

The Physics of pure reason
(Lecture 15th of the Course of Theoretical Physics)

M. Apostol

Department of Theoretical Physics, Institute of Atomic Physics,
 Magurele-Bucharest Mg-6, POBox Mg-35, Romania

email: apoma@theory.nipne.ro

Mechanics. At first sight the space is homogeneous and isotropic and the time flows uniformly. A body, viewed as a point, covers in the small duration Δt a small distance $\Delta \mathbf{r}$, where \mathbf{r} is its position at time t . Since the space and the time are uniform, in the next time interval Δt the point covers the same distance $\Delta \mathbf{r}$, with the same magnitude and the same direction as the previous one. The small quantities $\Delta \mathbf{r}$ and Δt may vary, but each time their ratio $\Delta \mathbf{r}/\Delta t$ is the same. In order to eliminate the arbitrary variation of $\Delta \mathbf{r}$ and Δt we take both $\Delta \mathbf{r}$ and Δt simultaneously to zero, *i.e.* they are so small in all cases, that we may neglect them. It follows that the motion is such that the velocity

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} \quad (1)$$

is a constant. This is Newton's Principle of Inertia. The position of the point is modified during the time, but its velocity is constant. This law identifies the velocity as the basic quantity which describes the motion. If the space, or the time, or both are not uniform, then the velocity changes in time. According to the notation introduced in equation (1) this change is written as $d\mathbf{v}/dt$. It is caused by a property of the space and the time, the presence of other bodies; we call it force, denoted by \mathbf{F} . The change may also depend on the properties of the moving body; we call this property the mass of the body, denoted by m . This is an unknown quantity. The law of the change of the velocity can therefore be written as

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F} . \quad (2)$$

This is the Newton's law of motion, known also as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} = d\mathbf{v}/dt$ is called acceleration. It is worth noting that, although the bodies are viewed in Mechanics as being point-like (or consisting of point-like bodies), the motion is made by infinitesimal distances and durations, not by points. This special contradiction permeates the whole Physics and calls for a special caution. If we give the position and the velocity at one point, then the motion is determined at the subsequent point. It follows that the equation (2) written as $m d^2\mathbf{r}/dt^2 = \mathbf{F}$ should be a second-order differential equation, *i.e.* the ratio \mathbf{F}/m should be a function of time, position and velocity at most; this equation gives the trajectory of the motion.

The Newton law of motion can be solved for a body in an external force field and for two interacting bodies; for more than two bodies it has, practically, no solution. This is an imperfection of the theory.

Electromagnetism. Let point-like electrical charges q be distributed in space with a uniform concentration n ; and let a rigid neutralizing background of charges $-q$ be also present. The

electrical charge q is an unknown quantity. Let \mathbf{u} be a local displacement; it produces a net charge density $\rho = -nq\mathit{div}\mathbf{u}$. Let us call $-4\pi nq\mathbf{u}$ electric field, denoted by \mathbf{E} . It follows

$$\mathit{div}\mathbf{E} = 4\pi\rho . \quad (3)$$

This is the Gauss law of the electric field and the first Maxwell equation.

By introducing fields, like the displacement \mathbf{u} and the electric field \mathbf{E} , the Electromagnetism departs from Mechanics. When mechanical point-like charges are used, with density of the form $\rho = q\delta(\mathbf{r})$, singular solutions are obtained, like $\mathbf{E} = q\mathbf{r}/r^3$ (Coulomb law). This contradictory framework calls for certain limitations.

We note that $\mathbf{p} = q\mathbf{u}$ is a dipole moment; $\mathbf{P} = n\mathbf{p}$ is a density of dipole moments, called polarization (and, here, $\mathbf{E} = -4\pi\mathbf{P} = -4\pi n\mathbf{p}$).

There exists a similar quantity. Indeed, apart from a displacement \mathbf{u} , a charge q with position \mathbf{r}_a may rotate about a point placed at \mathbf{r} with velocity \mathbf{v}_a . The quantity $q\mathbf{v}_a$ is a current, $\mathbf{j}_m = nq\mathbf{v}_a$ is a current density, and the quantity $(\mathbf{r}_a - \mathbf{r}) \times q\mathbf{v}_a$ would be similar to \mathbf{p} , provided we divide it by a velocity c .

It is worth noting that the velocity c is a universal velocity, independent of the charges, or the space, time, etc. It is an unknown quantity.

We call $\mathbf{m} = (\mathbf{r}_a - \mathbf{r}) \times q\mathbf{v}_a/2c$ magnetic moment, and $\mathbf{M} = (\mathbf{r}_a - \mathbf{r}) \times \mathbf{j}_m/2c$ magnetization. For small magnitudes of the rotation radius $\mathbf{r}_a - \mathbf{r}$ the magnetic moments are a distinct type of moments of point-like charges. The magnetization has the remarkable property $\mathit{curl}\mathbf{M} = \mathbf{j}_m/c$. Let us call $\mathbf{B} = 4\pi\mathbf{M} = 4\pi n\mathbf{m}$ magnetic induction. It satisfies the equation

$$\mathit{curl}\mathbf{B} = \frac{4\pi}{c}\mathbf{j}_m = 4\pi\mathit{curl}\mathbf{M} . \quad (4)$$

The magnetization current has the property $\mathit{div}\mathbf{j}_m = 0$. But there exists another current, corresponding to the displacement \mathbf{u} , given by $\mathbf{j} = nq\dot{\mathbf{u}}$. Therefore, the above equation should be written as

$$\mathit{curl}\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + 4\pi\mathit{curl}\mathbf{M} , \quad (5)$$

or, denoting $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$,

$$\mathit{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} ; \quad (6)$$

\mathbf{H} is called magnetic field. Equation (6) is the Ampere equation and, partially, the fourth Maxwell equation. At this moment there appears a contradiction between equations (3) and (6). Indeed, by the continuity equation, we get from equation (3) $\mathit{div}\dot{\mathbf{E}} = 4\pi\dot{\rho} = -4\pi\mathit{div}\dot{\mathbf{j}}$, while $\mathit{div}\mathbf{j} = 0$ from equation (6). The contradiction disappears if we add the term $\dot{\mathbf{E}}/c$ to the right of equation (6), which reads now

$$\mathit{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} . \quad (7)$$

This is Maxwell-Ampere equation and the fourth Maxwell equation. Again, a constant point-like current I leads to $\Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{I}\delta(\mathbf{r})$, where $\mathbf{H} = \mathit{curl}\mathbf{A}$, $\mathit{div}\mathbf{A} = 0$, and $\mathbf{H} = \frac{1}{c}\mathbf{I} \times \mathbf{r}/r^3$ (Biot-Savart law), which is singular in origin.

Moreover, from the definition $\mathbf{m} = (\mathbf{r}_a - \mathbf{r}) \times q\mathbf{v}_a/2c$ it follows $\mathit{div}\mathbf{m} = 0$, $\mathit{div}\mathbf{M} = 0$ and

$$\mathit{div}\mathbf{B} = 0 . \quad (8)$$

This is the Gauss equation for the magnetic induction and the second Maxwell equation. It shows that there is no magnetic charge.

We note that, if the charges would belong to a material medium, this medium disappears from equations (3) and (7), so we may view these equations as valid for vacuum; or an immaterial medium, which historically is called aether. In vacuum $\mathbf{B} = \mathbf{H}$ and equation (8) reads

$$\operatorname{div} \mathbf{H} = 0 . \quad (9)$$

Finally, we note that there exists a symmetry between \mathbf{E} and \mathbf{B} , since both are given by moments of charges (\mathbf{p} and \mathbf{m}). It is worth noting that while \mathbf{E} is related to $-4\pi n\mathbf{p}$, the magnetic induction is related to $4\pi n\mathbf{m}$. Therefore, in writing down this symmetry we should note that \mathbf{E} corresponds to $-\mathbf{B}$ and there is no magnetic charge and no associated current. Equation (7) reads

$$\operatorname{curl} \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j} + 4\pi \operatorname{curl} \mathbf{M} \quad (10)$$

for the couple \mathbf{B}, \mathbf{E} ; therefore, the corresponding equation for the couple \mathbf{E}, \mathbf{B} reads

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} . \quad (11)$$

This is the Faraday equation and the third Maxwell equation. In vacuum it reads $\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}$.

The Faraday equation can be derived in another way. According to the Helmholtz theorem the variation of the displacement \mathbf{u} is given by

$$du_i = \partial_j u_i dx_j = \frac{1}{2} (\partial_j u_i + \partial_i u_j) dx_j + \frac{1}{2} (\partial_j u_i - \partial_i u_j) dx_j . \quad (12)$$

The first bracket in equation (12) indicates a deformation, *i.e.* a change in the element of the distance $dl = (dx_i^2)^{1/2}$. The second bracket indicates a rotation; indeed, it can be written as

$$du_i^{\operatorname{rot}} = \frac{1}{2} (\partial_j u_i - \partial_i u_j) dx_j = \frac{1}{2} (\operatorname{curl} \mathbf{u} \times d\mathbf{r})_i , \quad (13)$$

or

$$d\mathbf{u}^{\operatorname{rot}} = \frac{1}{2} \operatorname{curl} \mathbf{u} \times d\mathbf{r} ; \quad (14)$$

we can see that $\operatorname{curl} \mathbf{u}$ is a rotation angle φ . It follows that $d\mathbf{u}^{\operatorname{rot}}$ is related to the rotation connected to the magnetic moment, such that the electric field is related to the magnetic induction. Indeed, from equation (14) we get

$$\operatorname{curl} \mathbf{u} = \frac{2}{dr^2} d\mathbf{r} \times d\mathbf{u}^{\operatorname{rot}} , \quad (15)$$

which can be written as

$$\begin{aligned} \operatorname{curl} \mathbf{u} &= \frac{2dt}{dr^2} d(\mathbf{r}_a - \mathbf{r}) \times \mathbf{v}_a = \frac{4cdt}{nqdr^2} d(\mathbf{r}_a - \mathbf{r}) \times \mathbf{j}_m / 2c = \\ &= \frac{cdt}{\pi nqdr^2} d\mathbf{B} . \end{aligned} \quad (16)$$

In this equation $d\mathbf{r} = d(\mathbf{r}_a - \mathbf{r})$; the variation of this quantity may be set equal to $2cdt$, such that we get

$$\operatorname{curl} \mathbf{u} = \frac{1}{4\pi nqcdt} d\mathbf{B} , \quad (17)$$

or

$$\mathit{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} . \quad (18)$$

It is worth noting that equations $\mathit{div}\mathbf{E} = 4\pi\rho$ and $\mathit{div}\mathbf{H} = 0$ (in vacuum) follow from equations

$$\begin{aligned} \mathit{curl}\mathbf{H} &= \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} , \\ \mathit{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \end{aligned} \quad (19)$$

and the continuity equation. It follows that only two (coupled) Maxwell equations are independent, which includes two unknowns vectors \mathbf{E} and \mathbf{H} (or, similarly, \mathbf{E} and \mathbf{B} , provided \mathbf{M} is known). This means that we have in fact only one unknown field vector.

As it is well known the standard way of solving the Maxwell equations is the introduction of the scalar potential Φ and the vector potential \mathbf{A} through

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \mathit{grad}\Phi , \quad \mathbf{B} = \mathit{curl}\mathbf{A} , \quad (20)$$

assuming \mathbf{M} is known. In vacuum $\mathbf{B} = \mathbf{H}$ and we get the wave equations

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\mathbf{j} , \quad \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = 4\pi\rho , \quad (21)$$

providing the Lorenz gauge

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \mathit{div}\mathbf{A} = 0 \quad (22)$$

is satisfied; this condition reduces the number of four unknowns of the potentials to three. The wave equations have also advanced solutions, which are unphysical; they are discarded. These solutions originate in the existence of the velocity c and are an imperfection of the theory.

The electrical charge is extraneous to the theory of electromagnetism. Similarly, the universal velocity c introduced in the definition of the magnetic moment, the magnetic induction and the magnetic field is arbitrary. Not only "the electrical charge is a stranger" in Electromagnetism (Einstein, Pauli), but also the universal velocity c is a stranger. The velocity c of the electromagnetic waves has been measured. It coincides with the measured speed of light, $c = 3 \times 10^{10} \text{ cm/s}$. This made Maxwell to suggest that light and electromagnetic radiation are the same thing.

The solution of the wave equations (21) is given by the Kirchhoff's (retarded) law

$$\Phi = \int d\mathbf{r}' \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|)}{|\mathbf{r} - \mathbf{r}'|} \quad (23)$$

(for the scalar potential); For $\rho = q\delta(\mathbf{r})f(t)$ we get $\Phi = qf(t - r/c)/r$, which, apart from being propagated, it is again singular at the origin.

In vacuum the Maxwell equations are

$$\begin{aligned} \mathit{div}\mathbf{E} &= 4\pi\rho , \quad \mathit{div}\mathbf{H} = 0 , \\ \mathit{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} , \quad \mathit{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} . \end{aligned} \quad (24)$$

A polarization $\mathbf{P} = nq\mathbf{u}$ appears in matter, from a charge density $-nq\mathit{div}\mathbf{u} = -\mathit{div}\mathbf{P}$; it carries a current density $\dot{\mathbf{P}}$. In addition a magnetization \mathbf{M} may appear, from a magnetization current

$c \cdot \text{curl} \mathbf{M}$, and a magnetic induction \mathbf{B} . According to the above discussion, the Maxwell equations read

$$\text{div} \mathbf{E} = 4\pi\rho - 4\pi \text{div} \mathbf{P}, \quad \text{div} \mathbf{B} = 0, \quad (25)$$

$$\text{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \text{curl} \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \dot{\mathbf{P}} + \frac{4\pi}{c} \mathbf{j} + 4\pi \text{curl} \mathbf{M},$$

or, introducing the electric displacement (induction) $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$ and using the magnetic field $\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}$,

$$\text{div} \mathbf{D} = 4\pi\rho, \quad \text{div} \mathbf{B} = 0, \quad (26)$$

$$\text{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \text{curl} \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j}.$$

Again, only the two equations (26) in the second row are independent, but now they include four unknown vectors. Additional information is necessary to reduce the number of these unknowns to two.

