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Einstein's Physics
(Lecture two of the Course of Theoretical Physics)

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Dedicated to our friend Ion Vitza in return of his many favours done upon Theoretical Physics.

1 Introduction

I will talk herein about the most basic concepts, and the most general ideas, of Physics. All of them are associated with Einstein's works, this is why talking about them amounts to talking about Einstein's works. Physics has been made along the time by Newton, Maxwell, Boltzmann, and Quantal Mechanics has been made by a dozen of scientists, but Einstein brought perhaps the most relevant contribution to all the basic concepts of Physics.

Albert Einstein was born in 1879 in Ulm, Germany, went to school in Munich and Aarau, Switzerland, graduated from the Federal Institute of Technology in Zurich (Eindgenossische Technische Hochschule), and got a job in the patent office in Bern. While in Bern he discussed much physics within his circle of friends, and in 1905 published a series of papers on Principle of Quanta, Statistical Principle and Principle of Relativity. He became professor in Zurich, Prague and Berlin, where finally he completed in 1916 his work on Principle of Inertia. In 1921 he has been awarded the Nobel Prize in Physics "for his services to theoretical physics and in particular for his discovery of the law of the photoelectric effect". Since 1933 Einstein settled in the United States of America, where he worked in Princeton on "Principle of Fields" until his death in 1955. Einstein was also much concerned with social issues, suggested the USA make the atomic bomb, and declined the presidency of Israel. Einstein believed in emotions, and, probably exceeded by his fellows' enquiries, left his brains to be studied after his death. They found nothing special.

2 Principle of Quanta

In the photoelectric effect (discovered by Hertz in 1887) the energy of the released electrons is proportional to the electromagnetic radiation frequency ν, above a certain threshold, while the electron flow is proportional to the radiation intensity. This prompted Einstein[1] in 1905 to infere that electromagnetic radiation consists of particles of energy

$$\varepsilon = h\nu \ , \tag{1}$$

where $h(= 6.62 \cdot 10^{-34}$Js) is Planck's constant. This is the Principle of Quanta. Planck introduced this quanta of energy in 1900, in connection to the black body (equilibrium) radiation spectrum. The radiation particles were later called photons, light quanta, gamma particles, or gamma quanta (in radioactive nuclei studies). This was perhaps for the first time that we have seen that radiation has a dual nature, that of waves and particles.

Einstein's implication was however reaching farther, suggesting that electrons motion, and hence atoms and matter motion, proceeds in such a way that it may have discrete energy levels, made out of quanta of energy, in quantal states of motion. Such an implication found its way in Bohr's work in 1913, and finally evolved in what we call today the Quantal Mechanics.

A general form of motion of an infinitesimal change $d\mathbf{r}$ in position and an infinitesimal lapse of time dt can be written as

$$\mathbf{k}d\mathbf{r} - \omega dt = d\Phi \ , \tag{2}$$

where $\mathbf{k}$ and ω are constants. This is the phase of a wave, with the wavevector magnitude $k = 2\pi/\lambda$ and pulsation (frequency) $\omega = 2\pi/T = 2\pi\nu$, where λ is the wavelength and T is the period. If such a wave is going to be associated with the mechanical motion, then we note that by multiplying (2) by Planck's constant $\hbar = h/2\pi$, we get εdt in the *lhs* of (2) according to (1), which means that $\hbar\mathbf{k}$ must be the momentum $\mathbf{p}$ and $\hbar d\Phi$ must be the mechanical action dS. The relationship

$$\hbar k = h/\lambda = p \tag{3}$$

was put forward by de Broglie in 1925 and is called the momentum quantization, while Planck-Einstein's relationship (1) is called the energy quantization. It folllows that a quantal motion is described by a wavefunction $\psi(\mathbf{r}, t)$, decomposable in a Fourier series of plane waves $\exp i[(\mathbf{pr} - Et)/\hbar]$, energy E and momentum $\mathbf{p}$, and hence all the physical quantities, are represented by operators like $E = i\hbar\partial/\partial t$ and $\mathbf{p} = -i\hbar\partial/\partial\mathbf{r}$ acting on the wave function, which have eigenvalues, mean values and dispersions; whence the wave function square is the probability of the motion, and the physical quantities obey Heisenberg's uncertainty principle, like, for instance, $\Delta p \cdot \Delta x \geq \hbar/2$, for one coordinate x. This is the whole Quantal Mechanics, with wavefunction, statistical nature and uncertainty. Operators are representable by matrices, which do not commute in general, as Heisenberg used to work in 1925. If we wish to get the energy E, then replace E in $E\psi = i\hbar\partial\psi/\partial t$ by hamiltonian $H = \mathbf{p}^2/2m + V$, for $\mathbf{p} = -i\hbar\partial/\partial\mathbf{r}$, as for a particle of mass m moving in a potential V, as Schrodinger did in 1926; this is Schrodinger's equation. The wavefunction evolves in space and time by a succession of infinitesimal plane waves, according to Huygens' principle, of quasi-classical phase $dS/\hbar$ extended along paths, as Feynman did in 1948 in his path integral formulation of Quantal Mechanics. Quantal motion reveals the wave nature of particles, the other half of duality. It is perhaps worth noting that quantal motion has no trajectory, according to (2), and space and time are separate in (2), as for non-relativistic motion.

