alpha}\mid 0_{\mathbf{k}\alpha}>=0$. The coefficients β are chosen such that equations (27) and (28) become

$$\begin{gathered} W=\sum_{\mathbf{k}\alpha}\hbar\omega\left(a^{+}_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+1/2\right) \ , \\ \mathbf{G}=\sum_{\mathbf{k}\alpha}\hbar\mathbf{k}\left(a^{+}_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+1/2\right) \end{gathered} \tag{31}$$

($\beta=c\sqrt{2\pi\hbar/\omega}$)); the field given by equation (19) becomes

$$\mathbf{A}=\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}c\sqrt{\frac{2\pi\hbar}{\omega}}\left(\mathbf{a}_{\mathbf{k}}e^{i\mathbf{kr}}+\mathbf{a}^{+}_{\mathbf{k}}e^{-i\mathbf{kr}}\right) \ ; \tag{32}$$

making use of the hamiltonian W given by equation (31), the operators $\mathbf{a}_{\mathbf{k}}$, $\mathbf{a}^{+}_{\mathbf{k}}$ may be seen in the Heisenberg representation, with the time dependence $e^{\mp i\omega t}$. The potential $\mathbf{A}$ becomes an operator in the space of the occupation numbers.

We note that making use of a transformation of the type

$$\begin{gathered} a_{\mathbf{k}\alpha}=\sqrt{\tfrac{\omega}{2\hbar}}q_{\mathbf{k}\alpha}+i\sqrt{\tfrac{1}{2\hbar\omega}}p_{\mathbf{k}\alpha} \ , \\ q_{\mathbf{k}\alpha}=\sqrt{\tfrac{\hbar}{2\omega}}(a_{\mathbf{k}\alpha}+a^{+}_{\mathbf{k}\alpha}) \ , \ p_{\mathbf{k}\alpha}=i\sqrt{\tfrac{\hbar\omega}{2}}(a^{+}_{\mathbf{k}\alpha}-a_{\mathbf{k}\alpha}) \ , \end{gathered} \tag{33}$$

where $[q_{\mathbf{k}\alpha},p_{\mathbf{k}'\alpha'}]=i\hbar\delta_{\mathbf{kk}'}\delta_{\alpha\alpha'}$, the energy and the momentum become

$$\begin{gathered} W=\sum_{\mathbf{k}\alpha}\left(\tfrac{1}{2}p^2_{\mathbf{k}\alpha}+\tfrac{1}{2}\omega^2q^2_{\mathbf{k}\alpha}+\hbar\omega/2\right) \ , \\ \mathbf{G}=\sum_{\mathbf{k}\alpha}\tfrac{\mathbf{k}}{\omega}\left(\tfrac{1}{2}p^2_{\mathbf{k}\alpha}+\tfrac{1}{2}\omega^2q^2_{\mathbf{k}\alpha}+\hbar\omega/2\right) \ , \end{gathered} \tag{34}$$

which show that the radiation can be expresesd as a sum of harmonic oscillators.

The quantization scheme described above is called the second quantization. It gives a sense to the notion of photon, because we can see that the energy W is a sum of integral multiples of $\hbar\omega$ and the momentum $\mathbf{G}$ is a sum of integral multiples of $\hbar\mathbf{k}$, and both $\hbar\omega$ and $\hbar\mathbf{k}$ are energy and momentum, respectively, according to the quantum-mechanical principles. However, we note that the choice of the coefficient β above (which leads to $\hbar\omega$ and $\hbar\mathbf{k}$) is arbitrary. The proper meaning of these quantities derives from the fact that in interaction processes the field $\mathbf{A}$ is multiplied by wavefunctions, which leads to factors of the form $e^{\frac{i}{\hbar}(\Delta E-\hbar\omega)t}$, $e^{\frac{i}{\hbar}(\Delta\mathbf{P}-\hbar\mathbf{k})\mathbf{r}}$, where ΔE and $\Delta\mathbf{P}$ are the change of energy and the change of momentum of the particles interacting with radiation, respectively; when integrated over time and coordinates these factors give the conservation of energy $\Delta E-\hbar\omega=0$ and momentum $\Delta\mathbf{P}-\hbar\mathbf{k}=0$. The first quantization with quantum-mechanical operators $i\hbar\frac{\partial}{\partial t}$ for energy and $-i\hbar grad$ for momentum may be used for photon, but the coordinates t and $\mathbf{r}$ are meaningless, both quantum-mechanically and relativistically.

The second quantization implies the existence of an angular coordinate φ, such that $\mid n>\sim e^{in\varphi}$, $n=-i\frac{\partial}{\partial\varphi}$ and

$$a=\sqrt{n+1}e^{-i\varphi} \ , \ a^{+}=e^{i\varphi}\sqrt{n+1} \tag{35}$$

(where the suffix $\mathbf{k}\alpha$ is left aside); we can see that the second quantization exists in an abstract space of an angular coordinate φ and an ocupation number n. The infinite energy $\sum_{\mathbf{k}\alpha}\hbar\omega/2$ (zero-point energy) is left aside, though in finite spatial regions its variation leads to the Casimir force. Similarly, the zero-point momentum is left aside.

We can write equation (32) as

$$\mathbf{A} = \sum_{\mathbf{k}\alpha} \left(\mathbf{A}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} + \mathbf{A}^*_{\mathbf{k}\alpha} a^+_{\mathbf{k}\alpha} \right) \quad , \tag{36}$$

where

$$\mathbf{A}_{\mathbf{k}\alpha} = c\sqrt{\frac{2\pi\hbar}{\omega V}} \mathbf{e}_{\mathbf{k}\alpha} e^{-i\omega t + i\mathbf{k}\mathbf{r}} \tag{37}$$

can be viewed as the photon field. This expression is not a relativist invariant expression, though the normalization condition (which includes $d\mathbf{k}/\omega$) is (up to a volume factor). Similarly, the energy-momentum tensor given by equations (31) is relativist invariant. The operators $a_{\mathbf{k}\alpha}$ can be transformed by a unitary transformation, such that their commutation relations are preserved under Lorentz transformations. The Schrodinger equation $i\hbar\dot{a} = [a, W]$ for the a-operators reduces to the trivial equality $\hbar\omega a = \hbar\omega a$. However, it is worth noting that as long as the quantized field is represented by operators it has not well-defined values as required by Relativity.

