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Plasmons and polaritons in a semi-infinite plasma

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Abstract

Plasmon and polariton modes are derived for a half-space (semi-infinite) plasma by using a general, unifying procedure as based on the equations of motion and suitable boundary conditions. Previous results are rederived in much a simpler manner and new ones are obtained. The approach is based on representing the charge disturbances by a displacement field in the positions of the moving particles (electrons). The dielectric response and the electron energy loss are computed. The propagation of an electromagnetic wave in the semi-infinite plasma is treated by using the retarded electromagnetic potentials, and the reflected and refracted waves are computed, as well as the reflection coefficient. It is shown that there exist two waves in the semi-infinite plasma, either damped or propagating. Although there is no singularity in the reflection coefficient, it exhibits an enhancement on passing from the propagating regime to the damped one.

1 Introduction

After the discovery of bulk plasmons in an infinite electron plasma,[1]-[3] there was a great deal of interest in plasmons propagating in structures with special geometries, like a half-space (semi-infinite) plasma, a plasma slab of finite thickness, a two-plasmas interface (two plasmas bounding each other), a slab with a cilindrical hole, structures with surface gratings or regular holes patterns, layered films, cilindrical rods and spherical particles, etc. There is a vast literature on various structures with special geometries exhibiting plasmon modes. These studies were aimed mainly at identifying new plasmon modes, like the surface plasmons,[4]-[7] accounting for the electron energy loss experiments and exploring the interaction of the electron plasma with electromagnetic radiation (plasmon-polariton excitations).[8]-[13] More recently, a possible enhancement of the electromagnetic radiation scattered on electron plasma with various geometries enjoyed a particular interest. In all these studies the plasmon and polaritons modes are of fundamental importance.[14, 15] The methods used in deriving such results were of great diversity, resorting often to particular assumptions, such that the basic underlying mechanism of plasmons or polaritons' occurrence is often obscured. The need is therefore felt of having a general, unifying procedure for deriving plasmons and polaritons modes in structures with special geometries, as based on the equation of motion of the charge density, Maxwell's equations and the corresponding boundary conditions. Such a procedure is presented in this paper for a semi-infinite plasma.

We represent the charge disturbances as $\delta n = -n div \mathbf{u}$, where n is the (constant, uniform) charge concentration and $\mathbf{u}$ is a displacement field of the mobile charges (electrons). This representation is valid for $\mathbf{q}\mathbf{u}_\mathbf{q} \ll 1$, where $\mathbf{q}$ is the wavevector and $\mathbf{u}_\mathbf{q}$ is the Fourier component of the displacement field. We assume a rigid neutralizing background of positive charge, as in the well-known jellium model. In the static limit, *i.e.* for Coulomb interaction, the lagrangian of the electrons can be written as

$$L = \int d\mathbf{r} \left[mn\dot{\mathbf{u}}^2/2 - \frac{1}{2}\int d\mathbf{r}' U(|\mathbf{r}-\mathbf{r}'|)\delta n(\mathbf{r})\delta n(\mathbf{r}') \right] + e\int d\mathbf{r}\Phi(\mathbf{r})\delta n(\mathbf{r}) \ , \tag{1}$$

where m is the electron mass, $U(r) = e^2/r$ is the Coulomb energy, $-e$ is the electron charge and $\Phi(\mathbf{r})$ is an external scalar potential. Equation (1) leads to the equation of motion

$$m\ddot{\mathbf{u}} = ngrad \int d\mathbf{r}' U(|\mathbf{r}-\mathbf{r}'|)div\mathbf{u}(\mathbf{r}') + egrad\Phi, \tag{2}$$

which is the starting equation of our approach. We leave aside the damping effects.

By using the Fourier transform for an infinite plasma it is easy to see that the eigenmode of the homogeneous equation (2) is the well-known bulk plasmon mode given by $\omega_p^2 = 4\pi ne^2/m$. On the other side, in the static limit, equation $\delta n = -ndiv\mathbf{u}$ is equivalent with the Maxwell's equation $div\mathbf{E}_i = -4\pi e\delta n$, where $\mathbf{E}_i = 4\pi ne\mathbf{u}$ is the internal electric field (equal to $-4\pi\mathbf{P}$, where $\mathbf{P}$ is the polarization). Making use of the electric induction $\mathbf{D} = -grad\Phi = \varepsilon(\mathbf{D}+\mathbf{E}_i)$, where ε is the dielectric constant, we get the well-known dielectric function $\varepsilon = 1-\omega_p^2/\omega^2$ in the long-wavelength limit from the solution of the inhomogeneous equation (2). Similarly, since the current density is $\mathbf{j} = -en\dot{\mathbf{u}}$, we get the well-known electrical conductivity $\sigma = i\omega_p^2/4\pi\omega$, by solving equation (2).