Einstein extended the quanta concept to the atomic vibrations' contribution to the heat capacity of solids in 1907,[2] paving the way toward field quantisation.

3 Statistical Principle

After a few attempts to derive, improperly, the statistical motion from mechanical motion, Einstein fully realized in 1905,[3] in connection to his studies on the Brownian motion, the chaotical nature of the statistical motion, as emphasized previously by Boltzmann in 1896 and 1898. This is the Statistical Principle. He introduced, on this occasion, the fluctuations in statistical motion, the time-like statistical distribution and suggested the kinetic equation which governs transport and approach to equilibrium. These subjects were further developed by Einstein in a subsequent series of paper, especially in connection with diffusion.

If chaotical, the statistical motion is described by a density of probability ρ on mechanical states (p, q) (phase space), where p and q are canonical coordinates (like momentum and position), the number of states being counted by cells of volume $(2\pi\hbar)^s$, where s is the number of the degrees of freedom; for a mechanical motion. While for quantal motion it is given by the diagonal elements $\rho_n = \rho_{nn}$ of the density matrix ρ_{nm} for quantal states n, m. They are called statistical distributions. For a wavefunction $\psi = \sum c_n\varphi_n$ the density matrix is given by $\rho_{nm} = c_n^* c_m$, according to the statistical nature of the quantal motion (any mean value $\bar{f} = \sum \rho_{nm} f_{mn} = tr(\rho f)$), but it is no more decomposable for statistical motion. Hence, the top ranking of the statistical motion, due to its generality. The practical realization of statistical motion is usually for quantal ensembles or classical non-relativistic ensembles with a great number of deegres of freedom, like ensembles of many particles.

If chaotical, the statistical motion reaches an equilibrium after a time long enough, where it is independent of time. This is the Principle of the Statistical (or Thermodynamical) Equilibrium. If chaotical, the statistical motion of one ensemble at equilibrium is independent of the statistical motion of another ensemble at the same equilibrium. This is the Principle of Statistical Independence, which makes additive the logarithm of statistical distributions. If chaotical, the additive $\ln\mathcal{N}$, where $\mathcal{N}$ is the number of states in the phase space, can only increase toward statistical equlibrium, where it stays stationary; this is the Law of the Increasing Entropy $S = \ln\mathcal{N}$, or its mean value $S = -(2\pi\hbar)^{-s}\int dpdq \cdot \rho\ln\rho$ (or $S = -tr(\rho\ln\rho)$), or the Second Law of Thermodynamics (or "Boltzmann H-theorem") (or the principle of "maximum probability"). Entropy is a measure of disorder, therefore. If statistical motion ceases, the ensemble at equilibrium must settle down with minimum of energy, which usually is the ground-state, which is physically unique, so the entropy vanishes under these circumstances, and we recover the mechanical motion at equlibrium; this is Nernst's Principle, or theorem, or the Third Law of Thermodynamics.

At equilibrium the additive $\ln\rho$ is a linear combination of constants of motion, like energy E, or number N of particles; or, taking the maximum of entropy under constant mean energy, number of particles, etc, one gets similarly