It is worth noting the normalization

$$< 1_{\mathbf{k}\alpha} \mid \mathbf{A}^*_{\mathbf{k}\alpha} a^+_{\mathbf{k}\alpha} \mathbf{A}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} \mid 1_{\mathbf{k}\alpha} >= c^2 \frac{2\pi\hbar}{\omega V} v \tag{38}$$

and the photon energy

$$< 1_{\mathbf{k}\alpha} \mid \mathbf{A}^*_{\mathbf{k}\alpha} a^+_{\mathbf{k}\alpha} \left(i\hbar \frac{\partial}{\partial t} \right) \mathbf{A}_{\mathbf{k}\alpha} a_{\mathbf{k}\alpha} \mid 1_{\mathbf{k}\alpha} >= c^2 \frac{2\pi\hbar}{\omega V} \cdot \hbar\omega \tag{39}$$

(and a similar expression for the photon momentum).

2.6 Time evolution

The time and the coordinates do not play any role in the quantization scheme described above; they are integrated out. The photon is an energy and momentum quantum in the space of the electromagnetic fields; radiation is delocalized. As long as we describe the states by means of energy and momentum, the time and the position are not determined, they are unphysical parameters, subject to Lorentz transformations. Indeed, space-like surfaces given by equation $n^\mu x_\mu = c\tau$, $n^2 = 1$ (time-like), where τ is a parameter, are unphysical, since $n^0 c\delta t - \mathbf{n}\delta\mathbf{r} = 0$ implies a velocity $v = c\frac{n^0}{n} = c\frac{\sqrt{1+|\mathbf{n}|^2}}{n} > 0$ higher than c. We may imagine that the four-dimensional space-time continuum is made of parallell space-like surfaces infinitesimally separated from one another. On such surfaces the operators may commute, *i.e.* they may be viewed as quantum-mechanical observables, but they are not relativist observables. The Quantum Mechanics remains separated from Relativity, by such an assumption.

We may imagine that the quantum-mechanical states depend on the time-position parameters. In order to preserve their scalar products the states should change by unitary (or anti-unitary) transformations; the unitay transformations are represented by hermitian operators. The operators change according to $O' = e^{-iF} O e^{iF}$, where the hermitian F is called the generator of the unitary operator $U = e^{iF}$. If F depends on coordinates, these equations may be called evolution equations. Noteworthy, they preserve the commutation (or anticommutation) relations. The dependence on coordinates can be realized by Lorentz transformations. Let us assume an infinitesimal translation δx_μ; then

$$\begin{aligned} O' = O(x_\mu + \delta x_\mu) = O + \partial^\mu O \delta x_\mu + ... = \\ = O - \tfrac{i}{\hbar}[P^\mu, O]\delta x_\mu + ... \quad , \end{aligned} \tag{40}$$

where F is written as $F = P^{\mu}\delta x_{\mu}/\hbar$. It follows

$$i\hbar\partial^{\mu}O = [P^{\mu}, O] \ , \tag{41}$$

whence $i\hbar\frac{\partial}{\partial t} = H$ and $-i\hbar grad = \mathbf{P}$, *i.e.* the quantum-mechanical operators for the hamiltonian (energy) and momentum. In particular, $i\hbar\frac{\partial O}{\partial t} = [H, O]$ is the Heisenberg evolution equation. It is preserved by the Lorentz transformations in the sense of equation (41) which is an identity. The energy and momentum make sense, as long as the time and the position are space-like. P^{μ} are called the translation generators of the Lorentz group; the components of the angular momentum (spin included) are the rotation generators.

We can now answer the question what is a photon? From equations (37) and (40) we write

$$\mathbf{A}^{*}_{\mathbf{k}\alpha}a^{+}_{\mathbf{k}\alpha} \mid 0 >= c\sqrt{\frac{2\pi\hbar}{\omega V}}\mathbf{e}^{*}_{\mathbf{k}\alpha}e^{i\omega t - i\mathbf{kr}} \mid 1_{\mathbf{k}\alpha} > \ , \tag{42}$$

whence one can see that a photon is a plane-wave electromagnetic field with a specified frequency ω, a specified wavevector $\mathbf{k}$ and a specified polarization α, which can be obtained from the wavefunction given by equation (42) by applying the quantum-mechanical operators $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar grad$, but where the time and the position have no meaning; the coordinates t and $\mathbf{r}$ are undetermined parameters on a space-like surface, where both Quantum Mechanics and the Relativity are meaningless. Both these theories are mutually "compatible" because they act in unphysical conditions. In particular, the wavefunction given by equation (42) may provide a probability current, identically conserved, which is meaningless because the coordinates are not measurable, they are undetermined parameters.

In addition, the photon is a purely quantum-mechanical "quantum" (particle), since, for instance, the wavefunction given by equation (42) goes to zero for $\hbar \longrightarrow 0$. The photons obey the Bose statistics, *i.e.* their operators satisfy commutation relations. If the electromagnetic field is high, then the occupation number is large and its variation by unity is irrelevant; then, the creation and destruction operators (q-numbers) may be viewed as (large) c-numbers and we recover the classical limit for $\hbar \longrightarrow 0$. The frequency and the wavelength being preserved, the coordinates remain undetermined parameters. Usually, such classical fields are superpositions with random phases, which account for the time moments and places of their origin; the field is incoherent. If the phases are equal (as with the laser fields), the field is coherent; in this case, it exhibits a much higher energy.

The number of phonon modes with frequency between ω and $\omega + \Delta\omega$ is of the order $V\omega^2\Delta\omega/c^3$; it follows

$$V\frac{\omega^2\Delta\omega}{c^3}\cdot\hbar\omega\cdot n \simeq V\cdot E^2 \ , \tag{43}$$

where n is the number of photons and E is th electric field (or magnetic field). If we know the electromagnetic field (energy) we can find out the number of photons; for monochromatic photons with a fixed wavevector, $\hbar\omega\cdot n \simeq V\cdot E^2$. If we measure a monochromatic radiation during a time Δt, we produce a perturbation $\Delta\omega$ which is at least of the order $\Delta\omega \simeq 1/\Delta t$; the most accurate measurement gives $n \simeq E^2c^3\Delta t/\hbar\omega^3$; for a classical field, $E^2\Delta t \gg \hbar\omega^3/c^3$, or $E^2\lambda^3 \gg \hbar/\Delta t$, where λ is the radiation wavelength; we recognize here the uncertainty in energy $E^2\lambda^3$.