We apply this approach to a semi-infinite plasma, and, after deriving the surface and bulk plasmon modes, compute the dielectric response and the electron energy loss. Further on, we consider the interaction with the electromagnetic field, as described by the usual term $(1/c)\int d\mathbf{r}\mathbf{j}\mathbf{A} - \int d\mathbf{r}\rho\Phi$ in the lagrangian, where $\mathbf{A}$ is the vector potential, $\rho = endiv\mathbf{u}$ is the charge density and Φ is the scalar potential. We limit ourselves to the interaction with the electric field (the non-relativistic limit), and compute the reflected and refracted waves, as well as the reflection coefficient. We find it more convenient to use the retarded potentials, which are equivalent with Maxwell's equations, instead of using directly the later. It is shown that there are two types of electromagnetic waves in a semi-infinite plasma, either damped or propagating, and the region in the wavevector space corresponding to their particular behaviour is determined. Although the reflection coefficient has no singularity, it exhibits nevertheless a enhancement on passing from the propagating to the damping regime, as expected. The present approach can be extended to various other structures with special geometries.

2 Plasma eigenmodes

We consider a semi-infinite plasma extending over the half-space $z > 0$. The displacement field $\mathbf{u}$ is then represented as $(\mathbf{v}, u_3)\theta(z)$, where $\mathbf{v}$ is the displacement component in the (x,y)-plane, u_3 is the the displacement component along the z-direction and $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function. In equation of motion (2) $div\mathbf{u}$ can then be replaced by

$$div\mathbf{u} = \left(div\mathbf{v} + \frac{\partial u_3}{\partial z} \right)\theta(z) + u_{30}\delta(z) \ , \tag{3}$$

where $u_{30} = u_3(\mathbf{r}, z=0)$, $\mathbf{r}$ being the in-plane (x,y) position vector. Equation (2) becomes

$$m\ddot{\mathbf{u}} = ne^2 grad \int d\mathbf{r}'dz' \frac{1}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}} \left[div\mathbf{v}(\mathbf{r}'.z') + \frac{\partial u_3(\mathbf{r}',z')}{\partial z'}\right] + $$
$$+ne^2 grad \int d\mathbf{r}' \frac{1}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}} u_3(\mathbf{r}',0) + egrad\Phi \tag{4}$$

for $z > 0$. One can see the depolarizing field occurring at the free surface $z = 0$ in equation (4). We use the Fourier transforms of the type

$$\mathbf{u}(r,z;t) = \sum_{\mathbf{k}} \int d\omega \mathbf{u}(\mathbf{k},z;\omega) e^{i\mathbf{kr}} e^{-i\omega t} \tag{5}$$

(for unit area in the plane), as well as the Fourier representation

$$\frac{1}{\sqrt{r^2+z^2}} = \sum_{\mathbf{k}} \frac{2\pi}{k} e^{-k|z|} e^{i\mathbf{kr}} \tag{6}$$

for the Coulomb potential. Then, it is easy to see that equation of motion (4) leads to

$$\omega^2 v = \frac{1}{2}\omega_p^2 \int_0^\infty dz' \left(kv - \frac{1}{k}\frac{\partial^2 v}{\partial z'^2}\right) e^{-k|z-z'|} - \frac{1}{2k}\omega_p^2 v_0' e^{-kz} - \frac{iek}{m}\Phi \tag{7}$$

and $iku_3 = \frac{\partial v}{\partial z}$, where we have dropped out for simplicity the arguments $\mathbf{k}$, z and ω and $v_0' = \left.\frac{\partial v}{\partial z}\right|_{z=0}$. The $\mathbf{v}$-component of the displacement field is directed along the wavevector $\mathbf{k}$ (in-plane longitudinal waves). This equation can easily be solved. Integrating by parts in its *rhs* we get

$$\omega^2 v = \omega_p^2 v - \frac{1}{2}\omega_p^2 v_0 e^{-kz} - \frac{iek}{m}\Phi \ , \tag{8}$$

hence

$$v = \frac{iek\omega_p^2}{2m} \frac{\Phi_0}{(\omega^2-\omega_p^2)(\omega^2-\omega_p^2/2)} e^{-kz} - \frac{iek}{m}\frac{\Phi}{\omega^2-\omega_p^2}$$
$$u_3 = -\frac{ek\omega_p^2}{2m} \frac{\Phi_0}{(\omega^2-\omega_p^2)(\omega^2-\omega_p^2/2)} e^{-kz} - \frac{e}{m}\frac{\Phi'}{\omega^2-\omega_p^2} \tag{9}$$

where $\Phi_0 = \Phi(\mathbf{k}, z = 0; \omega)$ and $\Phi' = \frac{\partial \Phi}{\partial z}$.