$$\rho = \frac{1}{Z}e^{-\beta(E-\mu N)} \ , \tag{4}$$

which is the Boltzmann distribution with Z the grand-canonical partition function (or canonical, leaving aside N). Or, $\rho = \exp[-\beta(H-\mu N)]/Z$, where H is the hamiltonian, N is the operator of the number of particles, and $Z = tr\exp[-\beta(H-\mu N)]$, and $\bar{f} = tr(\rho f)$ for the mean value of any quantity f. Computing S for this distribution one gets the reciprocal temperature $1/T = \beta = \partial S/\partial E > 0$, the chemical potential $\mu = \partial E/\partial N$ and the pressure $p = -\partial E/\partial V$, V being the volume, which all define equilibrium, the grand-canonical potential $\Omega = -T\ln Z = -pV$ introduced by Gibbs in 1902, and the First Law of Thermodynamics $dE = -pdV + dQ + \mu dN$, where $dQ = TdS$ is heat; and all the quantities are mean values, *i.e.* thermodynamic quantities. Similarly, leaving aside N one gets the free energy (Helmholtz free energy) $F = -T\ln Z = E - TS$ for the canonical partition function. In addition, the Gibbs free energy $\Phi = \mu N$, and $G = E + pV$ is the enthalpy. They are all thermodynamic potentials $E(V, S, N)$, $F(V, T, N)$, $\Omega(V, T, \mu)$, $G(p, S, N)$ and $\Phi(p, T, N)$, in the sense that all their changes are equal, and, for instance, dF, $d\Omega$, $d\Phi < 0$, reaching their minimum values at equilibrium, according to entropy increasing.

The distributions with partition function written as $Z = \exp(-\beta\Omega)$, or $Z = \exp(-\beta F)$, are also called Gibbs distributions. These partition functions can be computed by steepest descents, and we get distributions highly peaked on the mean, thermodynamic energy and particle number for large number N of particles; this is the "central limit theorem" or the "law of large numbers", and shows that thermodynamic fluctuations go like $1/\sqrt{N}$, as expected from statistical independence; indicating again the thermodynamic equilibrium as the "most probable state", and $N, V \to \infty$ at constant density $n = N/V$ as the thermodynamic limit; note that thermodynamic quantities, like energy, volume, number of particles, etc, are extensive quantities. For small temperatures the ensemble may go to a more ordered state, a "condensed state", and a corresponding lower energy, the partition function vanishing, which may lead to singularities in the derivatives of the thermodynamic potentials, and infinite fluctuations; the ensemble undergoes a phase transition, that "critical" temperature separating two distinct phases; a symmetry being broken in such phase transitions.

For a classical gas of identical particles the grand-partition function can be written as $Z = \sum(1/N!)Z_1^N = \exp(Z_1)$, where $1/N!$ is Gibbs' reduction factor for identical particles and $Z_1 = [V/(2\pi\hbar)^3]\int d\mathbf{p}\cdot\exp[-\beta(\varepsilon_\mathbf{p}-\mu)]$. This is Maxwell's distribution, with $\varepsilon_\mathbf{p} = p^2/2m$ the kinetic energy of a particle of mass m, perhaps the first statistical distribution, introduced by Maxwell in 1859. The number of particles $N = (1/\beta)\partial(\ln Z)/\partial\mu = [V/(2\pi\hbar)^3]\int d\mathbf{p}\cdot n_\mathbf{p}$, where

$$n_\mathbf{p} = \exp[-\beta(\varepsilon_\mathbf{p}-\mu)] \ , \tag{5}$$

is the classical occupation number. The condition of validity for classical behaviour $\exp(\beta\mu) \ll 1$ leads to thermal wavelength $\hbar/\sqrt{mT}$ much shorter than the mean inter-particle spacing $n^{-1/3}$, where $n = N/V$ is the gas density, which is indeed the quasi-classical regime. Let N_s identical particles be distributed among $\mathcal{N}_s$ states, then the entropy is gven by $S = -\ln\prod_s \mathcal{N}_s^N/N!$, or using Stirling's approximation $\ln N! = N\ln N - N$, one gets readily $S = \sum_s \mathcal{N}_s n_s \ln(n_s/e)$, where $n_s = N_s/\mathcal{N}_s$ is the occupation number (and $e \cong 2.73$); since $\mathcal{N}_s = \Delta\mathbf{r}\Delta\mathbf{p}/(2\pi\hbar)^3$ one gets the entropy of the classical gas $S = -\sum_\mathbf{p} n_\mathbf{p}\ln(n_\mathbf{p}/e)$.