3 Electrons

3.1 Electron quantum

From the classical theory we know the energy E of the relativist electron, given by

$$E^2 = p^2c^2 + m^2c^4 \quad , \tag{44}$$

where $\mathbf{p}$ is momentum and m is the electron mass. The energy should be positive, $E > 0$, but the Theory of Relativity requires a quadratic (homogeneous) form of energy-momentum. According to Quantum Mechanics, we may assign a wavefunction ψ to this electron, whose structure will be determined below, place its time-position coordinates on a space-like surface, and replace E and $\mathbf{p}$ by $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar grad$, respectively. We get the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\Delta\psi + \frac{m^2c^4}{\hbar^2}\psi = 0 \ . \tag{45}$$

This equation has an important particularity. In the classical limit $\hbar \longrightarrow 0$ there should exist a very large variation of the mechanical action, in comparison with $\hbar$, such that the corresponding energy to be comparable with the rest energy mc^2. The difficulty consists in the fact that the mechanical action changes by motion, while mc^2 is a rest energy. We can view the rest energy as arising from motion, in which case the limit $\hbar \longrightarrow 0$ may entail $m \longrightarrow 0$. Then, we should say that equation (45) would not have a classical limit. Therefore, the electron is a quantum (like the photon), and ψ should be viewed as a field. This is a basic construction in Quantum Electrodynamics and Quantum Field Theory. According to equation (45), there exist two frequencies $\pm\omega$, $\omega = \sqrt{c^2k^2+\omega_0^2}$, where $\omega_0 = mc^2/\hbar$, and the electron field should look like

$$\psi = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\left(\beta_{\mathbf{k}}c_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}} + \gamma_{\mathbf{k}}b_{\mathbf{k}}^{+}e^{i\omega t-i\mathbf{kr}}\right) \quad , \tag{46}$$

where V denotes the volume, $\beta_{\mathbf{k}}$ and $\gamma_{\mathbf{k}}$ are coefficients to be determined, $c_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are also coefficients to be specified, and ψ is, in general, complex.

Moreover, since there exist only two states for the electron - vacuum and one electron - it follows that there exist only two states, $\mid 0 >$ and $\mid 1_{\mathbf{k}} >$, and the coefficients $c_{\mathbf{k}}$ and $b_{\mathbf{k}}$ should be viewed as operators in the space of the occupation numbers $n_{\mathbf{k}} = 0, 1$ ($n_{\mathbf{k}} = c_{\mathbf{k}}^{+}c_{\mathbf{k}}$, $b_{\mathbf{k}}^{+}b_{\mathbf{k}}$) which satisfy the anticommutation relations

$$\begin{gathered}\{c_{\mathbf{k}}, c_{\mathbf{k}'}^{+}\} = \delta_{\mathbf{kk}'} \ , \ \{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = 0 \ , \\ \{b_{\mathbf{k}}, b_{\mathbf{k}'}^{+}\}a = \delta_{\mathbf{kk}'} \ , \ \{b_{\mathbf{k}}, b_{\mathbf{k}'}\} = 0\end{gathered} \tag{47}$$

and $\{c_{\mathbf{k}}, b_{\mathbf{k}'}\} = 0$, $\{c_{\mathbf{k}}, b_{\mathbf{k}'}^{+}\} = 0$, $c_{\mathbf{k}} \mid 0 >= 0$, $b_{\mathbf{k}} \mid 0 >= 0$; $c_{\mathbf{k}}$, $b_{\mathbf{k}}$ are destruction (annihilation) operators, while $c_{\mathbf{k}}^{+}$,$b_{\mathbf{k}}^{+}$ are creation operators. We can see that electrons obey the Fermi statistics (are fermions) and Pauli's exclusion principle is satisfied. The representaion $c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ can be used for fermion operators, with $\mid 0 >= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mid 1 >= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Quantum Mechanics deals with motion; including the rest mass (rest frequency ω_0) in quantum-mechanical processes has far reaching implications. Indeed, in these conditions an electron may be created from vacuum, by an energy mc^2; since the electric charge must be conserved, it follows that an accompanying electron with positive charge may also be created, simultaneously. We are led to the conclusion that electrons with positive charge should exists, *i.e.* positrons, or antielectrons (antiparticles) and electron-positron pairs can be created from vacuum, and mutually destroyed. The c-operators above are for electrons, while the b-operators are for positrons. Moreover, the c- (b-) operators may be viewed as c-numbers ("amplitudes of existence"), corresponding to fractions of electrons (positrons) in a macroscopic ensemble.

3.2 Electron spin

The electron has a 1/2-spin, according to experiment. This means that the field ψ introduced above should have two components (at least), which should transform under a Lorentz transformation as a spinor (an irreducible representation of the Lorentz group). This circumstance must be accommodated in the theory of the electron; equations (44) and (45) must include matrices. We can write equations (44) or (45) as

$$\gamma^\mu\gamma^\nu p_\mu p_\nu = \left\{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right\} p_\mu p_\nu = m^2c^2 \quad , \tag{48}$$

where γ^μ are some matrices; the antisymmetric commutator does not contribute, and we may require

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \quad , \tag{49}$$

where $g^{\mu\nu}$ is the metric tensor (with signature $(+,-,-,-)$); from equation (48) we recover the relativistic equation (44)

$$g^{\mu\nu}p_\mu p_\nu = p_\mu p^\mu = E^2/c^2 - \mathbf{p}^2 = m^2c^2 \ . \tag{50}$$

The ten conditions given by equation (49) can be satisfied by at least 4×4 matrices. We may choose Dirac's matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ ,$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \ , \ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ ; \tag{51}$$

with a shorthand notation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \ , \tag{52}$$

where $\boldsymbol{\sigma}$ are Pauli's matrices. It follows that ψ is a bispinor.