The displacement \mathbf{u} obeys the equation of motion

$$m\ddot{\mathbf{u}} + m\omega_c^2 \mathbf{u} + m\gamma \dot{\mathbf{u}} = q\mathbf{E}_0 + q\mathbf{E}_i \quad (27)$$

for a charge q with mass m in an external electric field \mathbf{E}_0 , where ω_c is a characteristic frequency, γ is a dissipation coefficient and $\mathbf{E}_i = -4\pi nq\mathbf{u}$ is the internal (polarization) electric field). This equation allows the definition of a polarizability α by $\mathbf{P} = \alpha\mathbf{E}_0$ and the definition of the electrical susceptibility χ_e by $\mathbf{P} = \chi_e \mathbf{E}_t$, where $\mathbf{E}_t = \mathbf{E}_0 + \mathbf{E}_i = \mathbf{E}$, as well as the definition of the dielectric function ε by $\mathbf{E}_0 = \varepsilon \mathbf{E}_t$; the external field \mathbf{E}_0 turns out to be the electric displacement \mathbf{D} . The polarizability may include also the atomic and molecular polarizability, the molecular vibrational polarizability, the molecular rotational polarizability and the Curie-Langevin orientational polarizability of the electric moments, proportional to the inverse of the temperature $1/T$. These relations establish a connection between \mathbf{D} and \mathbf{E} .

Similarly, the equation of motion of the magnetization is $\dot{\mathbf{M}} = \frac{q}{2mc} \mathbf{M} \times \mathbf{B}$, or $\dot{\mathbf{M}} = \gamma \mathbf{B} \times \mathbf{M}$, where $\gamma = \mu/\hbar$ is the gyromagnetic factor and μ is the magnetic moment. This equation does not establish a relation between \mathbf{M} and \mathbf{B} , but it indicates an eigenfrequency of the magnetization, given by the magnetic induction. The magnetic susceptibility χ_m defined by $\mathbf{M} = \chi_m \mathbf{H}$ and the magnetic permeability defined by $\mathbf{B} = \mu \mathbf{H}$ are derived by the induced magnetic moment in an external magnetic field (diamagnetic susceptibility) and the alignment of the statistical magnetic moments (Curie-Langevin law).

Relativity. The universal velocity c which appears in the Maxwell equations has far-reaching implications. First, we should note that in the time variation written as $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \frac{\partial}{\partial \mathbf{r}}$ we may view t and \mathbf{r} as independent variables, but we may also view t and \mathbf{r} as being related through the velocity \mathbf{v} . Second, the electromagnetic waves propagate such as $cdt - dl = 0$ (retarded waves), where dl is the infinitesimal distance, but they propagate also as $cdt + dl = 0$ (advanced waves), which is unphysical. Therefore, we must admit the relation $c^2 dt^2 - dl^2 = 0$ for electromagnetic waves, or $c^2 dt^2 - d\mathbf{r}^2 = 0$. This means that we may change the reference frame by a constant velocity \mathbf{v} and this relation is preserved. For bodies moving with velocity \mathbf{v} this relation should read $c^2 dt^2 - v^2 dt^2 = c^2 d\tau^2 > 0$, where τ is the time measured at rest (proper time). It follows that c is the maximum velocity in the universe. Also, it follows that the change of coordinates in referential frames, *i.e.* frames moving with constant velocities with respect to each other, should obey the condition $c^2 dt^2 - d\mathbf{r}^2 = ds^2 = \text{const}$; ds is called the element of the line of universe. The coordinate transformations which preserve this condition are the Lorentz transformations. Moreover, the Maxwell equations preserve their form under these transformations, such that the relativity principle for electromagnetic waves is satisfied. This can be seen very easily by using

four-vector notations: contravariant vectors like $x^\mu = (ct, \mathbf{r})$ and covariant vectors like $x_\mu = g_{\mu\nu}x^\nu$, where $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, otherwise zero. Similarly, the electromagnetic potentials are $A^\mu = (\Phi, \mathbf{A})$, the field tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Maxwell equations become

$$\begin{aligned}\partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} &= 0, \\ \partial_\nu F^{\mu\nu} &= -\frac{4\pi}{c}j^\mu,\end{aligned}\tag{28}$$

with the Lorentz condition $\partial_\mu A^\mu = 0$, where $j^\mu = (c\rho, \mathbf{j})$ is the current density (and $\partial_\mu = \partial/\partial x^\mu$). The equations for the potentials are $\partial_\nu \partial^\nu A^\mu - \partial^\mu (\partial_\nu A^\nu) = \frac{4\pi}{c}j^\mu$.

On the other hand, the Fermat (or Maupertuis') principle for a moving body with mass m is immediately generalized to the extremum of the action

$$S = -mc \int ds = -mc^2 \int \sqrt{1 - v^2/c^2} dt,\tag{29}$$

which leads to the lagrangian

$$L = -mc^2 \sqrt{1 - v^2/c^2},\tag{30}$$

the momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{r}} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\tag{31}$$

and the energy

$$\mathcal{E} = \mathbf{v}\mathbf{p} - L = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = mc^2 + \frac{1}{2}mv^2 + \dots,\tag{32}$$

whence we can see the rest energy mc^2 . In addition, $\mathcal{E}^2 = m^2c^4 + p^2c^2$ and $\mathcal{E}d\mathcal{E} = c^2\mathbf{p}d\mathbf{p}$, $d\mathcal{E} = \mathbf{v}d\mathbf{p}$. The variation of the action gives

$$\delta S = -mcu_\mu \delta x^\mu + mc \int \delta x^\mu \frac{du_\mu}{ds} ds,\tag{33}$$

where

$$u^\mu = \frac{dx^\mu}{ds} = \left(\frac{1}{\sqrt{1 - v^2/c^2}}, \frac{\mathbf{v}}{c\sqrt{1 - v^2/c^2}} \right)\tag{34}$$

is the four-dimensional velocity. Also, from equation (33), the four-dimensional momentum is

$$p_\mu = -\frac{\delta S}{\delta x^\mu} = mcu_\mu,\tag{35}$$

or

$$p^\mu = mcu^\mu = \left(\frac{mc}{\sqrt{1 - v^2/c^2}}, \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \right) = (\mathcal{E}/c, \mathbf{p})\tag{36}$$

(and $p^\mu p_\mu = m^2c^2$). The variation of the action in equation (33) gives also the equations of motion

$$\begin{aligned}\frac{dp^\mu}{ds} &= \frac{1}{c\sqrt{1 - v^2/c^2}} \left(\frac{1}{c} \frac{d\mathcal{E}}{dt}, \frac{d\mathbf{p}}{dt} \right) = \\ &= \frac{1}{c\sqrt{1 - v^2/c^2}} \left(\frac{1}{c} \frac{d\mathcal{E}}{dt}, \frac{d\mathbf{p}}{dt} \right) = \\ &= \frac{1}{c\sqrt{1 - v^2/c^2}} \left(\frac{1}{c} \mathbf{v}\mathbf{f}, \mathbf{f} \right) = g^\mu,\end{aligned}\tag{37}$$

where g^μ is the generalization of the force \mathbf{f} ; or $d\mathcal{E}/dt = \mathbf{v}\mathbf{f}$ and $d\mathbf{p}/dt = \mathbf{f}$. This way, the motion of the bodies satisfies the relativity principle. Therefore, the relativity principle is satisfied by both the electromagnetic equations and the mechanical equations; extended to all motions it is called the Einstein's Principle of Relativity. The specific way it works is called the Theory of Relativity. We note that, although the electromagnetic equations indicate the necessity of Lorentz transformations for coordinates, this requirement for other vectors, tensors follows from the Principle of Relativity.

As it is well known the Maxwell equations in vacuum lead to the Lorentz force $\rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{H}$ which acts upon charges and currents. Making use of $\mathbf{j}_\mu = \rho\mathbf{u}_\mu \frac{ds}{dt}$ and the field tensor $F^{\mu\nu}$ defined above this force density can be written as $\frac{1}{c}\rho F^{\mu\nu}u_\nu$, such that the equation of motion for the mass density μ is

$$\mu c \frac{du^\mu}{ds} = \frac{1}{c} \rho F^{\mu\nu} u_\nu . \quad (38)$$

We can see that the four-dimensional velocity determine the electromagnetic field through the Maxwell equations (28), and, conversely, the electromagnetic field determines the four-dimensional velocities through the equations of motion (38). Obviously, this is a self-interaction, which is unphysical. It should be avoided, by treating these equations separately, as if the charges, or the fields, would be external. The estimation of the self-interaction effects leads to the Lorentz damping, which indicates the limits of such an improper formulation of the problem. In particular, such restrictions lead to distances larger than the classical charge radius e^2/mc^2 , or Compton's wavelength $\hbar/mc = (e^2/mc^2) \frac{\hbar c}{e^2}$, where e is the charge, m is the particle mass, \hbar is Planck's constant and $e^2/\hbar c = 1/137$ is the fine structure constant. Also, the fields should be restricted to Schwinger limit $e/(e^2/mc^2)^2 = m^2 c^4/e^3$, or $mc^2/e(\hbar/mc) = (m^2 c^4/e^3) \frac{e^2}{\hbar c}$, which is a very high field. If e is the electron charge ($e = 4.8 \times 10^{10} esu$) and m is the electron mass ($m = 10^{-27}g$), the rest energy of the electron is $mc^2 = 0.5MeV$ ($1eV = 1.6 \times 10^{-12}erg$), the classical electron radius is $r_e = e^2/mc^2 = 2.8 \times 10^{-13}cm$, the Compton wavelength of the electron is $\lambda_e = \hbar/mc = 3.8 \times 10^{-11}cm$ and Schwinger's limit for the fields is $E_s = mc^2/e(\hbar/mc) = 6 \times 10^{13} esu$ ($1esu = 3 \times 10^4 V/m$). Beyond these limits the Electromagnetism, either classical or quantum-mechanical, becomes meaningless.

Quantum Mechanics. The world is made of many small, very small, particles, which we call atoms and molecules; which may have their own structure, consisting of subatomic particles, like the electron, the proton, the neutron, etc. Of course, such particles are much more numerous than the big bodies, *i.e.* bodies with dimensions at the macroscopic scale. Mainly the atomic particles move over limited small spatial regions, such that it may not only be impossible to characterize fully their motion, but it may often be not even desirable. An example is the small magnetic moments which served above to define the magnetic field. We may expect a certain indefiniteness, a certain uncertainty in characterizing the motion of the atomic and subatomic particles; for instance, we may not expect to be possible to define a trajectory of such moving particles. Rather, we may speak of waves (actually oscillations), which would characterize a mechanical state. However, the motion over a limited spatial region can be describable globally by local differential equations; consequently, it has a certain characteristic imposed by boundary conditions, which implies discrete values of the physical quantities. Indeed, the conditions satisfied at a portion of the boundary, should met the conditions satisfied at the opposite portion of the boundary. Of course, speaking in such terms we should have a wavefunction obeying certain differential equations, which defines mechanical states characterized by discrete values of the physical quantities.

Indeed, Planck realized that the electromagnetic radiation with frequency ω , exchanged by the atoms of the wall of a black body, has energies which are integral multiples of $\hbar\omega$, where $\hbar = 10^{-27}erg \cdot s$ is a universal constant, called the Planck's constant. This is an unknown. Einstein

gives individuality to this quantum of energy, which later was called photon. Rutherford suggested that the electrons moving about the nucleus in an atom have a discrete set E_n of energy levels, where n labels these levels. We arrive at speaking rather of a motion state for such particles, instead of definite mechanical quantities. An electromagnetic radiation with frequency $\omega = 10^{15} s^{-1}$ would have an energy quantum $\hbar\omega = 10^{-12} erg$, which may be exchanged with the electrons in an atom. Being so small, how would we measure it? We will measure it only indirectly, by measuring many such identical processes. It follows that the measurement has a statistical character and, also, during the measurement the measured mechanical motion would be perturbed, to a smaller or a larger extent. We can see that the motion of these atomic and subatomic particles implies quanta of energy. We say that this motion is quantized. This is the Quantum-Mechanical P

inciple of motion. The Quantum Mechanics introduces many new concepts for describing the motion, so it needs a longer discussion.

Motion is described by functions of time; any such function is decomposable in a Fourier series,

$$F(t) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} F(\omega) ; \quad (39)$$

according to the quantization idea, we can replace ω in this equation by $\omega = E/\hbar$, where $\hbar = h/2\pi$, h and \hbar being Planck's constants. However, these energies are discrete, then we should have $\omega_n = E_n/\hbar$, where n is an integer, and the integral should be replaced by a summation. We measure only energy differences, such that we should have

$$e^{-\frac{i}{\hbar}(E_m - E_n)t} \quad (40)$$

instead of $e^{-i\omega t}$, where n, m denote sets of integers; in this case $F(\omega)$ should be replaced by a quantity denoted F_{nm} , such that

$$F(t) = \sum_{n,m} e^{-\frac{i}{\hbar}(E_m - E_n)t} F_{nm} ; \quad (41)$$

this was the starting point of Heisenberg's matricial mechanics. The representation given by equation (40) is a new representation for time-dependent functions, with far-reaching implications. We note the principle that "only measurable quantities are admitted in theory" (which, in fact, is a relative principle) and the additional freedom brought by two labels instead of one in equation (41).

The quantities F_{nm} were recognised to be matrix elements. This implies that an operator F is associated to any physical quantity. F acts in a linear space of vectors φ, ψ, \dots with a scalar product of the form (φ, ψ) ; the operator F has eigenvectors φ_n and eigenvalues f_n , such that

$$F\varphi_n = f_n\varphi_n ; \quad (42)$$

the matrix elements are written as

$$F_{nm} = (\varphi_n, F\varphi_m) , \quad (43)$$

and the mean value is $F_{nn} = (\varphi_n, F\varphi_n)$, or, generally, $\overline{F} = (\varphi, F\varphi)$. We may assume that the mean values are measurable quantities and the eigenvalue f_n is the value the quantity F has in the state φ_n ; since they are real, the associated operators should be self-adjoint (hermitian). A hermitian operator has orthogonal eigenvectors for distinct eigenvalues. If there exist several eigenvectors for the same eigenvalue, that eigenvalue is degenerate, and we can orthogonalize the

degenerated eigenvectors; we may also orthonormalize the eigenvectors, *i.e.* we can impose the condition $(\varphi_n, \varphi_m) = \delta_{nm}$. The set of eigenvalues f_n is the spectrum of F (together with the corresponding eigenvectors); it may be discrete, or continuous; in the latter case the label n is continuous and we may write f_n as $f(n)$ and summations of the type \sum_n should be replaced by $\int dn$; the orthonormality condition reads in that case $(\varphi_n, \varphi_m) = \delta(n - m)$, where δ is Dirac's delta function. By \sum or \int we understand indistinctly either \sum , or \int , or both.