Quantal gases of identical particles are, essentially, gases of field quanta. Their wavefunctions are either symmetric or antisymmetric under permutation of any two particles, and their fields commute or anticommute, respectively. Consequently, their occupation number is either $n_\mathbf{p} = 0, 1, 2...$, or $n_\mathbf{p} = 0, 1$, for instance. For the latter it works an exclusion principle, introduced by Pauli in 1925, in the sense that these particles do not stay two or more on the same state. The partition function $Z = \Pi_\mathbf{p}(\Sigma_{n_\mathbf{p}} e^{-\beta(\varepsilon_\mathbf{p}-\mu)n_\mathbf{p}})$ can easily be calculated, and one finds the mean occupation number (occupancy) $n_\mathbf{p} = [\exp[\beta(\varepsilon_\mathbf{p}-\mu)] + 1^{-1}$, introduced by Fermi and by Dirac in 1926, which is also called the Fermi, or Fermi-Dirac, distribution for particles obeying Pauli's exclusion principle, which are called fermions; and $n_\mathbf{p} = [\exp[\beta(\varepsilon_\mathbf{p}-\mu)] - 1^{-1}$ which is called the Bose-Einstein distribution for particles with any occupation number, which are called bosons. The latter was introduced by Bose in 1924, and developed by Einstein,[4] who pointed out that it exhibits a condensation of particles on the lowest energy level, called the Bose-Einstein condensation; a feature underlying the superfluidity, and related phenomena in condensed matter. Later on, Pauli showed in 1940 that half-integral spin particles have a non-positive defined energy, unless they anticommute. While integral spin particles must commute. Therefore, fermions have a half-integral spin, bosons have an integral spin, which is called the spin-statistics theorem. The single-particle distribution function for quantal gases is $\rho_\mathbf{p} = \exp[-\beta(\varepsilon_\mathbf{p}-\mu)n_\mathbf{p}]/Z_\mathbf{p}$, where the single-particle partition function $Z_\mathbf{p} = \Sigma_{n_\mathbf{p}} e^{-\beta(\varepsilon_\mathbf{p}-\mu)n_\mathbf{p}}$, so the entropy $S = -\Sigma_{n_\mathbf{p},\mathbf{p}}\rho_\mathbf{p}\ln\rho_\mathbf{p}$ is readily obtained as $S = -\sum_\mathbf{p}[n_\mathbf{p}\ln n_\mathbf{p} + (1-n_\mathbf{p})\ln(1-n_\mathbf{p})]$ for fermions, and $S = \sum_\mathbf{p}[(1+n_\mathbf{p})\ln(1+n_\mathbf{p}) - n_\mathbf{p}\ln n_\mathbf{p}]$ for bosons. In the limit of low occupancy $n_\mathbf{p} \ll 1$ both go over to the classical entropy $S = -\Sigma_\mathbf{p} n_\mathbf{p}\ln(n_\mathbf{p}/e)$. N_s bosons are distributed together with $\mathcal{N}_s - 1$ "walls" between states, *i.e.* the number of distributions is $(\mathcal{N}_s + N_s - 1)!/(\mathcal{N}_s - 1)!N_s!$; while for fermions, with one particle at most in each state, the number of distributions is $\mathcal{N}_s!/N_s!(\mathcal{N}_s - N_s)!$. In both cases we get again the entropies given above. In general, quantal corrections add to the classical, or quasi-classical, regime.

In 1917 Einstein noticed[5] that $N_2 n$ absorbtion acts should equal at equilibrium $N_1(n+1)$ emission acts, where $N_{1,2} = e^{-\beta E_{1,2}}$ is the Boltzmann distribution of $N_{1,2}$ atoms, and n is the number of radiation quanta, $E_1 - E_2 = \hbar\omega$. Indeed, the Bose-Einstein distribution for photons, *i.e.* the black-body radiation, or thermal radiation distribution, is again recovered for n. This observation underlies the laser effect discovered much later, in 1960.

Statistical motion distributes the ensemble over its various mechanical, or quantal, states, among which the ensemble fluctuates. For statistical equilibrium, the statistical scale energy, which is the temperature T, must be much larger than the mechanical, or quantal (we are forced to talk quantally by the atomistic nature of the world) scale energy, say $\Delta\varepsilon$, so $T \gg \Delta\varepsilon$. This is why the statistical equilibrium is difficult to be attained at very low temperatures, or for small-size ensembles, whose quantal scale energy is rather large. Condensed matter, consisting of ensembles with a large number of degrees of freedom (macroscopic ensembles), has densely-distributed quantal states, so it behaves easier statistically and thermodynamically. Time $\tau_{th} = \hbar/T$ is called statistical fluctuations time, or thermal time, while $\tau_e \sim \hbar/\Delta\varepsilon$ is the externally observable time, or quantal fluctuations time; therefore, $\tau_{th} \ll \tau_e$. Statistical fluctuations are distributed by a probability $\sim e^S$, a distribution explicitly used by Einstein[6], where a dependence $\delta S = (1/2)(\partial^2 S/\partial x^2)(\delta x)^2$ of the entropy variation δS on any parameter change δx is implied. Therefore, fluctuations are distributed by a gaussian, of the form $\exp[-a(\delta x)^2]$, where $a > 0$, and, for instance, the fluctuation energy per particle is $\delta e \sim T\sqrt{c}$, where c is the heat capacity (at constant volume). For instance, taking the derivative of $\sum(E - \mathcal{E})e^{-\beta\mathcal{E}} = 0$ with respect to β one gets easily $\delta E^2 = T^2\partial E/\partial T$. Actually, the fluctuating time τ, denoted also τ_f, involves several quanta of action $\hbar$ (actually h, according to the cell volume in the phase space), say n, so that $\tau = nh/\delta e$, and $\tau_{th} \leq \tau(\tau_f) \ll \tau_e$.