The spin shows itself when an electromagnetic field is present, *i.e.* when $p_\mu \longrightarrow p_\mu - \frac{e}{c}A_\mu$ in equation (48), where e is the electron charge and A_μ are the components of the electromagnetic potential. We have

$$\begin{aligned} &\tfrac{1}{2}[\gamma^\mu,\gamma^\nu](p_\mu - \tfrac{e}{c}A_\mu)(p_\nu - \tfrac{e}{c}A_\nu) = \\ &= \tfrac{1}{4}[\gamma^\mu,\gamma^\nu][p_\mu - \tfrac{e}{c}A_\mu, p_\nu - \tfrac{e}{c}A_\nu] = \\ &= -\tfrac{ie\hbar}{4c}[\gamma^\mu,\gamma^\nu]F_{\mu\nu} \quad , \end{aligned} \tag{53}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field. We can check that

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} = (-\mathbf{E}, \mathbf{H}) \tag{54}$$

in a matricial notation for tensors. Similarly,

1

$$\frac{1}{2}[\gamma^\mu, \gamma^\nu] = \sigma^{\mu\nu} = (\boldsymbol{\alpha}, i\boldsymbol{\Sigma}) \quad , \tag{55}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad , \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad . \tag{56}$$

Finally, equation (48) gives

$$\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right) - \frac{ie\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu} = m^2c^2 \quad , \tag{57}$$

or

$$\left(p_\mu - \frac{e}{c}A_\mu\right)\left(p^\mu - \frac{e}{c}A^\mu\right) - \frac{ie\hbar}{c}\boldsymbol{\alpha}\boldsymbol{E} + \frac{e\hbar}{c}\boldsymbol{\Sigma}\mathbf{H} = m^2c^2 \quad , \tag{58}$$

or

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} - \frac{e}{c}\Phi\right)^2 - \left(i\hbar grad + \frac{e}{c}\mathbf{A}\right)^2 - \frac{ie\hbar}{c}\boldsymbol{\alpha}\boldsymbol{E} + \frac{e\hbar}{c}\boldsymbol{\Sigma}\mathbf{H} = m^2c^2 \quad , \tag{59}$$

where the electromagnetic potential is $A^\mu = (\Phi,\, \mathbf{A})$. The last two terms on the *lhs* represent the interaction of the spin with the electromagnetic field.

3.3 Dirac equation

It is easy to see that the quadratic equation (48) above,

$$\gamma^\mu\gamma^\nu p_\mu p_\nu\psi = m^2c^2\psi \quad , \tag{60}$$

is equivalent with the linear equation

$$\gamma^\mu p_\mu\psi = mc\psi \ ; \tag{61}$$

this is Dirac's equation. It has not a quasi-classical limit ($\hbar \longrightarrow 0$), it is a quantum-mechanical equation. The field ψ given by equation (46) must be a superposition of the eigenvectors of equation (61), where $p_\mu = i\hbar\partial_\mu = (\frac{i\hbar}{c}\frac{\partial}{\partial t},\, i\hbar grad)$. We look for solutions of the form

$$\begin{pmatrix} \varphi \\ \chi \end{pmatrix} \sim e^{-i\Omega t + i\mathbf{kr}} \quad , \tag{62}$$

and find out that $\Omega = \pm\omega$, where $\omega = \sqrt{c^2k^2 + \omega_0^2}$, $\omega_0 = mc^2/\hbar$. For $\Omega = \omega$ we get immediately

$$\beta_{\mathbf{k}} = \frac{1}{\sqrt{2\omega}}u_{\mathbf{k}} \ , \ u_{\mathbf{k}} = \begin{pmatrix} \sqrt{\omega + \omega_0}w \\ \sqrt{\omega - \omega_0}(\mathbf{n}\boldsymbol{\sigma})w \end{pmatrix} \quad , \tag{63}$$

where $\mathbf{n}=\mathbf{k}/k$ and w is a unit spinor $w^*w=1$. For $\Omega=-\omega$ it is convenient to take $-\mathbf{k}$ instead of $\mathbf{k}$; we get

$$\gamma_{\mathbf{k}}=\frac{1}{\sqrt{2\omega}}v_{\mathbf{k}} \ , \ \ v_{\mathbf{k}}=\left(\begin{array}{c}\sqrt{\omega-\omega_0}(\mathbf{n}\boldsymbol{\sigma})w^{'} \\ \sqrt{\omega+\omega_0}w^{'}\end{array}\right) \ \ , \tag{64}$$

where $w^{'*}w^{'}=1$. The two bispinors $u_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}}$ and $v_{\mathbf{k}'}e^{i\omega' t-i\mathbf{k}'\mathbf{r}}$ are orthogonal to one another. The electron field (equation (46)) becomes

$$\psi=\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega V}}\left(u_{\mathbf{k}}c_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}}+v_{\mathbf{k}}b_{\mathbf{k}}^{+}e^{i\omega t-i\mathbf{kr}}\right) \ \ . \tag{65}$$

We can see that there is a spin label $\sigma=1,2$ (or $\pm$) beside $\mathbf{k}$, which can be attached to the bispinors and to the operators $c_{\mathbf{k}}$ and $b_{\mathbf{k}}$;σ labels the eigenvectors of the operator $\mathbf{n}\boldsymbol{\sigma}$. It is worth noting that, besides these two spin labels, there exist two others, which distinguish between φ and χ; they reduce to one label in the non-relativistic limit $\omega\simeq\omega_0$. This additional two-valued label appears as a consequence of the negative frequencies, which are required by Relativity. The field ψ is relativist invariant, as a bispinor (up to the factor $1/\sqrt{V}$), with a unitary transformation for the second-quantization operators.

3.4 Energy, momentum, charge

Since

$$\frac{1}{\sqrt{2\omega V}}u_{\mathbf{k}}c_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}}\mid 1_{\mathbf{k};c}>=\frac{1}{\sqrt{2\omega V}}u_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}}\mid 0_{\mathbf{k};c}> \ \ , \tag{66}$$

we may view the formation

$$\psi_u=\frac{1}{\sqrt{2\omega V}}u_{\mathbf{k}}e^{-i\omega t+i\mathbf{kr}} \tag{67}$$

as the wavefunction of the electron; and, similarly,

$$\psi_v^*=\frac{1}{\sqrt{2\omega V}}v_{\mathbf{k}}^*e^{-i\omega t+i\mathbf{kr}} \tag{68}$$

may be viewed as the wavefunction of the b-particles; we call them positrons (the suffix c, b of the states means states corresponding to the c,b-operators). Obviously, their energy is $\hbar\omega$ and their momentum is $\hbar\mathbf{k}$; they can be obtained by applying the quantum-mechanical operators $i\hbar\frac{\partial}{\partial t}$ and $-i\hbar grad$, respectively. We note that $u_{\mathbf{k}}^*u_{\mathbf{k}}=v_{\mathbf{k}}^*v_{\mathbf{k}}=2\omega$ and $u_{\mathbf{k}}^*v_{-\mathbf{k}}=0$ (the orthogonality with respect to the wavectors is obtained by integrating over space). Actually, an "electron" (electron field) is a superposition of "pure" electrons and positrons, each including in its bispinor contributions from the other. Therefore, we may view