The solutions given by equation (9) exhibit two eigenmodes, the bulk plasmon $\omega_b = \omega_p$ and the surface plasmon $\omega_s = \omega_p/\sqrt{2}$, as it is well-known.[4, 5] Indeed, the homogeneous equation (8) ($\Phi = 0$) has two solutions: the surface plasmon $v = v_0 e^{-kz}$ for $\omega^2 = \omega_p^2/2$ and the bulk plasmon $v_0 = 0$ for $\omega^2 = \omega_p^2$. Making use of this observation we can represent the general solution as an eigenmodes series

$$v(\mathbf{k},z) = \sqrt{2k} v_0(\mathbf{k}) e^{-kz} + \sqrt{2} \sum_\kappa v(\mathbf{k},\kappa) \sin \kappa z \ , \tag{10}$$

for $z > 0$, where $v(\mathbf{k}, -\kappa) = -v(\mathbf{k}, \kappa)$, and $iku_3 w(\mathbf{k}, z) = \frac{\partial v(\mathbf{k},z)}{\partial z}$. Then, it is easy to see that the hamiltonian $H = T + U$ corresponding to the lagrangian $L = T - U$ given by equation (1) becomes

$$T = nm \sum_{\mathbf{k}} \dot{v}_0^*(\mathbf{k}) \dot{v}_0(\mathbf{k}) + nm \sum_{\mathbf{k}\kappa} \left(1 - \frac{\kappa^2}{k^2}\right) \dot{v}^*(\mathbf{k},\kappa) \dot{v}(\mathbf{k},\kappa)$$
$$U = 2\pi n^2 e^2 \sum_{\mathbf{k}} v_0^*(\mathbf{k}) v_0(\mathbf{k}) + 4\pi n^2 e^2 \sum_{\mathbf{k}\kappa} \left(1 - \frac{\kappa^2}{k^2}\right) v^*(\mathbf{k},\kappa) v(\mathbf{k},\kappa) \ , \tag{11}$$

where T is the kinetic energy and U is the potential energy. We can see that this hamiltonian corresponds to harmonic oscillators with frequencies $\omega_s = \omega_p/\sqrt{2}$ and $\omega_b = \omega_p$.

It is worth noting what happens if the surface of this semi-infinite plasma is covered with a dielectric with dielectric constant ε. It is easy to see that the dielectric brings a surface polarization proportional to $1/\varepsilon - 1$, so the surface displacement gets an additional contribution $(1/\varepsilon - 1)v_0$ in equation (8), leading to a total surface displacement v_0/ε. The surface plasmon is then modified to $\omega = \omega_p\sqrt{1 - 1/2\varepsilon}$.

Making use of $\mathbf{E}_i = 4\pi ne\mathbf{u}$ and of equations (9) we can write down the internal fields (polarization) as

$$\begin{aligned} \mathbf{E}_\perp(\mathbf{k}, z; \omega) &= \frac{i\mathbf{k}\omega_p^4\Phi_0(k,0;\omega)}{2(\omega^2-\omega_p^2)(\omega^2-\omega_p^2/2)}e^{-kz} - \frac{i\mathbf{k}\omega_p^2\Phi(\mathbf{k},z;\omega)}{\omega^2-\omega_p^2} \\ \mathbf{E}_\|(\mathbf{k}, z; \omega) &= -\frac{k\omega_p^4\Phi_0(k,0;\omega)}{2(\omega^2-\omega_p^2)(\omega^2-\omega_p^2/2)}e^{-kz} - \frac{\omega_p^2\Phi'(\mathbf{k},z;\omega)}{\omega^2-\omega_p^2} \end{aligned} \tag{12}$$

where $\mathbf{E}_\perp$ is directed along the in-plane wavevector $\mathbf{k}$ and $\mathbf{E}_\|$ is parallel with the z-axis (perpendicular to the surface $z = 0$).

We take an external potential of the form $\Phi(\mathbf{k}, z) = \Phi_0(\mathbf{k})e^{i\kappa z}$ (leaving aside the frequency argument ω), and get the electric induction $\mathbf{D}_\perp(\mathbf{k}, z) = -i\mathbf{k}\Phi_0(\mathbf{k})e^{i\kappa z}$ and $D_\|(\mathbf{k}, z) = -i\kappa\Phi_0(\mathbf{k})e^{i\kappa z}$ from $\mathbf{D} = -grad\Phi$. Then, we can see that the surface terms do not contribute to the response, as expected, and, making use of $\mathbf{E}_i = (1/\varepsilon - 1)\mathbf{D}$, we get the well-known dielectric function $\varepsilon(\kappa, \omega) = 1 - \omega_p^2/\omega^2$ in the long-wavelength limit.