Since the eigenvectors are linearly independent (and orthogonal), they may be taken as a basis of expansion for any other vector

$$\psi = \sum_n c_n \varphi_n \quad , \quad (44)$$

where c_n are some complex coefficients. It follows that the vectors represent the states of the "physical system", and any state can be written as a superposition of other states; the operators act on the state vectors; the eigenvectors are also called eigenstates. This is the superposition principle of the Quantum Mechanics. The mean value is

$$\bar{F} = (\psi, F\psi) = \sum_n f_n |c_n|^2 \quad , \quad (45)$$

which means that $|c_n|^2$ is the probability to have the state φ_n in the state ψ , and $c_n = (\varphi_n, \psi)$ is the amplitude of the corresponding probability. This observation was made by Born. We must have

$$(\psi, \psi) = |\psi|^2 = \sum_n |c_n|^2 = 1 \quad (46)$$

(since φ_n are orthonormal). Similarly, φ_n is the amplitude of probability of being in state n (or φ_n), and $(\varphi_n, \varphi_n) = |\varphi_n|^2 = 1$ is the probability of being in state n , which is unity. We can see that the vector ψ defines the state up to a phase. We note also that the quantum-mechanical operators are linear operators. The expansion given by equation (44) means that the eigenvectors φ_n form a complete (or closed) set of vectors, and equation (46) is the closure equation.

The matrices (operators) do not commute with one another, in general; we write

$$FG - GF = [F, G] \neq 0 \quad (47)$$

for two operators F and G that do not commute; $[F, G]$ is called the commutator of F and G (or the commutator of F with G). It is easy to see that if two operators commute, they have common eigenvectors, and if they have common eigenvectors, they commute. This means that they have well-defined values on the same state, they may be simultaneously measured. It follows that the quantum-mechanical state is defined as the common eigenvector of the maximum number of commuting operators. This would be the maximum information we may have about the quantum-mechanical motion. The hermitian operators associated to physical quantities are also called observables.

A matrix (operator) can be diagonalized, in general; hermitian operators always. The linear transform U which diagonalizes an operator F acts like

$$U^{-1}FU = \tilde{F} \quad , \quad (48)$$

where \tilde{F} is diagonal; if F is hermitian, then $U^{-1} = U^+$ and $U^+U = 1$; then, U is called an unitary operator; it may be written as $U = e^{iS}$, where S is hermitian. A unitary transformation preserves the commutation relations.

A convenient notation has been introduced by Dirac, who has written the scalar product as a "bracket" $(\varphi, \psi) = \langle \varphi | \psi \rangle$; then, Dirac called "kets" the vectors $|\psi\rangle$ and "bras" the vectors $\langle \varphi |$. Moreover, Dirac viewed functions of the form φ_n , or $\varphi(n)$ if n is continuous, as scalar products $\langle n | \varphi \rangle$; the closure equation (46) can be written as

$$\begin{aligned} (\psi, \psi) &= \sum_n |c_n|^2 = \sum_n (c, n)(n, c) = \\ &= \sum_n \langle c | n \rangle \langle n | c \rangle = 1 \quad , \end{aligned} \quad (49)$$

whence

$$\sum_n |n\rangle \langle n| = 1 \quad , \quad (50)$$

since $\langle c | c \rangle = 1$.

1. Equation of motion. Differentiating equation (41) with respect to the time we get

$$\frac{d}{dt} F_{nm} = \frac{i}{\hbar} (E_n - E_m) F_{nm} \quad . \quad (51)$$

We may imagine that the states n are eigenvectors of the hamiltonian H ,

$$H\varphi_n = E_n\varphi_n \quad (52)$$

(or $H |n\rangle = E_n |n\rangle$); then, removing the scalar products in equation (51), we get

$$\frac{dF}{dt} = \frac{i}{\hbar} (HF - FH) = \frac{i}{\hbar} [H, F] \quad ; \quad (53)$$

this equation is general; it can also be written as

$$F(t) = e^{\frac{i}{\hbar} Ht} F(0) e^{-\frac{i}{\hbar} Ht} \quad . \quad (54)$$

The resemblance of the equation (53) with the classical equation of motion

$$\frac{dF}{dt} = \{H, F\} \quad , \quad (55)$$

where $\{ \}$ is the Poisson bracket, is striking; to the same extent to which equation (55) gives the equations of motion of Newton's Mechanics, equation (53) gives the quantum-mechanical equations of motions. The quantum-mechanical equations of motion are given by the commutator with the hamiltonian. These are the canonical equations of motion; for this reason, the unitary transformations which diagonalize hermitian operators (the observables), and preserve the commutation relations, are also called canonical transformations.

The association

$$\frac{i}{\hbar} [H, F] \rightarrow \{H, F\} \quad , \quad [H, F] = -i\hbar \{H, F\} \quad (56)$$

was noted by Dirac. It is worth noting that in the limit $\hbar \rightarrow 0$ the commutator is vanishing: the operators become classical quantities. If the operators have an intrinsic time dependence, not given by the dynamics, then their full equation of motion reads

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{i}{\hbar} [H, F] \quad , \quad (57)$$

in full analogy with the classical equations of motion.

In scalar products of the form

$$(\varphi, F(t)\psi) = (\varphi, e^{\frac{i}{\hbar}Ht} F(0) e^{-\frac{i}{\hbar}Ht} \psi) \quad (58)$$

we may write

$$\begin{aligned} (\varphi, F(t)\psi) &= (\varphi, e^{\frac{i}{\hbar}Ht} F(0) e^{-\frac{i}{\hbar}Ht} \psi) = \\ &= (e^{-\frac{i}{\hbar}Ht} \varphi, F(0) e^{-\frac{i}{\hbar}Ht} \psi) \end{aligned} \quad (59)$$

and we may imagine that the operators do not depend on the time but, instead, the state vectors do; then we write simply $F(0) = F$ and

$$\varphi(t) = e^{-\frac{i}{\hbar}Ht} \varphi ; \quad (60)$$

which leads to the equation

$$i\hbar \frac{\partial}{\partial t} \varphi(t) = H \varphi(t) ; \quad (61)$$

this equation was later called the Schrodinger equation. Time-independent operators and time-dependent state vectors of the form given by equation (60) are said to belong to the Schrodinger picture (or Schrodinger representation); while time-dependent operators and time-independent state vectors of the form given by equation (58) are said to be in the Heisenberg picture (or Heisenberg representation). If the Heisenberg picture is performed only for the non-interacting part of the hamiltonian, while the interaction is left in the Schrodinger picture, then we say that we have the interaction picture.

We note here that in the above considerations H is independent of time, as for a closed, conservative motion, and the (constant) energies do exist. The corresponding states are called stationary states. If the hamiltonian depends on the time, then the energy eigenvalues $E_n(t)$ and the corresponding eigenvectors $\varphi_n(t)$ depend on the time; although, formally we may define time-dependent frequencies $\omega(t) = [E_n(t) - E_m(t)]/h$, the quantization idea of absorbing and emitting such energy differences raises difficulties, since the absorption and emission processes are not instantaneous. However, the Poisson brackets for the classical dynamics are valid, and their quantum-mechanical counterpart with commutators is their expression for finite amounts h of mechanical action; the classical limit is achieved by these equations of motion in the limit $h \rightarrow 0$, so we may maintain the quantum-mechanical equations of motion for time-dependent hamiltonians; the only difference is that expressions like Ht (in equation (54), for instance) are replaced by $\int^t dt' H(t')$ and expressions like $E_n t$ (in equation (41), for instance) are replaced by $\int^t dt' E_n(t')$. In addition, the time-dependent Schrodinger equation is satisfied by a linear combination of wavefunctions $\varphi_n(t)$ with time-dependent coefficients.

Let us consider a particle with mass m in a potential $V(\mathbf{r})$; the hamiltonian of its motion is

$$H = \frac{1}{2m} p^2 + V(\mathbf{r}) , \quad (62)$$

where \mathbf{p} is momentum. The motion of both the position \mathbf{r} and the momentum \mathbf{p} is given by

$$\dot{\mathbf{r}} = \frac{i}{\hbar} [H, \mathbf{r}] = \frac{i}{\hbar} \left[\frac{1}{2m} p^2, \mathbf{r} \right] = \frac{i}{\hbar} \frac{1}{2m} \{ \mathbf{p}[\mathbf{p}, \mathbf{r}] + [\mathbf{p}, \mathbf{r}]\mathbf{p} \} , \quad (63)$$

$$\dot{\mathbf{p}} = \frac{i}{\hbar} [H, \mathbf{p}] = \frac{i}{\hbar} [V(\mathbf{r}), \mathbf{p}] ;$$

if in the classical limit $\hbar \rightarrow 0$ we are going to recover the Newton's equations of motion

$$\dot{\mathbf{r}} = \mathbf{p}/m , \quad \dot{\mathbf{p}} = -\frac{\partial V}{\partial \mathbf{r}} \quad (64)$$

from the above equations (according to the correspondence principle), we should have

$$[p_x, x] = -i\hbar, \quad [p_x, y] = 0, \text{ etc}; \quad (65)$$

these commutation relations should be read $[p_i, x_j] = -i\hbar\delta_{ij}$, where i, j denote cartesian components. The first Newton's equation follows immediately from the first equation (63) and the second Newton's equation follows from the second equation (63) by a series expansion. It follows that the commutation relations between canonical conjugate variables are the fundamental elements of the quantum-mechanical dynamics.

2. Uncertainty principle. The eigenvalue f_n in the eigenvalues equation $F\varphi_n = f_n\varphi_n$ (or $F|n\rangle = f_n|n\rangle$) is the quantity F we are able to measure (exactly) in state n . We can also measure the mean value \overline{F} for a state φ , which is not an eigenstate of F , but, as any mean value, this is a statistical value; which means that in state φ the quantity F has a statistical deviation δF ; this is defined as

$$\delta F = \sqrt{\overline{F^2} - \overline{F}^2}; \quad (66)$$

for a general state n the mean value of F is $\overline{F} = F_{nn}$ and the mean square value is

$$\overline{F^2} = \sum_m F_{nm}F_{mn} = \sum_m |F_{nm}|^2; \quad (67)$$

we can see that $\overline{F^2}$ and the deviation δF are related to the off-diagonal matrix elements; and

$$\delta F = \sqrt{\sum'_m |F_{nm}|^2}, \quad (68)$$

where the prime on the summation sign means $m \neq n$.

If two operators do not commute they cannot be measured simultaneously, because they have not common eigenvectors; it follows that they have statistical deviations. These deviations can be measured, because we can measure mean values. How, in what sense, can we do that? We may imagine statistical ensembles of the same "physical system", *i.e.* either we prepare the system many times in the same conditions, or prepare many systems in the same conditions; then, we proceed to measure, say, the quantity F ; we shall get values f_n with frequency w_n ; the mean value will be $\overline{F} = \sum_n f_n w_n$; then we will know that the system is in state $\varphi = \sum_n c_n \varphi_n$, where $F\varphi_n = f_n \varphi_n$ and $w_n = |c_n|^2$. We will also know that each time we measure F we force the system to be in the eigenstate φ_n , *i.e.* each time we make a measurement the state vector of the system φ becomes the eigenvector φ_n ; each time we make a measurement the state vector of the system is destroyed, and is replaced by φ_n . This is called the collapse of the state vector, produced by the measurement process. The reason for this circumstance arises from the fact that the discrete quantum quantities imply the Planck's constant h , which at macroscopic scale is very small; therefore, during measurement, which is inevitably made by means of a macroscopic apparatus, we disturb appreciably the system, and force it to be in the macroscopically measurable state φ_n . We can see that the measuring subject, *i.e.* the human being, intervenes essentially in the dynamic behaviour of the quantum systems by the measuring apparatus, though it is not our subjectivity which dictates the measurement results, but the statistical nature of the measuring act. This is the so-called statistical interpretation (meaning) of the Quantum Mechanics, or Copenhagen interpretation of the Quantum Mechanics; it was put forward by Bohr in Copenhagen and his collaborators. In this context, the separate measurement of two non-commuting operators is called "complementarity".

In the measurement process described above the mean square of the quantity F is $\overline{F^2} = \sum_n f_n^2 w_n$, where $w_n = |c_n|^2$; the quantity F has a statistical deviation. The system has a state vector $\varphi = \sum_n c_n \varphi_n$ in this case. In general, φ_n are not necessarily the eigenvectors of F ; then, the mean value of F is $\overline{F} = \sum_{nm} c_n^* c_m F_{nm}$ (and a similar expression for $\overline{F^2}$); we may view $c_n^* c_m$ as the matrix element $w_{mn} = c_n^* c_m$ of some matrix (operator) w ; then $\overline{F} = \sum_m (wF)_{mm} = \text{tr}(wF)$, where tr is the notation for the trace (die Spur) of the matrix; it is called the density matrix; we note that $\text{tr}(w) = 1$ and $w^2 = w$. If the density matrix is given by a state vector, *i.e.* if $w_{mn} = c_n^* c_m$, the system is said to be a pure system (or to be in pure state); it is describable by a state vector, its state is "well defined" (quantum-mechanically). We may imagine that the system is not well (completely) describable by itself, it may depend on other systems for instance; then, the density matrix w_{nm} is not decomposable as a product $c_n^* c_m$, but the mean value is still definable as $\overline{F} = \sum_{nm} w_{mn} F_{nm} = \text{tr}(wF)$; in this case the system is said to be in a mixed state. We note that the property $\text{tr}w = 1$ is preserved, but not the idempotency property ($w^2 \neq w$). The time-dependence of the operators F can be transferred to the density matrix w ; then, it is worth noting that the equation of motion of the density matrix is

$$\dot{w} = \frac{i}{\hbar} [w, H] \tag{69}$$

(note the order of the operators in the commutator, in comparison with equation (53) for operators F). We owe the density matrix to Landau.

The position and momentum operators \mathbf{r} and \mathbf{p} of a particle do not commute (equation (65)). It follows that they cannot be measured simultaneously. If we measure the position with infinite accuracy, the momentum is completely undefinable; and viceversa. That means that when we measure the position of a particle we disturb it, by transferring to the particle an unknown momentum; and when we measure the momentum of a particle, we disturb its position. A state with well defined momentum is the state of an electron, for instance, accelerated in a finite space region by an electric field and then let free; a light ray reflected off an electron (in an ideal microscope) can indicate the electron position. The impossibility of knowing simultaneously position and momentum, *i.e.* position and velocity, of a quantum particle, means that the quantum particles have not a trajectory; we cannot (it is impossible to) define a trajectory for quantum particles. This is a surprising and far-reaching consequence of the quantization, in striking disagreement with Newton's Mechanics.