Condensed matter, either classical, or quantal, or consisting of many field quanta, has another remarkable property, in that it is always enclosed into a volume and has a number of particles. Therefore, it has a density, which is almost uniformly distributed in space and time (a concentration of particles), a property which together with its densely-distributed quantal states made the condensed matter be amenable to a quasi-classical description; which is precisely the quantal and statistical nature of matter. A volume v per particle can then be defined, and an inter-particle spacing, which fluctuates. These spatial fluctuations extend over a length a, where $a^2 \sim -(v^{4/3}\partial^2 s/\partial v^2)^{-1}$, where s is the entropy per particle, and a is of the order of the inter-particle spacing. The change in the particle density $n(x,t)$ can then be considered, over distances x and times t much larger than a and, respectively, τ, which reads

$$n(x, t+\tau) - n(x,t) = n(x+a,t) + n(x-a,t) - 2n(x,t) \ , \tag{6}$$

in its simplest form, or

$$\partial n/\partial t = (a^2/2\tau)\partial^2 n/\partial x^2 \ . \tag{7}$$

This kinetic equation was suggested by Einstein in connection with his studies on Brownian motion. In its simplest form above it is the difusion equation (whose well-known solution is a time-evolving gaussian), describing diffusion, dissipation, relaxation and approach to equilibrium. It involves a force of osmotic pressure, and in the atomic limit of hydrodynamics, it points out a quantal nature of viscosity, and hence of turbulence, by the fluctuating time, and, in general, originating in statistical fluctuations. Einstein's kinetic equation can be extended to space motion, to including momentum fluctuations $\delta\mathbf{p}$, and in the presence of external transport velocity $\mathbf{v}$ and forces $\mathbf{F}$, like

$$dn/dt = \partial n/\partial t + \mathbf{v}\partial n/\partial \mathbf{r} + \mathbf{F}\partial n/\partial \mathbf{p} = \delta n/\tau = \\ = (a^2/2\tau)\Delta n + (\delta p^2/2\tau)\Delta_{\mathbf{p}} n \ , \tag{8}$$

which is Boltzmann's equation, and $\delta n/\tau$ is Boltzmann's collision number per unit time. Making use of this equation, it is easy to see that the quasi-classical entropy $S = -\sum n \ln(n/e)$ increases, or stays stationary, in time. The approach to equilibrium go by fluctuations. The use of the quasi-classical entropy is legitimate here, as the density is simply the phase space density, which is very low for condensed matter. Actually, the usual description of the "collision integral" $\delta n/\tau$ by binary collisions in proving Boltzmann's H-theorem exhibits fluctuating transitions beneath.

Practical realization of statistical motion is made possible by interaction (a necessary but not sufficient condition), so the evolution of statistical probability ρ is described by the "master equation"

$$\partial \rho_n/\partial t = \sum_m (T_{nm}\rho_m - T_{mn}\rho_n) \ , \tag{9}$$

where T_{nm} are the well-known transition probabilities $T_{nm} = (2\pi/\hbar) \lfloor V_{nm} \rfloor^2 \delta(E_n - E_m + \hbar\omega)$ to the second-order of perturbation theory. It is easy to see that entropy $S = -\sum \rho \ln \rho$ increases in time, or stays stationary, and equation above reduces to Einstein's kinetic equation in the quasi-classical description of condensed matter. Actually, it is also easy to see that Einstein's kinetic equation involves "collisions" and introduced "transitions", by fluctuations.