$$W=\int d\mathbf{r}\psi^*(i\hbar\frac{\partial}{\partial t})\psi \ , \ \mathbf{P}=\int d\mathbf{r}\psi^*(-i\hbar grad)\psi \tag{69}$$

as the energy and momentum of the electron field, respectively. We get

$$\begin{aligned} W&=\sum\nolimits_{\mathbf{k}\sigma}\hbar\omega\left(c_{\mathbf{k}\sigma}^{+}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}^{+}\right) \ \ , \\ \mathbf{P}&=\sum\nolimits_{\mathbf{k}\sigma}\hbar\mathbf{k}\left(c_{\mathbf{k}\sigma}^{+}c_{\mathbf{k}\sigma}-b_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma}^{+}\right) \ \ , \end{aligned} \tag{70}$$

where the spin labels are introduced explicitly. Making use of the anticommutation relations, we get

$$W = \sum_{\mathbf{k}\sigma} \hbar\omega \left(c^+_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + b^+_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma} - 1\right) \ ,$$
$$\mathbf{P} = \sum_{\mathbf{k}\sigma} \hbar\mathbf{k} \left(c^+_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + b^+_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma} - 1\right) \ ; \tag{71}$$

W may be taken as the hamiltonian for the Heisenberg operators $c_{\mathbf{k}\sigma}$, $b_{\mathbf{k}\sigma} \sim e^{-i\omega t}$.

The formation

$$Q = \int d\mathbf{r}\psi^*\psi = \sum_{\mathbf{k}\sigma} \left(c^+_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}b^+_{\mathbf{k}\sigma}\right) =$$
$$= \sum_{\mathbf{k}\sigma} \left(c^+_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma} - b^+_{\mathbf{k}\sigma}b_{\mathbf{k}\sigma} + 1\right) \tag{72}$$

is a convenient representation for the electrical charge. We can see that the b-particles have an opposite-sign charge than the c-particles. Conventionally, we call the c-particles electrons and the b-particles positrons.

Beyond useful conventions, it remains that Relativity, combined with Quantum Mechanics, predicts the existence of the spin and the b-particles (positrons). The first quantization is limited (there is no hamiltonian), but it is supplemented with the second quantization, which provides a hamiltonian; it is not a space-time quantization. In addition, the space-time coordinates are, in fact, relegated to the status of undetermined, unphysical parameters. Integrating over undetermined space coordinates, in order to get the energy, momentum or charge, does not mean that we get necessarily meaningful results (only because the parameters are not present anymore); the results are acceptable, because they look reasonable. In any case, the electron field is global (delocalized).

The bispinors $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ are associated with the internal state of the electron. The internal state of the particles depends on the reference frame (the internal state is what we see as an internal state from a reference frame). Under a Lorentz transformation the bispinor components become a combination of themselves (they are an irreducible representation of the Lorentz group). The bispinors are responsible of the spin and the contribution of the negative frequencies. The spin is an internal angular momentum, associated with spatial rotations; the negative-frequency contribution is associated with proper Lorentz transformations. However, as long as the bispinors are quantum-mechanical, they are not determined and, consequently, they cannot be the object of relativist transformations. This is a basic contradiction between Quantum Mechanics and Relativity. We may use the electron field as a free field at most, which has well determined values, but we cannot use it in interaction. Actually, the relativist requirements are not applicable to interaction, and a consequent use of free fields in interaction leads to unphysical situations.

3.5 Interaction with radiation

If we multiply the Dirac equation (61) by γ^0 on the left, we get

$$p_0\psi = (\boldsymbol{\alpha}\mathbf{p} + mc\beta)\,\psi \ , \tag{73}$$

or

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar\boldsymbol{\alpha} grad + mc\beta)\,\psi \ , \tag{74}$$

where $\boldsymbol{\alpha}$ is given by equation (56) and β is another notation for γ^0. This equation may be viewed as a Schrodinger equation; however, with ψ derived above, equation (74) is an identity.

The hamiltonian of the field is W given by equations (69) and (71), which must be used for the Heisenberg representation of the field operators ψ ($c_{\mathbf{k}}$, $b_{\mathbf{k}}$).

In the presence of the radiation field, Dirac's equation (61) becomes

$$\gamma^{\mu}\left(p_{\mu}-\frac{e}{c}A_{\mu}\right)\psi=mc\psi \quad , \tag{75}$$

or

$$p_0\psi=(\boldsymbol{\alpha}\mathbf{p}+mc\beta)\,\psi+\frac{e}{c}\gamma^0\gamma^{\mu}A_{\mu}\psi \quad , \tag{76}$$

where A_{μ} is the potential of the electromagnetic field (and e is the electron charge); the last term in equation (76) is the energy of interaction of the electron with the radiation. In the spirit of W we get the interaction hamiltonian

$$V=\frac{e}{c}\int d\mathbf{r}\psi^{*}\gamma^0\gamma^{\mu}\psi A_{\mu} \quad . \tag{77}$$

We define the current $j^{\mu}=\psi^{*}\gamma^0\gamma^{\mu}\psi$ and introduce the notation $\overline{\psi}=\psi^{*}\gamma^0$, where γ^0 acts on the left; then

$$V=\frac{e}{c}\int d\mathbf{r}j^{\mu}A_{\mu} \quad ; \tag{78}$$

we note that j^{μ} is conserved ($\partial_{\mu}j^{\mu}=0$, from Dirac's equation); also, $\overline{u}_{\mathbf{k}}u_{\mathbf{k}}=2\omega_0$, $\overline{v}_{\mathbf{k}}v_{\mathbf{k}}=-2\omega_0$, $\overline{u}_{\mathbf{k}}v_{-\mathbf{k}}=-2c\mathbf{k}\boldsymbol{\sigma}$, $\overline{v}_{\mathbf{k}}u_{-\mathbf{k}}=2c\mathbf{k}\boldsymbol{\sigma}$. Making use of the Dirac equation (and $k_{\mu}k^{\mu}=\omega^2/c^2-\mathbf{k}^2=m^2c^2/\hbar^2$) we get

$$j_u^{\mu}=\overline{\psi}_u j^{\mu}\psi_u=\frac{1}{V}(1,\,\mathbf{v}/c) \tag{79}$$

and $j_v^{\mu}=j_u^{\mu}$, where $\psi_{u,v}$ are given by equations (67) and (68) and $\mathbf{v}=\partial E/\partial\mathbf{p}$ is the velocity. We can see that cj^{μ}may be viewed as a probability current, which, however, is conserved identically.