Let $\Delta F = F - \overline{F}$ and $\Delta G = G - \overline{G}$ for two operators F and G . Let

$$((\Delta F - i\lambda\Delta G)\varphi, (\Delta F - i\lambda\Delta G)\varphi) \geq 0 \quad , \tag{70}$$

where λ is a parameter; from this equation we get easily

$$(\delta F)^2 - i\lambda\overline{C} + \lambda^2(\delta G)^2 \geq 0 \quad , \tag{71}$$

where $C = [F, G]$ is the commutator of F with G ; this equation implies

$$\delta F \cdot \delta G \geq \frac{1}{2} |\overline{C}| \quad ; \tag{72}$$

for canonical conjugate variables \mathbf{p} and \mathbf{r} $C = -i\hbar$ and we have

$$\delta p \cdot \delta r \geq \frac{1}{2} \hbar \quad , \quad \delta p_x \cdot \delta x \geq \frac{1}{2} \hbar \quad , \quad \textit{etc} \quad . \tag{73}$$

These are Heisenberg's uncertainty (indeterminacy) relations. In the quasi-classical case $\delta F \cdot \delta G \simeq |\overline{C}|/2$; in particular, if $G = H$ (the hamiltonian) and F does not depend on time, we have

$$\delta F \cdot \delta E \geq \frac{1}{2} \hbar \left| \overline{F} \right| . \quad (74)$$

The mechanical motion proceeds with a fundamental indeterminacy, which is controlled by Planck's constant of mechanical action.

From equation (74) we may note that the most accurate measurement of F implies a certain time interval δt , such that $\delta E \cdot \delta t$ is of the order \hbar ; we write

$$\delta E \cdot \delta t \simeq \hbar ; \quad (75)$$

since the minimum uncertainty in energy is $\delta E \simeq \delta v \cdot \delta p$, where v is the velocity, we have $\delta v \cdot \delta p \cdot \delta t \simeq \hbar$; since the speed of light in vacuum c is the highest possible velocity, we have $\delta p \cdot \delta t \simeq \hbar/c$, which imposes an important constraint on the measurement process of the momentum. Similarly, from $\delta p \cdot \delta x \simeq \hbar$ we get $\delta x \simeq \hbar/mc$ (which is called the Compton wavelength of the particle); and $\delta p \simeq mc$ (which is the ultrarelativistic case); as well as, $\delta E \cdot \delta x \simeq c\hbar$, and $\delta E \simeq mc^2$, $\delta t \simeq \hbar/mc^2$; for higher relativistic momenta, or energies, the Quantum Mechanics becomes Classical Mechanics. We can see that the Theory of Relativity combined with the Quantum Mechanics exhibits fundamental constraints related to space, time and energy. This observation was made by Landau and Peierls.

3. Schroedinger equation. In equation (61) we associate the differential operator $i\hbar \frac{\partial}{\partial t}$ to the hamiltonian H ,

$$H \rightarrow i\hbar \frac{\partial}{\partial t} , \quad (76)$$

and write the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi ; \quad (77)$$

it follows that we consider the state vectors as being functions of the time t . The fundamental canonical quantum-mechanical commutation relations $[p_i, x_j] = -i\hbar \delta_{ij}$ suggest a further development; we may imagine that the position operator \mathbf{r} has a continuous set of eigenvalues, denoted by the same symbol \mathbf{r} , viewed as a classical quantity; then, we have the association

$$\mathbf{p} \rightarrow -i\hbar \frac{\partial}{\partial \mathbf{r}} ; \quad (78)$$

it follows that we are led to consider the state vectors as functions $\psi(t, \mathbf{r})$ of the time t and position \mathbf{r} ; then, the hamiltonian of a particle with mass m in a potential $V(\mathbf{r})$ becomes

$$H = -\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}) , \quad (79)$$

and the Schrodinger equation becomes

$$i\hbar \frac{\partial}{\partial t} \psi = \left[-\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}) \right] \psi ; \quad (80)$$

if ψ is an eigenvector of H , corresponding to energy E , then $\psi = e^{-\frac{i}{\hbar}Et} \varphi$ and

$$\left[-\frac{\hbar^2}{2m} \Delta + V(\mathbf{r}) \right] \varphi = E\varphi ; \quad (81)$$

equation (81) is called the (time-independent) Schroedinger equation, while equations (77) and (80) may be called time-dependent Schroedinger equations. The state vectors $\psi(t, \mathbf{r})$ are called wavefunctions. It follows that we associate waves $\psi(t, \mathbf{r})$ to material particles, as de Broglie hypothesized. We note that the momentum p is related to the wavelength λ through $p = \frac{2\pi\hbar}{\lambda}$, similarly to the energy $E = \frac{2\pi\hbar}{T}$, where T is the period ($\omega = 2\pi\nu = 2\pi/T$). We can see that the Schroedinger equation is an equation for eigenfunctions φ and eigenvalues E ; its spectrum may be both continuous and discrete. The wavefunctions φ corresponding to well-defined (constant) energies describe the stationary states. The eigenfunctions of \mathbf{r} corresponding to the eigenvalue \mathbf{r}_0 are $\delta(\mathbf{r} - \mathbf{r}_0)$.

With Dirac's notations we may write $\psi(t, \mathbf{r}) = \langle \mathbf{r} | \psi(t) \rangle$, and we may see that $\psi(t, \mathbf{r})$ is the probability amplitude to get a particle which is in the state ψ at position \mathbf{r} (at time t) (or to get at \mathbf{r} the particle which is in state ψ); $|\psi(t, \mathbf{r})|^2$ is the probability to get at \mathbf{r} the particle which is in state ψ (at time t). Integrals of the form

$$\int d\mathbf{r} \psi^*(t, \mathbf{r}) \varphi(t, \mathbf{r}) \tag{82}$$

can be viewed as scalar products; the closure relation requires

$$\int d\mathbf{r} |\psi(t, \mathbf{r})|^2 = 1 \quad , \tag{83}$$

which is called the normalization condition; the orthonormality relations can be written as

$$\begin{aligned} \int d\mathbf{r} \psi_E^*(t, \mathbf{r}) \psi_{E'}(t, \mathbf{r}) &= e^{\frac{i}{\hbar}((E-E')t)} \int d\mathbf{r} \varphi_E^*(\mathbf{r}) \varphi_{E'}(\mathbf{r}) = \\ &= e^{\frac{i}{\hbar}((E-E')t)} \delta(E - E') = \delta(E - E') \quad . \end{aligned} \tag{84}$$

We note that equally well we may consider the continuous set of eigenvalues \mathbf{p} for the momentum operator \mathbf{p} ; in that case the commutation relation would imply $\mathbf{r} \rightarrow i\hbar \frac{\partial}{\partial \mathbf{p}}$; this is called the momentum representation (or \mathbf{p} -representation), in contrast with the former representation $\mathbf{p} \rightarrow i\hbar \frac{\partial}{\partial \mathbf{r}}$ which is called \mathbf{r} -representation, or position representation (or q -representation, since q is reminiscent of the coordinate notation in Classical Mechanics). Also, we note that the time t is a parameter in Quantum Mechanics, such that a "representation" of the form $t \rightarrow -i\hbar \frac{\partial}{\partial E}$ has only a limited validity and usefulness.

We note here a point related to the adjointness of the differential operators; we have

$$\begin{aligned} \int d\mathbf{r} \psi^* (-i\hbar \frac{\partial}{\partial \mathbf{r}}) \varphi &= -i\hbar \psi^* \varphi |_S + \int d\mathbf{r} (i\hbar \frac{\partial}{\partial \mathbf{r}}) \psi^* \varphi = \\ &= -i\hbar \psi^* \varphi |_S + \int d\mathbf{r} (-i\hbar \frac{\partial}{\partial \mathbf{r}} \psi)^* \varphi \quad , \end{aligned} \tag{85}$$

where S denotes the boundary of the integration domain; the integration by parts gives the transposed operator, while the adjointness is ensured by taking the complex conjugation too. The self-adjointness (hermiticity) is ensured by zero contribution of the boundary; it follows that the wavefunctions must be vanishing at the boundary (at infinity, for instance), or be periodical over the integration domain (we must impose in that case the so-called cyclic boundary conditions). In addition, the wavefunctions, as solutions of the Schroedinger equation, which is a second-order differential equation, must be continuous and have continuous derivatives (they must be differentiable); special cases of discontinuous potentials require special considerations. Since the Schroedinger equation is linear, its solutions are superpositions of wavefunctions. Of course, the wavefunctions must be single-valued.

Let us multiply Schroedinger equation (80) by ψ^* on the left and the the Schroedinger equation for ψ^* by ψ on the right, and subtract the results; we get

$$\begin{aligned} \psi^* \mid i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} \Delta \psi + V \psi , \\ -i\hbar \frac{\partial \psi^*}{\partial t} &= -\frac{\hbar^2}{2m} \Delta \psi^* + V \psi^* \mid \psi \end{aligned} \quad (86)$$

and

$$\frac{\partial}{\partial t} |\psi|^2 + \frac{i\hbar}{2m} \operatorname{div} [\operatorname{grad} \psi^* \cdot \psi - \psi^* \operatorname{grad} \psi] ; \quad (87)$$

we introduce the notations

$$\rho = |\psi|^2 \quad (88)$$

for the probability density and

$$\mathbf{j} = \frac{i\hbar}{2m} [\operatorname{grad} \psi^* \cdot \psi - \psi^* \operatorname{grad} \psi] \quad (89)$$

for the particle flow, *i.e.* the number of particles per unit time per unit area perpendicular to \mathbf{r} ; equation (87) is the continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0 , \quad (90)$$

which ensures the conservation of the number of particles; ρ may be viewed as the density of particles (at \mathbf{r} and t), *i.e.* the number of particles per unit volume (or the amount of a particle per unit volume); note that the normalization condition given by equation (83) tells that $\psi \sim 1/\sqrt{V}$, where V is the volume of the integration domain (*i.e.*, the domain where the motion exists); the quantum-mechanical wavefunction provides a fluid picture for the quantum particles. The continuity and derivability of the wavefunction ensure the physical condition of the conservation of matter.

4. Quasi-classical Quantum Mechanics. A free particle has the wavefunction $e^{-\frac{i}{\hbar}Et + \frac{i}{\hbar}\mathbf{p}\mathbf{r}}$, where E denotes its energy and \mathbf{p} is the momentum of the particle. We note that $-Et + \mathbf{p}\mathbf{r}$ is the mechanical action. The motion of the macroscopic bodies proceeds with large amounts of mechanical action, such that their wavefunction varies rapidly over distances and times. Their wavelengths and periods are very small. Moreover, the motion of the macroscopic bodies involves high potentials which vary slightly and slowly over distances and times, such that this small-wavelength and small-period characteristic of the motion of the macroscopic bodies is preserved in their motion. More, large values of \mathbf{p} have large deviations, while the \mathbf{p} -dependence of E is small. We may expand the action around a value \mathbf{p}_0 and introduce the group velocity $\mathbf{v}_0 = \partial E / \partial \mathbf{p}|_{\mathbf{p}_0}$; the superposition of the wavefunctions leads to

$$\int d\mathbf{p} e^{-\frac{i}{\hbar}Et + \frac{i}{\hbar}\mathbf{p}\mathbf{r}} \sim e^{-\frac{i}{\hbar}E_0t + \frac{i}{\hbar}\mathbf{p}_0\mathbf{r}} \delta(\mathbf{r} - \mathbf{v}_0t) , \quad (91)$$

which indicates the localization of the body. Such a superposition is called a wavepacket.

Such a picture can be applied to small particles, moving over small distances and times, providing their energy and momentum are sufficiently large, and do not change too much over distances and times. The wavefunction is a complex quantity, so its free-particle form is preserved inside a potential barrier, either spatial or temporal, where the action $-\int E dt + \int \mathbf{p} d\mathbf{r}$ becomes a complex quantity; the main contribution to the wavefunction comes from the zeros of the action, so we may estimate the imaginary part of the action, which gives the amplitude of probability for the

particle to tunnel through a potential barrier or to make a transition in an external time-varying potential. This is the tunnel effect (due to Landau, suggested by Gamow and others). We can see that it is based on the continuity of the wavefunction.

The picture described above is the quasi-classical Quantum Mechanics. In this picture \hbar is sufficiently small that it can be neglected; the commutators go to zero and the particle may be characterized by both a position and a momentum. The dynamics of the particle is purely classical, except for the tunnel effect which is reminiscent of the Quantum Mechanics. The tunnel effect may occur for macroscopic bodies too, but their action is so large that the tunneling probability is extremely small.

5. Measurement process. Let

$$\psi = \sum_n a_n \psi_n \tag{92}$$

be an expansion of ψ in a series of (orthogonal) wavefunctions ψ_n ; then

$$\int d\mathbf{r} \psi_n^* \psi = a_n ; \tag{93}$$

on the other hand,

$$\int d\mathbf{r} |\psi|^2 = \sum_n |a_n|^2 \int d\mathbf{r} |\psi_n|^2 ; \tag{94}$$

if $\int d\mathbf{r} |\psi|^2$ is the probability of being in the state ψ in the volume of integration (with proper normalization), and $\int d\mathbf{r} |\psi_n|^2$ is the probability of being in the state ψ_n in the same volume, then $|a_n|^2$ is the probability that state ψ_n be included in state ψ in the volume of integration. It follows that we can take a_n as the amplitude of probability that state ψ_n be in state ψ ; or, the extent to which the state ψ_n is included in state ψ . Therefore, the quantity

$$a = \int d\mathbf{r} \psi_1^* \psi_2 \tag{95}$$

is an expression of a measurement process: its squared modulus indicates the amount to which the state ψ_1 is measured by state ψ_2 . In a more descriptive sense, ψ_2 is the instrument (the apparatus) by which we measure the presence and the extent of ψ_1 , and the process of measurement is the integration in equation (95). Obviously, these statements are valid for any generalized coordinate.