Einstein's kinetic equation describes also transport in condensed matter. In macroscopic transport small amounts of matter, energy, heat, electric charge, spin, etc are put on excited states slightly above the equilibrium, and follow their space and time evolution through the ensemble. The excited states of the interacting condensed matter are elementary excitations, as Landau pointed out in 1941 and 1957, and the macroscopic transport proceeds by such elementary excitations having a characteristic velocity v. The elementary excitations form an ideal, quasi-classical gas of particles, obeying therefore Einstein's kinetic equation, with the only difference that they have a finite lifetime τ_{life} and a mean free path $\Lambda = v\tau_{life}$. This lifetime is also called "collison" time τ_{coll}, for classical ensembles of particles. In order to be well-resolved the elementary excitations must be so that $\tau_{th} \leq \tau(\tau_f) \ll \tau_{life}(\tau_{coll}) \ll \tau_e$; in some instances, elementary excitations are constructed on elementary excitations, so the meaning of the lifetime and quantal time is to be observed. In small-size, mesoscopic or nanoscopic, structures, transport proceeds ballistically, or by tunneling, coherently or incoherently when thinish, in small amount, in contrast to the statistical, macroscopic transport. While macroscopic transport may proceed both by smooth diffusion or by inertial pulses.

4 Principle of Relativity

In 1905 Einstein[7] noticed that the space-time transforms employed by Lorentz in 1985 to keep Maxwell's equations of the electromagnetic field invariant under a change of inertial frames are also relevant for mechanical motion. On this occasion, he recognized the character of universal constant of the speed of light c, and Galilei's Principle of Relativity as the general principle of motion. Maxwell introduced the equations of the electromagnetic field in 1873, and Galilei is credited with the principle of relativity around 1632.

Motion proceeds in space and time, and it is relative, *i.e.* it depends on the reference frame with respect to which we talk about position $\mathbf{r}$ and time t. If time t is to be compared with distance

r, then a universal, constant, velocity c there must exist, such that ct is a distance. It turned out that this universal, constant, velocity is the speed of light, $c \simeq 3 \cdot 10^8$m/s, as shown by Michelson in 1881. Under the change of inertial frames, *i.e.* passing to another frame moving with constant velocity $\mathbf{v}$, the element ds of the length of the universe line must be an invariant, acording to Galilei's Principle of Relativity, which means that the "euclidean" distance square

$$ds^2 = -d\mathbf{r}^2 + c^2 dt^2 = -dx_i dx_i \tag{10}$$

is invariant under such a change, where $i = 1, 2, 3, 4$ $\mathbf{r} = (x_1, x_2, x_3)$, $x_4 = \tau = ict$. This is Lorentz' invariance. Since $d\mathbf{r} = \mathbf{v}dt$, it follows that speed of light is the maximum velocity in the Universe ($ds^2 \geq 0$). Lorentz's transforms, length contraction, time dilation, the proper time at rest, Doppler effect and the invariance of the phase of a plane wave, all come out straightforwardly from the invariance of the length of the universe line. The Universe turns out to be a four-dimensional space-time, the "velocities" are $u_i = dx_i/ds$, the Principle of Inertia reads $du_i/ds = 0$, Newton's Law is $mc^2 du_i/ds = f_i$, where f_i are forces, $f_4 = i\mathbf{v}\mathbf{f}/c$ is a temporal force, the relativistic motion is curvilinear in space-time and accelerations may generate inertial forces in non-inertial frames; the latter being Einstein's Principle of General Relativity, or Einstein's Principle of Equivalence, or Einstein's Principle of Inertia. For $v \ll c$ the length ds of the universe line reduces to ct, space $d\mathbf{r}$ is practically decoupled from time, and the non-relativistic motion is recovered.

In addition, since $dS = pdr$ and $dS = -Edt$, where dS is the infinitesimal action and E is the energy, then (10) leads straightforwardly to $E^2 = const + p^2c^2$, where $const = m^2c^4$, hence the inertia of energy $E = \sqrt{m^2c^4 + p^2c^2}$ and the mass-energy equivalence

$$E = mc^2 \ , \tag{11}$$

introduced by Einstein.[8] One can see that there is not a distinction anymore between potential and kinetic energy in relativistic motion. The mass-energy equivalence above opened the way towards releasing the nuclear energy in nuclear fission, fusion, and in fuelling the "shining" stars, as Bethe shown in 1939. Mechanical motion and electromagnetism are "unified" by relating time to space within Galilei's Principle of Relativity, and this is Einstein's Principle, or Theory, of Relativity.

There is a certain discrepancy between the "linear" quantal motion (2) and the "quadratic" relativistic motion (10), which has been solved by Dirac in 1928 by introducing "relativistic" matrices and fields for particles. We learnt on this occasion that particles may have spin, like the electron that has indeed a 1/2-spin, as Goudsmit and Uhlenbeck discovered in 1926, and that there exist antiparticles, like the positron for the electron; and, in general, that particles are quanta of relativistic fields; whose statistical motion may co-exist all right with their relativistic character.