4 Interaction

4.1 Interaction of electrons with radiation

In the space of the occupation numbers we define the hamiltonian of the photons

$$H_p=\sum_{\mathbf{q}\alpha}\hbar\omega a_{\mathbf{q}\alpha}^{+}a_{\mathbf{q}\alpha} \quad , \tag{80}$$

the hamiltonian of the electrons

$$H_e=\sum_{\mathbf{k}\beta}\hbar\Omega\left(c_{\mathbf{k}\beta}^{+}c_{\mathbf{k}\beta}+b_{\mathbf{k}\beta}^{+}b_{\mathbf{k}\beta}\right) \quad , \tag{81}$$

the interaction hamiltonian

$$V=\frac{e}{c}\int d\mathbf{r}\overline{\psi}\gamma^{\mu}\psi A_{\mu} \quad , \tag{82}$$

the photon field

$$A_{\mu}=\sum_{\mathbf{q}\alpha}c\sqrt{\frac{2\pi\hbar}{\omega V}}\left(e_{\mu\mathbf{q}\alpha}a_{\mathbf{q}\alpha}e^{-i\omega t+i\mathbf{q}\mathbf{r}}+e_{\mu\mathbf{q}\alpha}^{*}a_{\mathbf{q}\alpha}^{+}e^{i\omega t-i\mathbf{q}\mathbf{r}}\right) \quad , \tag{83}$$

the electron field

$$\psi = \sum_{\mathbf{k}\beta} \frac{1}{\sqrt{2\Omega V}} \left(u_{\mathbf{k}\beta} c_{\mathbf{k}\beta} e^{-i\Omega t + i\mathbf{kr}} + v_{\mathbf{k}\beta} b^+_{\mathbf{k}\beta} e^{i\Omega t - i\mathbf{kr}} \right) \tag{84}$$

and the current $j^\mu = \overline{\psi}\gamma^\mu\psi$. In the formulae written above $\omega = cq$, $\Omega = \sqrt{c^2k^2 + \omega_0^2}$, $\omega_0 = mc^2/\hbar$, $\alpha = 1, 2$ is the polarization label, $\beta = 1, 2$ is the spin label and the other notations are the usual ones. We introduce additional notations

$$\gamma^\mu e_{\mu\mathbf{q}\alpha} = \gamma_{\mathbf{q}\alpha} \ , \ \ \gamma^\mu e^*_{\mu\mathbf{q}\alpha} = \gamma'_{\mathbf{q}\alpha} \tag{85}$$

and $\mathbf{k}^\pm = \mathbf{k} \pm \mathbf{q}$, $\Omega^\pm = \sqrt{c^2k^{\pm 2} + \omega_0^2}$. We view all the second-quantization operators in the Heisenberg picture (representation) with the free hamiltonians $H_{p,e}$; this is called the interaction picture. We introduce also the notations

$$\begin{aligned} U_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}') &= \overline{u}_{\mathbf{k}\beta}\gamma_{\mathbf{q}\alpha}u_{\mathbf{k}'\beta'} \ , \\ V_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}') &= \overline{v}_{\mathbf{k}\beta}\gamma_{\mathbf{q}\alpha}v_{\mathbf{k}'\beta'} \ , \\ W_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}') &= \overline{u}_{\mathbf{k}\beta}\gamma_{\mathbf{q}\alpha}v_{\mathbf{k}'\beta'} \ , \\ S_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}') &= \overline{v}_{\mathbf{k}\beta}\gamma_{\mathbf{q}\alpha}u_{\mathbf{k}'\beta'} \end{aligned} \tag{86}$$

and the same notations with prime for $\gamma_{\mathbf{q}\alpha}$ replaced by $\gamma'_{\mathbf{q}\alpha}$. The interaction has eight terms, given below:

$$\begin{aligned} V_1 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^- V}}U_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}^-)c^+_{\mathbf{k}\beta}c_{\mathbf{k}^-\beta'}a_{\mathbf{q}\alpha} \ , \\ V_2 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^- V}}W_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};-\mathbf{k}^-)c^+_{\mathbf{k}\beta}b^+_{-\mathbf{k}^-\beta'}a_{\mathbf{q}\alpha} \ , \\ V_3 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^+ V}}U'_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}^+)c^+_{\mathbf{k}\beta}c_{\mathbf{k}^+\beta'}a^+_{\mathbf{q}\alpha} \ , \\ V_4 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^+ V}}W'_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};-\mathbf{k}^+)c^+_{\mathbf{k}\beta}b^+_{-\mathbf{k}^+\beta'}a^+_{\mathbf{q}\alpha} \ , \\ V_5 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^+ V}}S_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};-\mathbf{k}^+)b_{\mathbf{k}\beta}c_{-\mathbf{k}^+\beta'}a_{\mathbf{q}\alpha} \ , \\ V_6 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^+ V}}V_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}^+)b_{\mathbf{k}\beta}b^+_{\mathbf{k}^+\beta'}a_{\mathbf{q}\alpha} \ , \\ V_7 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^- V}}S'_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};-\mathbf{k}^-)b_{\mathbf{k}\beta}c_{-\mathbf{k}^-\beta'}a^+_{\mathbf{q}\alpha} \ , \\ V_8 &= e\sqrt{\tfrac{\pi\hbar}{2\omega\Omega\Omega^- V}}V'_{\beta\alpha\beta'}(\mathbf{k},\mathbf{q};\mathbf{k}^-)b_{\mathbf{k}\beta}b^+_{\mathbf{k}^-\beta'}a^+_{\mathbf{q}\alpha} \ . \end{aligned} \tag{87}$$

It is worth noting that we have to solve, in fact, the electron-photon coupled equations of motion

$$\begin{aligned} \gamma^\mu p_\mu\psi - mc\psi &= \tfrac{e}{c}\gamma^\mu A_\mu\psi \ , \\ \tfrac{1}{c^2}\ddot{A}_\mu - \Delta A_\mu &= \tfrac{4\pi}{c}\overline{\psi}\gamma_\mu\psi \ , \end{aligned} \tag{88}$$

which amount to non-linear equations.