Let us assume that we try to gauge the state ψ_1 by its momentum; then we use

$$\psi_2 = \frac{1}{\sqrt{\Delta V}} e^{\frac{i}{\hbar} \mathbf{p}' \cdot \mathbf{r}} ; \tag{96}$$

it is conceivable that our meter has several values of \mathbf{p}' distributed within the range $\Delta \mathbf{p}'$ (around some value \mathbf{p}'), so we may use

$$\psi_2 = \frac{1}{\Delta \mathbf{p}' \sqrt{\Delta V}} \int d\mathbf{p}' e^{\frac{i}{\hbar} \mathbf{p}' \cdot \mathbf{r}} ; \tag{97}$$

we get

$$a = \frac{1}{\Delta \mathbf{p}' \sqrt{\Delta V}} \int d\mathbf{p}' \int d\mathbf{r} \psi_1^* e^{\frac{i}{\hbar} \mathbf{p}' \cdot \mathbf{r}} ; \tag{98}$$

let us assume that ψ_1 has a certain \mathbf{p} to some extent $a_{\mathbf{p}}$,

$$\psi_1 = \frac{1}{\sqrt{\Delta V}} a_{\mathbf{p}} e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{r}} ; \tag{99}$$

then,

$$a = \frac{h^3}{\Delta \mathbf{p}' \Delta V} \int d\mathbf{p}' a_{\mathbf{p}'}^* \delta(\mathbf{p} - \mathbf{p}') = \frac{h^3}{\Delta \mathbf{p} \Delta V} a_{\mathbf{p}}^* . \quad (100)$$

If $\Delta \mathbf{p} \Delta V$ is very large, *i.e.* the accuracy of the measurement is very low, and our apparatus ψ_2 is a quantum-mechanical apparatus, then a is extremely small, and we have no definite result; our measurement process fails. On the contrary, if the apparatus is classical, *i.e.* $\Delta \mathbf{p} \Delta V \simeq h^3$, then $a \simeq a_{\mathbf{p}}$, which means that, indeed, we succeed to measure the state ψ_1 defined by its momentum. However, it is worth noting that the quantum-mechanical character of the product $\Delta \mathbf{p} \Delta V$ is valid for finite $\Delta \mathbf{p}$ and ΔV and large values of the product $\Delta \mathbf{p} \Delta V$.

Similarly, we can use an apparatus

$$\psi_2 = \frac{1}{\Delta E'} \int dE' e^{-\frac{i}{\hbar} E' t} \phi_2(\mathbf{r}) \quad (101)$$

for measuring a state

$$\psi_1 = e^{-\frac{i}{\hbar} E t} \phi_1(\mathbf{r}) \quad (102)$$

defined by its energy E ; it is conceivable that the measuring process takes a finite time Δt , during which we collect the results; we get

$$a = \frac{1}{\Delta E' \Delta t} \int dt \int dE' e^{\frac{i}{\hbar} (E - E') t} \int d\mathbf{r} \phi_1^*(\mathbf{r}) \phi_2(\mathbf{r}) , \quad (103)$$

or, leaving aside the spatial factor,

$$a = \frac{h}{\Delta E' \Delta t} \int dE' \delta(E - E') = \frac{h}{\Delta E \Delta t} ; \quad (104)$$

if the apparatus is quantum-mechanical, there will be large uncertainties in energy and time, and the measurement process fails (it gives a vanishingly small, indefinite result); if the apparatus is classical, the uncertainties are of the order $\Delta E \Delta t \simeq h$, and we get a definite result a ; which shows to what extent a state defined by its energy is present in the measured state ψ_1 .

6. The limits of the Quantum Mechanics. The wavefunction is not defined over regions smaller than its local wavelength or in times shorter than its momentary period. Over larger distances or longer times it is periodic, or quasi-periodic, or its period or its wavelength change in a definite way, which gives a meaning to the wavefunctions over larger distances and longer times.

The extent to which we probe an energy E' by an energy E is given by the integral

$$\int_{-\Delta t}^{\Delta t} dt e^{\frac{i}{\hbar} (E - E') t} = \frac{2 \sin[(E - E') \Delta t / \hbar]}{(E - E') / \hbar} \quad (105)$$

(similarly for probing a momentum, or other physical quantities). In the limit $\Delta t \rightarrow \infty$ this integral is proportional to $\delta(E - E')$, which would mean an exact measurement. The error is $\delta E \simeq \hbar / \Delta t$, or $\delta E / E \simeq T / \Delta t$, where T is the period. Since the quantum-mechanical periods are much shorter than the measuring times, the relative error is small.

The uncertainty principle brings an important limitation in the context of the Relativity. The relativistic energy of a particle with mass m at rest is mc^2 and the relativistic momentum of this particle at rest is mc . It follows that a particle at rest has a duration of life of at least \hbar / mc^2 and a spatial extension of at least \hbar / mc (its Compton wavelength). Below this duration and this distance the physical processes are meaningless. This argument was advanced by Landau and

Peierls. It follows that the Quantum Electrodynamics and the field theories, which attempt to describe the relativistic motion at the quantum-mechanical level, encounter a serious drawback.

Statistical Physics. Let us consider a macroscopic amount of a gas, consisting of N , $N \gg 1$, identical and independent atoms or molecules enclosed in a macroscopic volume V . These atomic particles are in a permanent motion, which may be more or less energetical. This motion may belong either to the particles viewed as material points, or to their internal structure. A similar picture may hold for other macroscopic bodies. The corresponding energy may be transferred from a body to other. We say that the macroscopic bodies have a temperature, and their constituents are in thermal agitation. The thermal agitation was pointed out by Bernoulli.

The particles in thermal agitation may collide with one another, and exchange energy in elastic collisions, or elastic processes of energy transfer. The coordinates of motion in this case vary very erratically, both in space and time, such that we cannot speak of a trajectory, or a well determined mechanical motion. The thermal agitation is not a motion in the sense given by Mechanics. We can characterize it by a probability. It is a statistical motion. The probability was introduced by Maxwell in order to describe the statistical motion.

Therefore, we should view the coordinate q and momentum p of a particle in thermal agitation as statistical variables, and associate a probability density $\rho(p, q)$ to this motion; the integration should imply $dpdq/2\pi\hbar$, according to Quantum Mechanics, which indicates a state in the phase space, *i.e.* in the (q, p) -space. If the motion is quantum-mechanical the integration should be replaced simply by a summation over states. For several degrees of freedom, or for many particles, the element of volume in the phase space is the product of the elements of volume for each degree of freedom. If normalized to number of particles, ρ is a density distribution. The ensemble of particles described by a density of probability in the phase space is a statistical ensemble.

The physical quantities of a statistical ensemble are mean values generated by the probability density. They have statistical deviations and fluctuations. Since the particles (sub-ensembles) are independent, the statistical deviation of an ensemble is given by

$$\overline{\delta f^2} = \sum_{ij} \overline{\delta f_i \delta f_j} = \sum_i \overline{\delta f_i^2} \sim N , \quad (106)$$

such that $\sqrt{\overline{\delta f^2}/\bar{f}} \sim \sqrt{N}/N = 1/\sqrt{N}$. We can see that the physical quantities are very close to their mean values for large N . This observation is called the principle of statistical independence (or law of large numbers); it gives a measure to the extent to which a statistical description is valid. It is worth noting that the sub-ensembles (particles should be independent, *e.g.* without an interaction. The interaction should be solved first, before employing a statistical description.

For independent sub-ensembles the probability is multiplicative. It is convenient to work with its logarithm, which is additive. Therefore, we introduce the entropy $\mathcal{S} = -\ln \rho$. We note that if we have many equally-probable states, the probability is small, and the entropy is large; on the contrary, if we have a small number of states, the entropy is small. Therefore, the entropy is a measure of the degree of disorder (molecular chaos). This observation was made by Boltzmann. In general, the probability may depend on the time. Let us assume that initially we prepare an ensemble in certain conditions. These conditions impose certain restrictions upon the states of the ensemble, *i.e.* only a certain, small number of states satisfy these conditions. We may say that initially we are in a state with a small entropy. Let the ensemble be set free of these conditions. For sufficiently long times its entropy, if it exists, *i.e.* if it is finite and non-vanishing, it will acquire a value which is independent of time and larger than the initial value. Therefore, the ensemble tends to a state of thermal equilibrium where its entropy is stationary and increased.

This is the Law of Increase of Entropy. It is the Statistical Principle. Temporal ensembles were pointed out by Einstein. We note that the sub-ensembles may be at thermal equilibrium but the whole ensemble, subject to external conditions, may not be, as in transport phenomena. Also, it is worth noting that all these considerations are valid as long as the sub-ensembles are independent; the interaction should be solved first, and the particles may be subject, at most, to an external field.

A statistical ensemble is defined by its volume, number of particles and other parameters, like its energy. The energy plays a special role, since it is an additive integral of motion, so the entropy should be a linear function of energy; a similar role is played by the other additive integrals of mechanical motion, like the momentum and the angular momentum (there are no more). For a given energy there exist many equally-probable states, so we may define an entropy. Then, the probability is $e^{-S(\mathcal{E})}$ (leaving aside the other variables), and we may expand in series this probability for small variation $\delta\mathcal{E}$ of the energy; we get

$$\rho = e^{-S_0 - (\partial S/\partial \mathcal{E})_0 \delta \mathcal{E} + \dots} \quad (107)$$

We may imagine that $\delta\mathcal{E}$ is the energy of a small ensemble which is in contact with the larger one, denoted by the suffix 0; we denote it by \mathcal{E} . Also, we may assume that these ensembles are at thermal equilibrium. Then, there exists a mean value of $(\partial S/\partial \mathcal{E})_0$, corresponding to the mean entropy $S = -\int \rho \ln \rho$ and the mean energy $E = \int \rho \mathcal{E}$, which is independent of time and the same for the whole ensemble, consisting of the smaller one and the larger one. We denote it by $1/T = \partial S/\partial E$ and call T temperature. It is here where the thermal equilibrium and its associated law of increase of entropy are employed. The Statistical Physics is constructed upon the Principle of Thermal Equilibrium, or upon its associated Law of Increase of Entropy. It follows that we get the probability distribution

$$\rho = \text{cons} \cdot e^{-\mathcal{E}/T} \quad (108)$$

which is called the canonical, or Gibbs, distribution. The distribution of constant energy $\rho \sim \delta(\mathcal{E} - \mathcal{E}_0)$ is called the microcanonical distribution. If we include the number of particles, we get the grand-canonical distribution $\rho \sim e^{-\mathcal{E}/T + \mu N/T}$, where μ is called the chemical potential. We note that the canonical and grand-canonical distributions can also be derived from the stationarity of the mean entropy under the conditions of a constant mean energy and a constant mean number of particles.

Let us write the normalization condition as

$$\int e^{c - \beta \mathcal{E}} = 1 \quad (109)$$

where c is a constant and $\beta = 1/T$ ($\int e^{-\beta \mathcal{E}}$ is called partition function); we note that $c - \beta E = -S$. The variation of this condition gives

$$dc - E d\beta - \beta \frac{\partial E}{\partial \lambda} d\lambda = 0 \quad (110)$$

where λ is a parameter like the volume V ; in which case $\partial E/\partial \lambda = -p$, where p is the pressure. The condition given by equation (110) can also be written as

$$d(c - \beta E) = -dS = -\beta dE - \beta p dV \quad (111)$$

or

$$dE = -p dV + T dS \quad (112)$$

We can see that the change of the entropy, *i.e.* a change in the degree of disorder, gives a change in energy $dQ = TdS$, which is called heat. For high temperatures there exists a large amount of energy which can be exchanged as heat; the thermal agitation is high. In the limit $T \rightarrow 0$ the thermal agitation ceases, *i.e.* the ensemble occupies only one state, which makes the entropy go to zero. This is known as the Nernst Principle.

Equation (112) defines an equilibrium transformation. If the entropy is constant these processes are reversible. A body with a constant entropy is thermo-isolated. The processes where the entropy increases are irreversible processes. A statistical ensemble cannot pass by itself in a state with a lower entropy. If the process depends on a parameter α which varies slowly in time, then the change dS/dt in entropy is proportional to $(d\alpha/dt)^2$ at equilibrium, since $dS/dt > 0$, such that these processes are also reversible; they are called adiabatic processes. Of course, a reversible process is not necessarily adiabatic.

A typical thermo-isolated process involves a body which receives the heat $Q + W$ at the hot temperature T_h , performs a mechanical work W and releases the heat Q at the cold temperature T_c . The entropy being constant (it is called *Verwandlungsinhalt* by Clausius) we have $(Q + W)/T_h = Q/T_c$, whence $Q = W \frac{T_c}{T_h - T_c}$, $Q + W = W \frac{T_h}{T_h - T_c}$ and the efficiency coefficient is $\eta = W/(Q + W) = 1 - \frac{T_c}{T_h}$. This is called the Carnot process, performed by a thermal engine, which cannot transform completely the heat in mechanical work (in a reversible process).

If we include the number of particles the equilibrium transformations are defined by

$$dE = -pdV + TdS + \mu dN \quad , \quad (113)$$

where N denotes here the mean number of particles. From this equation we get

$$d(E + pV - TS - \mu N) = N(vdp - sdT - d\mu) \quad , \quad (114)$$

where $v = V/N$ and $s = S/N$ are the specific volume and the specific entropy. This equation shows that

$$E + pV - TS - \mu N = 0 \quad . \quad (115)$$

$F = E - TS$ is the Helmholtz free energy, $W = E + pV$ is the enthalpy, $\Phi = \mu N$ is the Gibbs free energy and $\Xi = E - TS - \mu N = pV$ is the grand-canonical potential. From the above equations (*e.g.*, equations (109) and (111)) we get $e^{-\beta F} = \int e^{-\beta \mathcal{E}}$ and $e^{-\beta \Xi} = \int e^{-\beta \mathcal{E} + \beta \mu N}$.

A next-order expansion of the entropy in equation (107) gives a factor $e^{-\delta S}$; since the variation δS is at equilibrium, it is given by $\Delta E + p\Delta V - T\Delta S = T\delta S$, and has the negative sign. It follows that

$$\rho_f = const \cdot e^{-\beta(\Delta E + p\Delta V - T\Delta S)} \quad (116)$$

gives the probability of fluctuations; the number of particles is absent, because its fluctuations are defined at constant volume, such that they are related to the volume fluctuations. In equation (116) the energy is a function of S and V , and the non-vanishing contributions occur in the second-order expansion of E . The second-order variations of the quantity $\Delta E + p\Delta V - T\Delta S$ should be positive, in order to ensure the thermal equilibrium. Moreover, a change in an equally-probable distribution is $\delta\rho = -(1/\delta\mathcal{N}^2)\delta\mathcal{N} \sim e^{\delta S}$, where \mathcal{N} is the number of states; we get again $\rho_f \sim e^{\delta S} = e^{-\beta(\Delta E + p\Delta V - T\Delta S)}$. The fluctuation distribution was pointed out by Einstein.