There is, however, a certain limitation for the dynamics of the quantal relativistic fields, coming out from the relativistic mixing up of space and time, as shown by Landau and Peierls in 1930. Indeed, from $\Delta p \Delta x \sim \hbar$ one can also get $\delta v \Delta p \Delta t \sim \hbar$, where δv is the change in velocity. Since the velocities are bound from above by the speed of light c one obtains $\Delta p \Delta t \sim \hbar/c$, *i.e.* momentum is not well-defined instantaneously. Similarly, $\Delta x \sim \hbar/mc$ at least, at rest, which means that position is intrinsically not well-defined. Or, since $\Delta x \sim \hbar c/\varepsilon$ in motion, and $\varepsilon = cp$ in the ultra-relativist limit for instance, then $\Delta x \sim \hbar/p$, *i.e.* the position extends at least over the quantal wavelength. In Quantal Relativity fields move, neiher physical quantities, nor space, nor time. These limitations have far-reaching implications, in that the evolution of the relativistic fields can not be followed anymore in time, perturbation series are plagued with divergencies (removable by "renormalizing" the interacting objects), and we have to be content with probability transition amplitudes, as Heisenberg indicated for the scattering (S) matrix in 1938.

5 Principle of Inertia

From 1907 onwards Einstein insisted to view gravitation as a free motion in a curved space, *i.e.* as an inertial force. This is called Einstein's Principle of Equivalence, or Principle of General Relativity, or Principle of Inertia. In 1916 he finally wrote down the equations of the gravitational field.[9] On this occasion he removed the embarassing distinction between the gravitational mass and inertial mass in Newton's mechanics.

Motion proceeds by infinitesimal displacements $d\mathbf{r}$ in infinitesimal lapses of time dt. A free particle moving in homogeneous, uniform, absolute space and time preserves the same $d\mathbf{r}$ during the same dt, *i.e.* its velocity $\mathbf{v} = d\mathbf{r}/dt = const$, providing it exists, *i.e.* providing there is a trajectory of the motion. This is the great Principle of Inertia as formulated by Newton around 1686, and suggested by Descartes in 1630, or by Galilei in 1632. It also reads $d\mathbf{v}/dt = 0$, whence Newton's Law of motion $md\mathbf{v}/dt = md^2\mathbf{r}/dt^2 = \mathbf{F}$ for a force $\mathbf{F}$ acting on a mass m. Newton's law is invariant under Galilei's space-time transforms, which is Galilei's Principle of Relativity. The reference frames where Newton's law holds are called inertial frames. They move with respect to each other with a constant velocity. Motion in non-inertial frames, like circular motion, or motion in curved spaces, develops an inertial force in Newton's law, by the "own motion of the space". Among many other examples of forces, Newton shown in 1686 that gravitation force $\mathbf{F} = GmM\mathbf{r}/r^3$ between any two masses m and M is universal and keeps planets in motion around the Sun, for instance. The constant $G = 6.67 \cdot 10^{-11}\mathrm{Nm}^2/\mathrm{Kg}^2$ is the constant of gravitation, as measured by Cavendish in 1796-1798. The mass m in Newton's gravitation law is the same as the mass m in Newton's Law of motion, though obviously they have distinct natures, the former is called gravitational, the latter inertial. If gravitation is viewed as an inertial force in curved spaces, then we get rid of this unnatural distinction.

The whole trick is to write down the infinitesimal distance in the universe as $ds^2 = -g_{ij}dx^idx^j$ this time, where g_{ij} is the metric tensor for contravariant and covariant vectors, like the space-time coordinates x^i, for instance. This infinitesimal lenght of the universe line is Lorentz invariant. The metric tensor is, in general, associated to curved spaces, much alike the elastic deformations in elastic bodies. The velocities are $u^i = dx^i/ds$, and the Principle of Inetia and Newton's Law would imply similar changes in these velocities, with the length of the universe line ds. The difference is only that in curved spaces the change in vectors involve not only their own changes but also the changes due to the parallel transport along curved lines. This way, the derivatives acquire an additional contribution from the curved geometry itself, which can easily be related to inertial forces in such non-inertial frames. Moreover, such a parallel transport brings a quadratic contribution, such that the ensuing equations are non-linear. For instance, the Principle of Inertia reads

$$du^i/ds + \Gamma^i_{kl}u^ku^l = 0 \ , \tag{12}$$

where Christofell's symbols are related to the derivatives of the metric tensor, say $\Gamma \sim g\partial g/\partial x$, with a symbolic notation. On the other hand, it is easily to see that gravitational potential φ is related to the metric tensor g, from Lorentz invariance, such that (12) above implies gravitational forces incorporated in the geometric trajectory of a free particle in motion. A similar equation holds also for light rays, which are therefore bent in a gravitational field, as Einstein shown in 1911.