4.2 Interaction effects

Let us imagine a state with electrons and photons without interaction; it is an eigenstate of the free hamiltonian $H_0 = H_p + H_e$. The interaction creates and destroys photons and electrons, such that its effect is another state with free photons and electrons in various other individual states; this final state is also an eigenstate of the free hamiltonian. During the interaction process the interacting particles are free particles; the interaction, the particles and their states are the same thing. It follows that the electron-photon interaction is meaningless. This is the sense of nullifying (zeroing) the electron charge by Landau's pole. This circumstance arises from the description of the quantum-mechanical interaction in terms of (relativist) free fields.

In such scattering experiments (decay including) we are interested in time effects. We assume that at the initial moment of time $t \longrightarrow -\infty$ the interaction is absent and we have an initial state of free electrons and photons ψ_i (incoming wave). This state evolves gradually to the output state ψ_o at the final moment of time $t \longrightarrow +\infty$, when the interaction is again absent and the output state ψ_o is a state of free electrons and photons (outgoing wave). This is a typical scattering experiment. We endow the interaction $V(t)$ with an exponential factor $e^{-\alpha|t|}$,

$$V(t) = e^{\frac{i}{\hbar}H_0 t} V e^{-\frac{i}{\hbar}H_0 t} e^{-\alpha|t|} \quad , \tag{89}$$

where $\alpha \longrightarrow 0^+$; this is the adiabatic introduction (and removal) of the interaction. Then we ask what is the probability of finding a final state ψ_f of free electrons and photons in the output state ψ_o, which is a superposition of free states (particles); the amplitude of this probability is (ψ_f, ψ_o). The interpretation of $\alpha = 1/T$ as the inverse of the (long) duration T of the interaction is essential for getting formally meaningful results. We emphasize that the scattering problems, where the presence of the adiabatic exponent α is necessary, is very different from the problem of stationary solutions; its treatment is possible only within the framework of the interaction picture.

The state ψ evolves in time according to the equation

$$i\hbar \frac{\partial \psi}{\partial t} = V(t)\psi \ ; \tag{90}$$

its solution (with the initial condition ψ_i) is

$$\psi(t) = \psi_i - \frac{i}{\hbar} \int_{-\infty}^{t} dt_1 V(t_1)\psi(t_1) \ ; \tag{91}$$

the output state is given by

$$\psi_o = \psi_i - \frac{i}{\hbar} \int_{-\infty}^{\infty} dt_1 V(t_1)\psi(t_1) \ ; \tag{92}$$

or, by iteration,

$$\begin{gathered} \psi_o = \psi_i - \tfrac{i}{\hbar} \textstyle\int_{-\infty}^{\infty} dt_1 V(t_1)\psi_i + \\ + \left(-\tfrac{i}{\hbar}\right)^2 \textstyle\int_{-\infty}^{\infty} dt_1 V(t_1) \int_{-\infty}^{t_1} dt_2 V(t_2)\psi_i + ... \ ; \end{gathered} \tag{93}$$

we can see that ψ_o is given by a perturbation series. The transition amplitude (ψ_f, ψ_o) is the matrix element S_{fi} of the scattering matrix S obtained from equation (93).

The perturbation series is a series in powers of e; the corresponding energy is $e^2/(\hbar/mc)$, where $\hbar/mc$ is electron's Compton wavelength; we get

$$e^2/(\hbar/mc) = \frac{e^2}{\hbar c} mc^2 \quad , \tag{94}$$

so that we may view $\alpha = \frac{e^2}{\hbar c} = 1/137$ as a measure of the stength of the electromagnetic interaction; it is called the constant of fine structure. Similarly, the effect upon the photons is

$$\frac{e^2}{(c/\omega)} = \frac{e^2}{\hbar c}\hbar\omega = \frac{e^2}{\hbar c}(\hbar\omega/mc^2)mc^2 \ . \tag{95}$$

Since $\alpha \ll 1$ we may limit ourselves to the second order of the perturbation theory.

4.3 Ambiguities and infinities

Equation (90) is not relativist invariant. It would be if the effect of $V(t)$ on ψ would be of the form $\psi \sim e^{-i\omega t}$, which would make the scattering matrix a unitary matrix; but this would mean an energy brought by interaction, which is unlikely as long as the interaction is withdrawn adiabatically, after its adiabatical introduction. We are left with the only possibility $\omega = 0$, which would nullify again the interaction. This raises serious doubts about the validity of finite-order calculations by using the perturbation series given by equation (93).

Since time t is an undetermined parameter we may write down equation (90) with a parameter τ instead of t, corresponding to space-like surfaces. Then, the formal relativist invariance is not a problem anymore (or it is already solved by such a procedure). The invariant perturbation theory can be obtained in this manner, with the very convenient Feynman propagators and diagrams. However, the parameter τ spoils any significance of time evolution.

Instead of equation (90) we can use $(E - H_0)\varphi = V\varphi$ (with relativist invariance ensured by a unitary transformation) and its Lippmann-Schwinger solution

$$\varphi_o = \varphi_i + \frac{V}{E - H_0 + i0^+}\varphi_i + ..; \tag{96}$$

this shows that the time evolution is in fact meaningless. However, equation (96) implies a unitary operator, whose determinant is equal with unity, such that the energies E are given by this unitarity condition. If the original state is degenerate, the interaction removes this degeneracy and the output state is a superposition of states, each with its own temporal factor, a combination which is not relativist invariant. This factor is neglected in the limit $t \longrightarrow +\infty$ in the matrix element which gives the transition amplitude. The computations can be restricted to processes which conserve the energy, but this would nulify again the interaction, except, possibly, in finite orders of the perturbation series. If the spectrum is continuous, the unitarity condition is an identity and the full computation of the effects of the interaction gives zero.

If we extend the t-integration to infinity, as in equation (92), the result looks as being relativist invariant, but in the integration process, which involves both the time t and the position $\mathbf{r}$, we include both space-lke points, which are legitimate, and time-like points, which are not. This would invalidate the invariant scheme of perturbations in the interaction representation.

Actually, the evolution equation (90) is not relativist invariant. It should not be, since the quantum-mechanical interaction is not subject to relativist trasformations, but the use of relativist free fields in the perturbation series associated with this equation leads to divergencies. It is impossible to describe fully interactions effects in terms of free particles.