Various processes which are present in a statistical ensemble are classified according to the energies involved and the corresponding times, by

$$\begin{aligned} \varepsilon_{eq} > T > \varepsilon_f > \delta\varepsilon_{ex} \gg \delta\varepsilon_q > \delta\varepsilon_{obs} \quad , \\ \tau_{eq} < \tau_{th} < \tau_f < \tau_{lf} \ll \delta\tau_q < \tau_{obs} \quad . \end{aligned} \quad (117)$$

In these equations ε_{eq} is the energy shared by the sub-ensembles to achieve equilibrium; the equilibrium is achieved in the shortest time τ_{eq} . τ_{th} is the time which determines the equilibrium (the thermal time), with an uncertainty T in energy. This time is shorter than the fluctuation time τ_f , which involves a fluctuation energy ε_f . The elementary excitations have a longer lifetime τ_{lf} , with a smaller energy uncertainty $\delta\varepsilon_{ex}$. A long time $\delta\tau_q$ is necessary to determine the quantum levels, which are very dense in condensed matter, and requires a very small uncertainty; and, finally, the time τ_{obs} of observation of all these processes is very long, in order to have a very small uncertainty in determining their energy.

The temperature is measured in Celsius degrees ($^{\circ}C$) by a thermometer. The coefficient of thermal expansion was measured for ideal classical gases and the law $pV = NT$ was found, where $T = 273.15 + t^{\circ}C$; it follows that the zero temperature corresponds to $-273.15^{\circ}C$. The temperature T is the Kelvin temperature, and the Celsius degree became the Kelvin degree K . The law of the ideal gases $pV = NT$ and the energy law $E = \frac{3}{2}NT$ for these gases have been derived both experimentally and theoretically from the molecular-kinetic theory of gases of Maxwell. The Kelvin degree is $1K = 1.38 \times 10^{-16}erg$ (Boltzmann constant). The law of simple and multiple proportions in chemical reactions induced the idea of molecules composed of integral numbers of atoms. An amount of a substance with mass equal with its molecular mass in grams is called mol. Mols have been identified comparatively, viewing the hydrogen as the simplest substance. In normal conditions, *i.e.* at $T = 300K$ and pressure $p = 10^6 dyn/cm^2$, any mol has a volume $22.4l$. It follows that we have $N = 6 \times 10^{23}$ atoms (molecules) in any mol. This is the Avogadro's number.

Elasticity and Fluids. Condensed matter consists of macroscopic bodies like gases, liquids, solids (plasma has a special place). At thermal equilibrium these bodies, which are statistical ensembles, may be viewed as consisting of statistical sub-ensembles, each at their own thermal equilibrium. If their thermal equilibrium is the same, then the whole ensemble is at a global thermal equilibrium; otherwise, we may have local equilibrium, but not a global one. The thermal equilibrium may be pushed down to atomic and molecular constituents, although, obviously, the condition of a large number of particles is not satisfied. They are so inasmuch as they exchange continuously energy with their neighbours, to a smaller or a larger spatial extent. At the atomic level the molecular agitation is erratical, and the physical quantities vary abruptly and indeterminately in space and time. We can average this statistical motion over small spatial regions and small durations. Thereby, we get domains, which microscopically may be sufficiently large, but macroscopically are so small that we may view them as points in space, at a given time. Then, we may consider the motion of these point-like domains, keeping in mind the spatial and temporal restrictions, *i.e.*, we may study their motion for slight variations of forces in space and slow variations of forces in time; this amounts to be aware of a spatial cutoff and a temporal one.. In this way we get a continuous medium, as it is usual for liquids and solids, and gases.

The domains described above may be viewed as statistical sub-ensembles, *i.e.* they may be endowed with statistical properties like entropy, energy, etc. They define a large homogeneous statistical ensemble, though not necessarily isotropic. For instance, if the domains are reduced to the unit cells in a crystalline solid, they form an anisotropic solid, though homogeneous. Viewing the domains as sub-ensembles allows one to consider the transfer of energy between domains, which proceeds by thermoconduction and viscosity. The thermal conduction is due to collisions, the viscosity arises from local friction. In liquids these phenomena are important, in solids they are of a lesser extent. Indeed, most of the phenomena in solids proceeds sufficiently fast as to be considered adiabatical, *i.e.* thermo-isolated, where the entropy is not changed (adiabatic transformations are slow in comparison with the statistical motion; in fact, it is pretty fast); so, there is no thermo-conduction; in addition, the viscosity is very small in usual solids.

The variation of the volume due to the atomic extent of the particles is

$$\frac{\delta V}{V} = \frac{a_0^3}{a^3} , \quad (118)$$

where a_0 is the atomic dimension and a is the mean separation distance between the particles; it follows $\delta V = (a_0/a)^3 a^3 N$, where N is the number of particles. On the other hand, the volume variation of a domain is $\delta V = Sa = N^{2/3} a^3$, where S is the area of the surface which encloses the domain. These two relations lead to $N^{1/3} = (a/a_0)^3$, which indicates a dimension $R = N^{1/3} a = (a/a_0)^3 a$ for a domain. Making use of typical values $a_0 = 1\text{\AA}$ and $a = 4\text{\AA}$, we get $R = 64a = 256\text{\AA}$ and $N = (R/a)^3 \simeq 10^5$. This is an estimation of the size of the domains.

The displacement \mathbf{u} of the domains produces translations, rotations and deformations (this is known as the Helmholtz theorem). The deformations are produced by the strain tensor $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. In liquids (leaving aside the viscosity) this tensor is diagonal. On the other hand, forces arising from the neighbours act upon the surface of a domain; therefore, these forces (per unit volume) are given by $f_i = \partial_j \sigma_{ij}$, where the symmetrical tensor σ_{ij} is the stress tensor. Forces \mathbf{P} acting upon the unit area of the surface of a body are given by $P_i + n_j \sigma_{ij} = 0$, where \mathbf{n} is the vector normal to the surface. This ensures the equilibrium of the surface. Similarly, the equilibrium in the volume is ensured by $g_i + \partial_j \sigma_{ij} = 0$, where \mathbf{g} is the external force. A linear relation should exist between σ_{ij} and u_{ij} , which, for homogeneous and isotropic bodies, is given by $\sigma_{ij} = 2\mu u_{ij} + \lambda u_{kk} \delta_{ij}$. This is known as the Hooke's law, and λ and μ are the Lamé elastic coefficients. Ut vis, sic tensio, for elastic bodies. The equation of motion of the displacement is given by the Newton's law, which reads

$$\rho \ddot{u}_i = \partial_j \sigma_{ij} + g_i , \quad (119)$$

or, for homogeneous and isotropic bodies,

$$\rho \ddot{\mathbf{u}} - \mu \Delta \mathbf{u} - (\lambda + \mu) \text{grad div} \mathbf{u} = \mathbf{g} , \quad (120)$$

where ρ is the density. These are known as the Navier-Cauchy equations for the elastic solid. They are solved with boundary conditions, which may include forces on the surface, or wave conditions at infinity; initially conditions are also needed.

The elastic forces in a solid produce, beside the dilatation u_{ii} , shear forces, associated with the coefficient μ . In liquids (and gases) the shear is absent. The stress tensor in fluids is diagonal; it gives rise to a force given by $f = -\text{grad} p$, where p is the pressure. In contrast with the elastic solids, the domains may flow in fluids, with a velocity $\mathbf{v} = d\mathbf{u}/dt$. In this case, the change in time of the velocity involves the transport contribution, so that the above equation (120) becomes

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \text{grad}) \mathbf{v} = -\text{grad} p + \mathbf{g} , \quad (121)$$

for incompressible fluids. This is known as Euler's equation. The viscosity is included by

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \text{grad}) \mathbf{v} = -\text{grad} p + \eta \Delta \mathbf{v} + (\zeta + \eta/3) \text{grad div} \mathbf{v} + \mathbf{g} , \quad (122)$$

similarly to the elastic solids, where η and ζ are the viscosity coefficients. These equations are known as the Navier-Stokes equations.

A particular case appears from a local displacement \mathbf{u} , which produces a local change of volume $\delta V = V \text{div} \mathbf{u}$ (and a change of density $\delta \rho = -\rho \text{div} \mathbf{u}$). In a fluid we have $\delta \sigma_{ij} = -\delta p \delta_{ij} = \lambda \delta V / V \delta_{ij}$. Therefore, $\lambda = -V(\partial p / \partial V)$ is the compression coefficient. It follows that equation (120) becomes

$$\rho \ddot{\mathbf{u}} - \lambda \text{grad div} \mathbf{u} = \mathbf{g} ; \quad (123)$$

the same equation is obtained from equation (121) for small velocities. Leaving aside the external force and taking the *div*, we get a wave equation for compression and dilatation waves, which propagates with velocity $\sqrt{\lambda/\rho}$; this is the sound equation. Since the sound waves are faster than the statistical motion, we need to take the adiabatic compressibility coefficient for the sound velocity. We note that the sound is longitudinal ($div\mathbf{u} \neq 0$, $curl\mathbf{u} = 0$). There exist also transverse waves of the density in condensed matter, propagating with a velocity related to the coefficient μ ; they are long-wavelength waves and have $curl\mathbf{u} \neq 0$ (and $div\mathbf{u} = 0$).

Physical Kinetics. A statistical ensemble at equilibrium, consisting of many statistical sub-ensembles (particles, domains), is characterized by a statistical distribution F , which is a function of the position \mathbf{r} and the velocity \mathbf{v} of the particles. An external field or a change in various parameters determine a change in this distribution, which may depend on the time t . For small variations, which depends slightly on the position and slowly on the time, the local equilibrium is preserved, but not the global one. A particular case is the transport phenomena. To what extent is such a super-imposed mechanical motion compatible with a statistical description?

Let $\mathbf{u}(t, \mathbf{r}, \mathbf{v})$ be a local displacement, which obeys the equation of motion $\partial\dot{\mathbf{u}}/\partial t + (\mathbf{v} + \dot{\mathbf{u}})\partial\dot{\mathbf{u}}/\partial\mathbf{r} = \mathbf{K}/m$, where \mathbf{K} is an external force and m denotes the mass of the particles; the effect of the collisions remains to be included. We note the occurrence of the transport term $(\mathbf{v} + \dot{\mathbf{u}})\partial\dot{\mathbf{u}}/\partial\mathbf{r}$. In solids and liquids the motion is ascribed to small, but macroscopic, regions of matter at equilibrium, such that the velocity \mathbf{v} is absent; moreover, in solids the velocity $\dot{\mathbf{u}}$ is much smaller than the velocity of propagation of the disturbance, such that the term $\dot{\mathbf{u}}\partial\dot{\mathbf{u}}/\partial\mathbf{r}$ is absent too; however, both in solids and liquids there exists an interaction with the neighbouring "particles", similar to the molecular collisions (friction), which contributes to the dissipation.

In thermal motion we still consider sufficiently large regions of matter, in order to be able to define the concentration, for instance; however, such regions are small enough to allow for fluctuations and molecular collisions; positions and velocities (in general phase-space variables) are continuously and statistically distributed up to the uncertainty of defining these regions; the thermal dynamics is defined in terms of probabilities. It follows that we must consider small displacements \mathbf{u} , with small variations in space and time, as well as small effects of the molecular collisions, in order to preserve the basic assumptions of concentration, probability, deviations, uncertainties, fluctuations and, ultimately, the Euler approach. This limitation is in accordance with the external perturbations, which, usually, are small and vary slightly and slowly in space and time, in the sense that their effects are small at our space and time scales. In particular, it follows that we may limit ourselves to $\mathbf{v}\partial\dot{\mathbf{u}}/\partial\mathbf{r}$ for the transport term in gases.

The displacement velocity may suffer small changes due to collisions, or friction; therefore, an additional term $m\ddot{\mathbf{u}} \simeq m\dot{\mathbf{u}}/\tau$ must be added to the equation of motion, where τ is a large relaxation time; being large, it may be considered constant; therefore, the equation of motion reads

$$\frac{\partial\dot{\mathbf{u}}}{\partial t} + \mathbf{v}\frac{\partial\dot{\mathbf{u}}}{\partial\mathbf{r}} + \gamma\dot{\mathbf{u}} = \frac{\mathbf{K}}{m}, \quad (124)$$

where $\gamma = 1/\tau$ is the collision frequency. We note that the collision term gives a force which opposes the mechanical motion; this force tends to establish the thermal equilibrium. Under the conditions presented here, equation (124) describes the accommodation of the mechanical motion ("mechanical equilibrium") with the thermal equilibrium; it describes mechanical motion compatible with thermal equilibrium. The relaxation time τ defines a molecular freepath $l = v\tau$ and a mean freepath $\Lambda = v_{th}\tau$. Close to equilibrium we may neglect the parameter γ , as if, formally, we put $\tau = \infty$. Therefore, we must have $\omega\tau \gg 1$, where ω is the frequency of variation of the displacement \mathbf{u} . The limit $\tau \rightarrow \infty$ ($\gamma \rightarrow 0$) is called the collisionless limit. In addition, we must have $\omega u \ll v$.

Equation (124) can be recast in an interesting form. Mechanical motion generates a variation $f = \delta F$ of the equilibrium distribution; it is given by

$$f = \delta F = -\dot{\mathbf{u}} \frac{\partial F}{\partial \mathbf{v}} ; \quad (125)$$

the minus sign implies a motion with velocity $-\dot{\mathbf{u}}$ ("observer's motion"); we note that $\dot{\mathbf{u}}$ is a local velocity and equation (125) is at local equilibrium. We may view $F(v)$ as a velocity density of particles; when $\mathbf{v} \rightarrow \mathbf{v} + \dot{\mathbf{u}}$ this density becomes $F(\mathbf{v} + \dot{\mathbf{u}})$ and the change in density is $\delta F = -[F(\mathbf{v} + \dot{\mathbf{u}}) - F(v)] = -\dot{\mathbf{u}} \partial F / \partial \mathbf{v}$; the minus sign accounts for the fact that this amount of particles is lost. In fact, the process is $F(\mathbf{v}) \rightarrow F(\mathbf{v} - \dot{\mathbf{u}})$. Multiplying equation (124) by $\partial F / \partial \mathbf{v}$ we get

$$\frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{K}}{m} \frac{\partial F}{\partial \mathbf{v}} = -\gamma f ; \quad (126)$$

this is the Boltzmann's kinetic equation. It can also be written as

$$\frac{df}{dt} + \gamma f = 0 , \quad (127)$$

which shows that the total change brought by the mechanical motion plus the change due to collisions is zero; *i.e.*, the thermal equilibrium is consistent with the mechanical motion, in the terms discussed here. This is the general meaning of the Kinetics. Equation (127) may be viewed as the general relaxation equation for any physical quantity. It is worth noting that the change in temperature is given by $\delta T = \overline{m\mathbf{v}\dot{\mathbf{u}}} = T f / F$. We note that $m\mathbf{v}\dot{\mathbf{u}} = \delta(mv^2/2) = U$ and $f = -\beta U F$, where U is the external potential energy and $\beta = 1/T$.