It is well-known that the variation of the action for matter, electromagnetic interaction included, goes with the energy-momentum tensor T_{ik}; it is easily to see that a similar variation for the gravitation action is related to the curvature of space, more precisely it is related to the Ricci tensor R_{ik}, which goes, generically, like gR, where R is the curvature, or Riemann's, tensor, which in turn goes like $R \sim (\partial\Gamma/\partial x, \Gamma^2)$. Riemann's tensor accounts for a non-vanishing change in a vector transported along a closed circuit in curved spaces, similar with the geometric phase pointed out much latter by Berry in 1984. From the vanishing of the total change in the action one gets straightforwardly Einstein's equations for the gravitational field

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik} \ , \tag{13}$$

where $R = g^{ik}R_{ik}$ is the scalar curvature and G is the universal constant of gravitation. Equations are non-linear, being given matter they provide the corresponding gravitational field, via the curvature of the space-time. Equations give also gravitational waves, gravitational radiation (which can be quantized!), corrections to Newton's law of gravitation for moving bodies, etc.

Einstein's theory of gravitation has implications for cosmology. On the asumption that matter fills space uniformly, time is the same for the whole Universe, and the space metric indicates either an open or a closed, expanding, Universe, in which case light coming in from space shifts towards red, according to Hubble observation made in 1929; which led to the original Big Bang hypothesis on the birth of the Universe. There exists a relict microwave background of cca 3K, as Penzias and Wilson seen in 1964, and its anisotropy may even give hints about the shape of the Universe. Singularities in gravitation solutions, related perhaps to black holes, do not seem to be physical, their origin might be similarly mysterious as the mysteriously missing matter and radiation in the Universe, like the "dark" matter and the "dark" energy. The dynamics of galactic objects, like stars, pulsars, giant reds, quasars, novae and supernovae, white dwarfs and neutron stars, seems to obey Einstein's equations of gravity.

6 "Principle of Fields"

Starting 1920, probably, till death, Einstein entertained his belief of "unifying" electromagnetism, and later on weak and strong interactions, with gravitation in a geometric picture of fields. The main difficulty in such an enterprise is the limitation relativity and quantal motion put on each other, in that there is no well-defined space, nor an instantaneously well-defined momentum, nor a well-defined time for a well-defined motion, for relativistic quantal fields, as gravity requires. From $Gm/\lambda = c^2$ and $\lambda = h/mc$ one gets Planck's length $\lambda = \sqrt{\hbar G/c^3} \sim 10^{-33}$cm, below which quantal motion and space-time curvature are conflictual. Einstein himself felt for long unsatisfied with the statistical nature of the quantal motion. The way out might be the more recent string theory, which assumes that all the matter constituents are small, quasi-one-dimensional strings, attached perhaps to branes, vibrating, involving more dimensions than four, and unifying the symmetries in supersymmetries; suggesting therefore a Theory of Everything.

Meantime, there have been known weak interactions, with their gauge fields bosons, and leptons, like neutrinos, whose dynamics is mixed up with that of photons and electrons, so that a unified electroweak picture emerged in the so-called standard model, suggested by Glashow, Weinberg and Salam around 1967. Similarly, strong interactions associated with nucleons and mesons, or hadrons and baryons in general, indicated a sub-nuclear structure of quarks, interacting via gluons in a quantal chromodynamics, and a grand unification of the strong interaction with the electroweak one is suggested. Search for more fundamental fields, like Higgs' boson, wherefrom massive particles may come about by breaking symmetries hidden in additional dimensions continues. On a comparative scale, with the "fine" constant of the fine structure $e^2/\hbar c = 1/137$ for the electromagnetic interaction (where $-e$ is the electron charge), the weak interaction is 10^{-5}, the strong one is 1, and the gravity is 10^{-39}. These strengths might be comparable and unifying for higher and higher energies, over smaller and smaller lengths, and on shorter and shorter times. "Quantizing" the gravitation however still remained as ellusive as ever. Ironically, while looking for one, single principle, we learnt about too many particles, fields, dimensions, properties, quantal numbers, etc. What we finally learn from Physics is that there is motion with numbers.

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