The scattering matrix is computed by using the perturbation series given by equation (93) in finite orders. The Feynman diagrams provide a guide for computing such matrix elements. The most difficult point in such calculations is related to the quantities defined by equation (86). The basic processes investigated by such methods are Compton scattering (photons by electrons), pair production (either by electron-photon, or by photons by photons (Breit-Wheeler process)), electron-electron scattering (Moller process), electron-positron scattering (Bhabha process), including bremsstrahlung, electron-positron annihilation, electron-positron bound state (positronium), photon-photon scattering. The lowest order of the scattering matrix for most of these processes is second order (except for pair production in electron-photon scattering and for photon-photon scattering, the lowest order of the latter being four). In higher orders of the perturbation series divergencies (infinities) appear.

First, in calculating vacuum-vacuum S-matrix elements divergencies (for large momentum transfer) appear as a consequence of vacuum fluctuations, *i.e.* the creation and the absorption of particles which do not conserve energy in elementary acts of interaction; they are called virtual particles. This is a direct indication that free particles are not suitable for describing interaction effects. These divergencies, which are present already in the second order, look like infinite phase factors, and may be left aside.

Equation (82) includes self-interaction, which, in computing the S-matrix elements, should be avoided. Such contributions, which are present already in the second order, are infinite (for large momentum transfer). There is not an unambiguous way of avoiding them; on the other hand, these divergencies appear as an electron mass renormalization (electron self-energy), which leaves behind a finite result. A similar situation appears for the self-interaction of the photon, which is made finite by charge renormalization.

The profound reason for getting finite results from infinities, by renormalization, is the quantization of the fields, the existence of the commutation and anticommutation relations.

The self-energy divergencies preserve formally the electrons and the photons. There is another type of divergencies (also for large momentum transfer) which dress with interaction the elementary interaction given by equation (82); they are called vertex-part divergencies. These divergencies (which are present already in the second order), when summed up, imply a vanishing electron charge (a null interaction), known as Landau's pole.

All the divergencies discussed above are ultraviolet divergencies, associated with a large momentum transfer. There exist also infrared divergencies, associated with a low momentum transfer, which require an indefinite number of photons.

All the ultraviolet divergencies appear as a result of the asymptotic behaviour of the free-fields Green functions (propagators), which go like $1/q$ $(1/k)$ for electrons and $1/q^2$ for photons (where q is the momentum of the virtual states); it is easy to see that the contribution of the type $\int dq \cdot q^2(1/q^2)(1/q)$ which appears in the lowest order is logarithmically divergent.

Extracting finite results by renormalization in finite orders of the perturbation series amounts to treat approximately equation (90), which acquires an approximate solution of the form $\psi \sim e^{-i\omega t}$; then, the relativist invariance is fulfilled, as a trivial identity; this is the sense of the fact that covariance is a guiding principle of renormalization. Very likely, the perturbation series is divergent (an asymptotic series). In any case, the occurrence of a finite ω from a nullifying interaction is the effect of approximations which are beyond control; such results may appear for approximate transition probabilities.

4.4 External field

An external eletromagnetic field with a potential A^μ can be introduced in Dirac equation by the substitution $A^\mu \longrightarrow A^\mu + a^\mu$, where a^μ is the radiation field of the electron. The (self-) energy of the electron is modified by radiative corrections. In a static magnetic field the magnetic moment of the electron is slightly modified (anomalous magnetic moment, Schwinger); in the Coulomb field of the hydrogen atom the levels are slightly split (Lamb shift, Feynman). Both calculations involve mass renormalization.

It is worth noting that, instead of computing directly the S-matrix from equations (90) and (93), we can solve (by means of the perturbation theory) the coupled Dirac and wave equations (where the second quantization operators do not appear; their effect is taken by energy and momentum conservation). Then, the Green functions for these equations are needed (invariant functions, Schwinger method). However, such a method is suitable for stationary states, and special attention should be paid to it for scattering problems.

4.5 Conclusion

It is usually claimed that the main problem of Quantum Electrodynamics and Quantum Field Theory is the ocurrence of infinities. The origin of some of these infinities is the confusion between charges and currents, on one hand, and fields, on the other hand. This ambiguity is present in classical Maxwell equations and an infinite electron self-energy occurs also in classical theory. Therefore, this ambiguity is inescapable, it cannot be circumvented. It appears also in working with equations, not only in working with the perturbation series of the S-matrix. However, a special type of infinities appear as a consequence of working with relativist free fields in interaction problems.

Much more interesting is the possibility oferred by the quantization of extracting finite results from infinities, by renormalization; there is not an equivalent classical procedure. The running coupling constants associated with the renormalization techniques is only a modification of the problem to make it compatible with a desired solution.

The free-fields scheme leads to at least two fundamental difficulties with Quantum Electrodynamics and Quantum Field Theory, both originating in field quanta. A field quanta has a definite frequency and wavevector, therefore it is described by a plane wave. We can have a quantum-mechanical energy and momentum, a Schrodinger equation or a time evolution for Heisenberg operators are identically satisfied, we can also define a probability amplitude, by estimating the mutual content of the eigenfunctions. However, if we view the plane waves as wavefunctions, then the time and the position are completely undetermined. It is assumed that such undetermined parameters should be space-like, *i.e.* they should be meaningless in Relativity. The Relativity remains an empty scheme in these circumstances. We may attempt to give up viewing the plane waves as wavefunctions, and view them as classical functions. Then Relativity would make sense, and we may attempt to relegate the quantization to the second quantization. Unfortunately, on one hand, this is not possible, since the first quantization is still present; on the other hand, the quantum-mechanical indeterminacy remains with the field operators, which, consequently, cannot be subject to relativist transformations. This basic difficulty can be seen very clearly in the time evolution of the states with interaction (equation (90)), which either is not relativist invariant, or has unacceptable solutions.

We could give up the time evolution of the wavefunction and even the second quantization, and work with field equations (the quantization would be ensured by energy and momentum conservation). However, the interpretation of the results will re-open the problems described above. The profound origin of these problems is the fundamental incompatibility between Quantum Mechanics and Relativity, when working with relativist free fields in quantum-mechanical interaction problems. It may appear as curious that the purely quantum-mechanical spin arises from Relativity and the negative-frequency part of the electron vanishes either in the non-relativistic limit $c \longrightarrow \infty$ or in the classical limit $\hbar \longrightarrow 0$; however, this is valid for free fields. The reason is that free fields cannot be subject to quantum-mechanical measurements.

Although the experiments do not reflect the conflictual nature of Quantum Mechanics and Relativity, the current approaches insist upon uniting these two theories, because we interpret the results of the field interaction in terms of free fields.