We note that the equilibrium is local in the Boltzmann kinetic equation; leaving aside the collisions (and the force term), we get the transport equation; the compatibility of this equation with the statistical distribution is ensured locally by equation (125); but not globally. In this context, we note also that we may use equally well equation (124) for kinetic phenomena, or its transport part for transport, each time in the context of local thermal equilibrium.

We note also that the collision term γf is relevant for the change f of the distribution function; though present, the collisions do not appear in the equation of motion of the equilibrium function F , which is stationary, *i.e.* $dF/dt = 0$ (this is related, formally, to Liouville's theorem). The dependence on the thermal velocities \mathbf{v} and the occurrence of $\partial F / \partial \mathbf{v}$ in equation (126) ensures the thermal equilibrium in the presence of the mechanical motion.

In view of the small contributions in equation (124), we should limit ourselves to first-order (small) variations f , as in equation (126). The attempt of including higher-order variations in equation (126), or non-linearities, is not warranted. The entropy is maximal at equilibrium, such that it is affected only by higher-order variations of the distribution function, not by first-order variations. Therefore, we must consider the change f in the distribution function as an equilibrium transformation.

In general, achieving the thermal (statistical) equilibrium may be viewed as an open problem in Statistical Physics. The existence of the thermal equilibrium and the increase of entropy towards equilibrium, as well as its stationarity at equilibrium, are, in fact, the principles of statistical motion. The existence of γ (and $\gamma > 0$) in equation (124) indicates a dissipation of motion, which does not mean, necessarily, equilibrium. The evolution of the statistical distribution towards equilibrium implied by equation (127) is generated by the assumption in equation (125), which means, in fact, this evolution. Transport phenomena, plasmas or chemical reactions may offer instances of, at least limited, special thermal equilibrium.

It is often attempted to derive the dissipation coefficient γ from the structure of the collisions, which would lead, it is claimed, to the increase of entropy (the tendency of the ensemble to go to equilibrium); this approach is called the Boltzmann's H -theorem. However, all the arguments brought in support of this claim are valid at equilibrium, which, consequently, is not a proof of the law of increase of entropy.

As long as we derive the time dependence of the change in distribution we need to observe the causality principle, which should give results in the future from causes in the past. After this dependence is established, we may omit the small parameter γ in calculations. The causality principle gives rise to a special kind of energy dissipation, imposed by the thermal relaxation to equilibrium. This dissipation effect is known as the Landau damping.

The Boltzmann kinetic equation can be extended to various types of statistical ensembles, consisting of independent statistical sub-ensembles. We note that it is essential to have independent sub-ensembles, *i.e.* the interaction problem should be solved first; of course, the solution should be given in the statistical context. The most direct approach is to solve for the elementary excitations, either single-particle or collective, with limitations imposed by their lifetime and mean freepath. Corrections to the elementary excitations generated by interaction can be estimated by various perturbation-theory techniques, including diagrammatic analysis. Attempts of determining the single-particle distribution function by the hierarchy of higher-order distribution functions, which imply multiple correlations, lead to non-integrable equations, which, usually, are truncated by uncontrollable techniques or hypotheses.

Finally, we note that the dependence of the distribution function on the position or the velocity may occur through second-order derivatives, which leads to diffusion equations, like the Fokker-Planck equation. Typically, such approaches appear in the brownian motion, where Langevin equation accounts for a stochastic force.

The Boltzmann equation may be applied to delocalized waves too, especially in a quantum-mechanical context. It is worth noting, however, that the displacement \mathbf{u} introduced above is a field; the extension to a wavefunction requires a special discussion, especially in the context of the quantum-mechanical condensates.

Quantum Field Theory. The idea appeared to describe the quantum-mechanical motion of a relativistic particle. A relativistic particle has a large momentum and a large energy, such that over reasonable distances and times it may acquire large amounts of mechanical action. Therefore, it is problematic from the start if such an idea would be reasonable.

The most direct way to quantize the motion of a relativistic particle is to start with the relativistic energy-momentum relationship $E^2 = c^2 p^2 + m^2 c^4$ and replace E by $i\hbar \frac{\partial}{\partial t}$ and \mathbf{p} by $-i\hbar \frac{\partial}{\partial \mathbf{r}}$. We get the Klein-Gordon equation

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0 . \quad (128)$$

What is this ψ ? As it is well known, it is not a wavefunction, because the above equation does not conserve the probability; because it is a second-order differential equation in time. It is necessary to give a meaning to ψ .

The solution of this equation is a superposition of plane waves $a_{\mathbf{k}} e^{-i\varepsilon_{\mathbf{k}} t + i\mathbf{k}\mathbf{r}}$ and $b_{\mathbf{k}} e^{i\varepsilon_{\mathbf{k}} t - i\mathbf{k}\mathbf{r}}$, where $\varepsilon_{\mathbf{k}} = c\sqrt{k^2 + m^2 c^2/\hbar^2}$. On the other hand, equation (128) can be derived from a lagrangian, and a hamiltonian can be set up for it, by the Euler-Lagrange rules. Making use of the superposition of plane waves, this hamiltonian acquires a quadratic form. In order to simplify the presentation, we assume $b_{\mathbf{k}} = a_{\mathbf{k}}^*$. With well-chosen coefficients in front of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$, the hamiltonian becomes

$$H = \sum_{\mathbf{k}} \hbar \varepsilon_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} . \quad (129)$$

We may imagine that $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^*$ are canonical conjugate coordinates, actually quantum-mechanical operators; which satisfy commutation relations $[a_{\mathbf{k}}, a_{\mathbf{k}'}^*] = \delta_{\mathbf{k}, \mathbf{k}'}$; we may even for linear combinations of them which looks like the coordinates q and the momenta p of quantum-mechanical harmonic oscillators. Then, these operators satisfy the Heisenberg equations of motion

$$\dot{a}_{\mathbf{k}} = \frac{i}{\hbar} [H, a_{\mathbf{k}}] = \varepsilon_{\mathbf{k}} a_{\mathbf{k}} \quad , \quad (130)$$

which, of course, mean Schroedinger equations for wavefunctions of time and the coordinates q . All these, in an abstract space. Moreover, the commutation relations lead us to admit the existence of a number of occupation with integral values $n = 0, 1, 2, \dots$, a vacuum state with $n = 0$, states denoted by n , with n quanta with energies $\hbar\varepsilon_{\mathbf{k}}$; in this space $a_{\mathbf{k}}$ is a destruction operator and $a_{\mathbf{k}}^*$ is a creation operator. This is called the canonical quantization, the procedure is called the second quantization, these quanta are called bosons. A similar scheme was devised for $n = 0, 1$, where the quanta are called fermions, and the commutation relations are $\{a_{\mathbf{k}}, a_{\mathbf{k}'}^*\} = \delta_{\mathbf{k}, \mathbf{k}'}$. The second quantization was devised by Dirac, Wigner and Jordan, and extended to non-relativistic particles by Schwinger and Martin. And, still, what is the meaning? All what we can say is that these a 's mean the extent to which a particle (viewed as a quantum) exists, or does not exist. This quantum-mechanical motion is not a motion in time and space, it is a motion in time of the existence or non-existence of the particle. We are far way from the original idea of combining Quantum Mechanics with the Relativity. Making use of such a framework we can only say what is the probability to have what kinds of particles after an interaction process is consumed in a long time, which proceeds from a far way distance and ends at the other far away end of a collision experiment, where we detect the particles.

The time evolution of the second-quantization operators, *i.e.* of the states defined by them, is governed by the scattering S -matrix, which is the solution of the Heisenberg's equation of motion. This solution has an iterative form, so it gives perturbation-theory results; its currently-used exponential form is not valid. This form may probably hold only if the interaction process is viewed as consisting of successive point-like collisions in time; the evolution in-between these collisions moments being relegated to so-called virtual particles, which means a self-interaction, assumed to be without no effect upon the calculations. This very obscure Feynman-diagram procedure is a very problematic procedure. In fact, as it is well known, it leads to divergences, which are simply dismissed by the so-called renormalization techniques.

The canonical quantization was extended to electromagnetic field and to electrons. It turned out that there is not possible to have a lagrangian theory for the electromagnetic field, so we have not a hamiltonian, and the canonical quantization cannot be done, in relativistic (covariant) form. The reason resides in the fact that we have scalar and longitudinal degrees of freedom, which do not describe a motion, for which the lagrangian theory is devised, but an instataneous interaction. The scalar and longitudinal degrees of freedom are removed in an (self-) interaction problem, *i.e.* in the problem where the charge and the currents interact with their own field. We remain only with the transverse degrees of freedom (the photons), wich can be quantized, as, in fact, was done initially by Dirac. The formal covariant-like quantization of the electromagnetic field introduces redundant degrees of freedom, whose consequences are difficult to be estimated. Actually, they lead to infrared divergences in the Quantum Electrodynamics, which are dismissed on various, more or less convincing, physical grounds (the so-called Bloch-Nordsieck theorem). Their effect on the remaining finite results is difficult to be assessed.

We arrive at the Dirac equation. The Dirac equation reads

$$\gamma^{\mu} p_{\mu} \psi = mc \psi \quad , \quad (131)$$

where γ^μ are the Dirac matrices, $p_\mu = i\hbar\partial_\mu$ are the electron momenta, $\partial_\mu = (\partial/c\partial t, grad)$, m is the electron mass and ψ is a four-component spinor ($\mu = 0, 1, 2, 3$). Applied twice and using the anti-commutation relations for the matrices γ^μ , we get the Klein Gordon equation. It shows that the electron has a one-half spin and that there exist the anti-electron (the positron). In order the energy be positive definite, we need to impose anti-commutation relations for the associated creation and destruction operators, which shows that the electron is a fermion, with occupation number 0 and 1. All these are great merits of the Dirac equation.

The equation can also be written as

$$i\hbar\frac{\partial\psi}{c\partial t} = (-i\hbar\boldsymbol{\alpha}grad + mc\beta)\psi, \quad (132)$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$. However, a relativistic electron has an energy pc of the order mc^2 , which leads to a mechanical action of the order $mc^2\tau$; if $mc^2\tau < \text{a few } \hbar$, which means τ of the order of the Compton time $\tau_c = \hbar/mc^2$ and distances of the order of the Compton wavelength $\lambda_c = \hbar/mc$, then the Dirac equation is valid. On the contrary, if $\tau \gg \tau_c$ and distances d are such that $d \gg \lambda_c$, then we are in the quasi-classical limit and the Dirac equation becomes

$$E/c = \boldsymbol{\alpha}\mathbf{p} + mc\beta, \quad (133)$$

which is not a valid equation. The Dirac equation has not a classical limit. It follows that the Dirac equation cannot be used to describe the motion of a relativistic quantum-mechanical electron in this range of parameters.

Moreover, for distances shorter than the Compton wavelength, or for shorter times than the Compton time, the electromagnetic field becomes meaningless. This can be seen, for instance, for an electron in the Coulomb field of a heavy nucleus with atomic number Z , when the field $Ze/(\hbar/mc)^2$ is comparable with Schwinger's field given by $eE(\hbar/mc) = mc^2$; we get in this case $Z = \hbar c/e^2 = 137$, which shows that such atoms cannot exist. For distances and times shorter than the Compton values, although the Dirac equation is valid, it is not appropriate to describe the electromagnetic interaction. It remains that the great merit of the Dirac equation is the fact that it puts the electron equation in a relativistically invariant form, which leads to predicting the fermionic nature of the electrons, the one-half spin of the electron and the anti-electron (the positron), all consequences of the Relativity in the quantum-mechanical description. The quantum-mechanical motion of the relativistic electron over distances and times of the order of the Compton values is the Zitterbewegung. The Zitterbewegung is not a physical motion. It is associated with the destruction and creation of the electron quantum. In the S -matrix processes, when absorbing or emitting a photon, the electron is destroyed and created, such that the uncertainty in energy is of the order mc^2 . These unphysical processes cannot be described by any equation. If it could, it would amount to describing by equations the quantum-mechanical uncertainty, which is non-sensical. Therefore, we must content ourselves with a basic cutoff in working with electron equations, of the order of the Compton wavelength for distances, and, correspondingly, the Compton time for durations. The effects associated with shorter distances and times than a cutoff of the order of the Compton cutoff are uncertainties. This makes the Quantum Field Theory pointless.

We note also that the magnetic moment $\boldsymbol{\mu} = \frac{1}{2c}\mathbf{r} \times q\mathbf{v}$ can be written as $\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{r} \times \mathbf{p} = \frac{q}{2mc}\mathbf{L}$, where \mathbf{L} is the orbital angular momentum. As it is well known the orbital angular momentum is quantized in integral multiple of \hbar , such that the quanta of magnetic moment is $q\hbar/2mc$. For electrons it is called Bohr magneton ($e\hbar/2mc = 0.9 \times 10^{-20} \text{ erg/Gs}$). In general, for composite particles this simple proportionality is altered, and the relation is $\boldsymbol{\mu} = g\frac{q}{2mc}\mathbf{L}$, where g is called

the gyromagnetic ratio. We may imagine an intrinsic angular momentum of an electron moving with velocity c and distributed over the Compton wavelength, given by $s = \frac{\hbar}{2mc}mc = \hbar/2$; it is half the Planck's constant. It is called the electron spin. The Dirac equation gives $g = 2$ for the electron. In general, g depends on the (unknown) structure of the particle.

Boson Quantum Electrodynamics. The motion of the relativistic quantum-mechanical electron proceeds by a change δx^μ in the coordinates x^μ , denoted u^μ , according to the scheme $x^\mu \longrightarrow x^\mu + \delta x^\mu$, $\delta x^\mu = u^\mu$, $x^\mu \longrightarrow x^\mu + u^\mu$. We will take the first-order variations with respect to u^μ of the Dirac equation

$$\gamma^\mu \partial_\mu \psi = \frac{mc}{i\hbar} \psi . \quad (134)$$

According to the Dirac equation the Zitterbewegung implies that the coordinates should be viewed as matrices (like γ^μ). We write $x^\mu = s^\mu \cdot 1$, where $s^0 = ct$, $\mathbf{s} = \mathbf{r}$ and 1 denotes the unit matrix; we have $x_\mu x^\mu = s_\mu s^\mu \cdot 1 = s^2 \cdot 1$, where $s^2 = c^2 t^2 - \mathbf{r}^2$. For δx^μ we need $\delta x_\mu \delta x^\mu = u_\mu u^\mu = u^2 = ds^2$; the (non-trivial) solution of this equation is

$$\delta x^\mu = u^\mu = \frac{1}{2} u \gamma^\mu \quad (135)$$

(since $\gamma_\mu \gamma^\mu = 4$). The first-order expansion of the Dirac equation is

$$\gamma^\mu (\partial_\mu \psi + u^\nu \partial_\nu \partial_\mu \psi) = \frac{mc}{i\hbar} (\psi + u^\nu \partial_\nu \psi) , \quad (136)$$

or

$$\partial^\mu \partial_\mu \psi = -\frac{m^2 c^2}{\hbar^2} \psi , \quad (137)$$

which corresponds to the Klein-Gordon equation $p_\mu p^\mu \psi = m^2 c^2 \psi$. We note that u is absorbed into ψ , hence the bosonic character of the latter. We write

$$\psi = \sum_{\mathbf{k}} c \sqrt{\frac{\hbar}{2\varepsilon_k}} (\psi_{\mathbf{k}\alpha}) e^{i\mathbf{k}\mathbf{r}} , \quad (138)$$

where $\varepsilon_k = c\sqrt{k^2 + m^2 c^2 / \hbar^2}$,

$$(\psi_{\mathbf{k}\alpha}) = \begin{pmatrix} c_{\mathbf{k},+1} \\ c_{\mathbf{k},-1} \\ b_{-\mathbf{k},-1}^* \\ b_{-\mathbf{k},+1}^* \end{pmatrix} , \quad (139)$$

and the c 's and the b 's satisfy usual bosonic commutation relations for four distinct types of bosons, corresponding to $c_{\mathbf{k}\sigma}$ and $b_{\mathbf{k}\sigma}$; $\sigma = \pm 1$ is the spin label and the c 's and the b 's correspond to positive and negative energies (frequencies), respectively. We attribute this field to the motion of the electrons. This bosonic field has an energy

$$H_e = \sum_{\mathbf{k}\sigma} \hbar \varepsilon_k (c_{\mathbf{k}\sigma}^* c_{\mathbf{k}\sigma} + b_{\mathbf{k}\sigma}^* b_{\mathbf{k}\sigma}) , \quad (140)$$

a momentum and a charge, similar to the Dirac field, and a current density given by

$$\mathbf{j}^\mu = (c\rho, \mathbf{j}) = \frac{ie}{\hbar} (\bar{\psi} \partial^\mu \psi - (\partial^\mu \bar{\psi}) \psi) , \quad (141)$$

or

$$\rho = \frac{ie}{\hbar c^2} (\bar{\psi} \dot{\psi} - \dot{\bar{\psi}} \psi) , \quad j_i = -\frac{ie}{\hbar} (\bar{\psi} \partial_i \psi - \partial_i \bar{\psi} \psi) , \quad (142)$$

where the adjoint $\bar{\psi}$ is the transposed and conjugate to ψ . This current density can be written immediately in terms of the c and b operators. In addition, the Klein-Gordon equation can be obtained from a lagrangian theory. The electron motion is the motion of the Zitterbewegung.

The linear approximation used in the expansion above spoils the effects of the electromagnetic field, such that, in the presence of the electromagnetic field, we need to use the covariant derivative $D_\mu = \partial_\mu - \frac{e}{i\hbar}A_\mu$, according to $\psi \rightarrow \psi + u^\mu D_\mu \psi$; we get the Klein-Gordon equation

$$\left(p_\mu - \frac{e}{c}A_\mu\right) \left(p^\mu - \frac{e}{c}A^\mu\right) \psi - \frac{ie\hbar}{2c}\sigma^{\mu\nu}F_{\mu\nu}\psi = m^2c^2\psi \quad (143)$$

with electromagnetic field. In addition, the limits of the Electromagnetism imply a lower bound upon the interaction (photons) wavelength, of the order of the Compton wavelength (as well as upon the electron wavelength). We can see that it amounts to applying twice the Dirac equation. We can see that the interaction terms are

$$v_p = \frac{1}{c}j_\mu A^\mu = \frac{ie}{c\hbar} [\bar{\psi}(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\psi] A^\mu \quad (144)$$

(where the gauge condition $\partial_\mu A^\mu = 0$ is used),

$$v_d = -\frac{e^2}{c^2\hbar^2}(\bar{\psi}\psi) A_\mu A^\mu \quad (145)$$

and

$$v_H = -\frac{e}{2c\hbar}\bar{\psi}(\Sigma + \Sigma^*)\psi\mathbf{H}, \quad (146)$$

$$v_E = \frac{ie}{2c\hbar}\bar{\psi}(\boldsymbol{\alpha} - \boldsymbol{\alpha}^*)\psi\mathbf{E},$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad (147)$$

and $\boldsymbol{\sigma}$ are the Pauli matrices. The interactions v_H and v_E arise from the $\sigma^{\mu\nu}F_{\mu\nu}$ -term in equation (143). This is known as the "Pauli term". The symmetrization and antisymmetrization in equations (146) ensure the consistency of the equations of motion (for ψ and $\bar{\psi}$ and real energy).

Now, we can proceed to do perturbation theory and S -matrix expansion (using equation (39)) making use of the free electron hamiltonian H_e given by equation (140), the free photon hamiltonian $\mathcal{E}_t = \sum_{\mathbf{k}} \frac{1}{2}\hbar\omega_{\mathbf{k}}(a_{\mathbf{k}}a_{\mathbf{k}}^* + a_{\mathbf{k}}^*a_{\mathbf{k}})$, $\omega_{\mathbf{k}} = ck$, where $a_{\mathbf{k}}$ are the operators for the transverse photons, and the interactions terms given above. Obviously, we must use a cutoff wavelength of the order of the Compton wavelength $\lambda_c = \hbar/mc$. Similarly, we retain in v_p only the transverse part.

We note that the ψ in the Klein-Gordon equation has now a well defined meaning: it is a displacement, whose motion can be quantized. The quantum-mechanical motion of the relativistic electron interacting with the electromagnetic field is limited by the divergences of the field at small distances (and times) and by the limitations imposed upon the time and the distance by the relativistic energy mc^2 and the relativistic momentum mc .

The calculations performed by using the Dirac field imply using twice the Dirac equation, which amounts to using the Klein-Gordon equation. In the lowest order of the perturbation theory such calculations do not imply the electron self-energy, nor the vacuum polarization. The results are the same as those obtained by the Dirac theory of radiation, or by using the quantum-mechanical theory of perturbation, as described by Heitler. To this approximation, the interaction does not occur over regions with dimension smaller than the Compton wavelength. These results are in good agreement with experimental measurements. Higher-order calculations in the perturbation

theory, involving electron self-energy and vacuum polarization, imply interactions over distances of the order of the Compton wavelength (self-interaction), and lead to divergences (ultraviolet divergences). Apart from their theoretical interest, such calculations were prompted by two experimentally measured effects, namely the Lamb shift and the anomalous magnetic moment of the electron.

In order to eliminate these divergences the Quantum Electrodynamics devised a renormalization and regularization scheme. The resulting finite quantities are extremely small corrections. It is claimed that these results are in very good agreement with the experimental measurements.

The renormalization and regularization procedure is based on arbitrary modifications of the divergent integrals and on an obscure claim that the interaction effects should act upon bare particles with an infinite mass and an infinite charge, such that, cancelling out these two infinities, the one arising from interaction and the other arising from the bare particles, we would get finite results. This procedure is almost impossible to be applied in practice, and many errors were discovered in calculations of the Lamb shift and the anomalous magnetic moment of the electron. Moreover, it seems that the experimental measurements are affected by unknown, unexplainable errors. Indeed, according to the boson theory described above, the results must depend on the cutoff used, which is of the order of the Compton wavelength but it has no definite value. These calculations, as well as the corresponding experimental results only tell that we are dealing with uncertainties; we can assess their order of magnitude, but, of course, we cannot give a definite value.

Gravitational field. The mass m , the electrical elementary charge e of the electron, the speed of light in vacuum c and the Planck's constant \hbar are extraneous quantities to the theory. Einstein attempted to give a theoretical meaning of the mass.

Let us assume, as usually, that an ensemble of point-like masses m with the concentration n suffers a local displacement \mathbf{u} ; the mass density varies by $\mu = mndiv\mathbf{u}$. If we define a field $\mathbf{g} = -4\pi mn\mathbf{u}$, it satisfies the equation

$$div\mathbf{g} = -4\pi\mu ; \tag{148}$$

the potential φ given by $\mathbf{g} = -grad\varphi$, satisfies the equation

$$\Delta\varphi = 4\pi\mu . \tag{149}$$

If $\mu = m\delta(\mathbf{r})$, we get $\varphi = -m/r$ and $\mathbf{g} = -m\mathbf{r}/r^3$. This spatial dependence is in accord with the law of gravitation of Newton. Moreover, an attractive force acts upon the mass M , given by $\mathbf{g}M$, and the energy of this mass in the gravitational field is $M\varphi$. Einstein's idea was to associate this energy to the rest energy Mc^2 , or at least, to a variation of this energy. In this case, the potential φ would be a variation of c^2 ; since this is not permissible, it remains that we should view the energy $Mc^2 + M\varphi$ as related to a time variation which would lead to Mc^2 ; this would mean a change in time $t \rightarrow t(1 - \varphi/c^2)$. The metric would change then to $c^2(1 + 2\varphi/c^2)dt^2 - d\mathbf{r}^2$. It would follow that the mass, which determines the gravitational field, is associated with a change of metric; Einstein viewed the mass density μ in equation (149) as proportional to $\Delta\varphi$; since $\mathbf{g} \sim 1/r^2$, which is the curvature of a surface, it follows that the mass density is in fact, the curvature of the four-dimensional time-space, *i.e.* the change in the metric. We note that it may appear that a similar reasoning seems, at first sight, to apply to the electric field. This is not so, because the Coulomb potential q/r has not the dimension of a squared velocity.

Actually, neither the Newton's gravitational potential φ has the dimension of c^2 . Mass appeared in the equations of motion of Mechanics, and we have established a certain convention about its units, as well as about the energy units. This mass is called inertial mass. If the gravitational potential of an inertial mass is to have dimension of c^2 it needs a gravitational constant G with

dimensions $cm^3/g \cdot s^2$. Such a constant is not at all conventional. If Einstein's idea is to be promoted, the occurrence of this constant should be made reasonable. Or, the Einstein's theory would apply to a gravitational mass, which would be different from the inertial mass. Of course, this later option would mean a theory about nothing. The two masses are in fact equal. And the gravitational constant is $G = 6.67 \times 10^{-8} cm^3/g \cdot s^2$.

It remains that the gravitational field (constant G included) may be described as a variation of the metric, which leads to the Poisson equation above for the static field; for time-varying fields a wave equation of small variations of the potential is derived immediately from the metric $c^2 dt^2 - d\mathbf{r}^2 = 0$ as $\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0$; this equation describes gravitational waves. They are transverse waves, with two independent components, due to the presence of the time as an additional degree of freedom.

We may replace the mass with the gravitational constant. We are left with the other three constants, electron charge e , speed of light in vacuum c and the Planck's constant \hbar . The energy relation $mc^2 + e\varphi$, where φ is the electric potential, can be written as $mc^2(1 + e\varphi/mc^2)$, whence we may define a mass variation $\delta m = e\varphi/c^2$. The potential can be estimated as $\varphi = e/\Delta r$, where $\Delta r \cdot mc = \hbar$. We get $\delta m = m \frac{e^2}{\hbar c}$, which defines an electromagnetic mass, through the strange ratio $\frac{e^2}{\hbar c}$ which is independent of units; this is the fine structure constant; it has the value $\frac{1}{137}$. δm is the mass correction estimated within the boson theory of Quantum Electrodynamics. We can see that Relativity defines a limit of Quantum Mechanical uncertainty which occurs in Electromagnetism. This may explain the occurrence of the fine structure constant (which was a great problem to Pauli).

Concluding remarks. Mechanics introduced point-like particles, but showed that their motion consists of continuous parts, albeit infinitesimal. The unknown mass was introduced on this occasion. Forces distributed continuously in space and time, produced by another unknown called electrical charge, were introduced by Electromagnetism. These forces were called fields. When produced by point-like charges, they become singular. A certain limitation should be exercised in these situations. A fundamental unknown was introduced on this occasion, which is the speed of light in vacuum. An undetermined motion described by a probability was introduced by Statistical Physics. Similarly, Quantum Mechanics recognized a fundamentally undetermined motion and pointed out basic limitations, by the uncertainty relations, related to another unknown, which is Planck's constant. The existence of the universal speed of light in vacuum led to Relativity, which gives a determined meaning to the limitations required by motion. These limitations are effective in the quantum-mechanical electromagnetic interaction, called Quantum Electrodynamics, and, in general, in all field theories. An attempt to derive the mass from the curvature of the time-space, indicates rather an additional unknown, which is the gravitational constant.

We can see that Physics is constructed by pure reason, in relation with indications received from experiment. Most of its construction is reasonable, but its starting points remain unknown, mysterious and irrational. These unknown things are mass, gravitational constant, electron charge, speed of light and Planck's constant; one of the last three may be replaced by the fine structure constant.

The modern Physics brought the Physics of Elementary Particles as a new realm of investigation. First, two new forces have been added to the gravitational force and the electromagnetic force; they are the weak force and the strong force. Corresponding coupling constants have been devised, which increased the number of unknowns. The large number of elementary particles have been classified by using symmetries and elementary constituents with many additional intrinsic characteristics, almost as many as the number of the elementary particles. These additional constituents increased enormously the number of unknowns. In addition, there is no reasonable circumstance for symmetries, especially the internal ones. Moreover, some time-space symme-

tries are violated, they are not symmetries. It is believed that at least the three electromagnetic, weak and strong forces derive from the same unique force, at very high energy. These forces have been constructed very similarly with the electromagnetic force, by field theories, which include the relativistic-quantum-mechanical limitations. No new idea was introduced, all these are sterile generalizations, no progress in understanding the Nature has been recorded by the modern physical theories.