3 Electron energy loss

It is well known that the energy loss per unit time (stopping power) is given by

$$P = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = -e\mathbf{v}\mathbf{E}_i \ , \tag{13}$$

for an electron moving with velocity $\mathbf{v} = (\mathbf{v}_\perp, v_\|)$, where the field $\mathbf{E}_i$ is taken at $\mathbf{r} = \mathbf{v}_\perp t$ and $z = v_\| t$ for $t > 0$ ($z > 0$). It is assumed that the electron energy is sufficiently large and the energy loss is small enough to use a constant $\mathbf{v}$ in estimating the *rhs* of equation (13). The potential created by this electron is given by the Poisson equation $\Delta\Phi = 4\pi e\delta(\mathbf{r} - \mathbf{v}_\perp t)\delta(z - v_\| t)$, whence, by making use of the Fourier representation (6), we get

$$\Phi(\mathbf{k}, z; \omega) = -\frac{2ev_\|}{(\omega - \mathbf{k}\mathbf{v}_\perp)^2 + k^2v_\|^2}e^{-i(\mathbf{k}\mathbf{v}_\perp - \omega)z/v_\|} \ . \tag{14}$$

We introduce this potential in equation (12) and compute the energy loss. It contains two contributions, one associated with the bulk plasmons,

$$P_b = e^2\omega_p^2\sum_{\mathbf{k}}\int d\omega\frac{i\omega}{\omega_p^2 - \omega^2}\cdot\frac{2v_\|}{(\omega - \mathbf{k}\mathbf{v}_\perp)^2 + k^2v_\|^2} \ , \tag{15}$$

and another arising from surface effects,

$$P_s = e^2\omega_p^4\sum_{\mathbf{k}}\int d\omega\frac{1}{(\omega^2 - \omega_p^2/2)(\omega^2 - \omega_p^2)}\cdot\frac{v_\|(i\mathbf{k}\mathbf{v}_\perp - kv_\|)}{(\omega - \mathbf{k}\mathbf{v}_\perp)^2 + k^2v_\|^2}e^{-kv_\| t}e^{i(\mathbf{k}\mathbf{v}_\perp - \omega)t} \ . \tag{16}$$

Both contributions can be calculated straightforwardly. We get the well-known result $P_b = \left(-e^2\omega_p^2v_\|/2v^2\right)\ln(vk_0/\omega_p)$, where k_0 is a maximum cut-off wavevector (associated with the ionization energy, or with the inverse of the mean inter-particle spacing), and

$$P_s = -2\pi\frac{e^2\omega_p v_\|}{v^2t}\left(\sqrt{2}\sin\omega_p t/\sqrt{2} - \sin\omega_p t\right) \tag{17}$$

for $\omega_p \ll vk$. We can see the oscillatory behaviour of the stopping power arising from the surface effects in the transient regime near the surface.

4 Electromagnetic radiation

We assume a plane wave incident on the plasma surface under angle α. Its frequency is given by $\omega = cK$, where c is the velocity of light and the wavevector $\mathbf{K} = (\mathbf{k}, \kappa)$ has the in-plane component $\mathbf{k}$ and the perpendicular-to-plane component κ, such as $k = K\sin\alpha$ and $\kappa = K\cos\alpha$. In addition, $\mathbf{k} = k(\cos\varphi, \sin\varphi)$. The electric field is taken as $\mathbf{E} = E_0(\cos\beta, 0, -\sin\beta)e^{i\mathbf{kr}}e^{i\kappa z}e^{-i\omega t}$, such as $\cos\beta\sin\alpha\cos\varphi - \sin\beta\cos\alpha = 0$ (transversality condition $\mathbf{KE}_0 = 0$). In spite of the fact that the electrons are acted by a propagating wave, we still use the Coulomb interaction between them (static limit), which is unphysical. Consequently, some features of the results given in this section are unphysical. Nevertheless, we present here such a treatment, in order to show the main technical points involved in the next section, where the correct treatment is given.

Starting from equation (2) and using the Fourier representation (6), we write down the equations of motion similar with those given by equation (7). It is convenient to use the projections of the in-plane displacement field $\mathbf{v}$ on the vector $\mathbf{k}$ and on the vector $\mathbf{k}_\perp = k(-\sin\varphi, \cos\varphi)$. We denote these components by $v_1 = \mathbf{kv}/k$ and $v_2 = \mathbf{k}_\perp\mathbf{v}/k$. Leaving aside the irrelevant arguments $\mathbf{k}$ and ω (and making an integration by parts in order to remove some explicit surface terms) we get the equations of motion

$$\begin{gathered}\omega^2 v_1(z) = \tfrac{1}{2}k\omega_p^2\int_0^\infty dz' v_1(z')e^{-k|z-z'|}+\\ +\tfrac{i}{2}\omega_p^2\int_0^\infty dz' u_3(z')\tfrac{\partial}{\partial z'}e^{-k|z-z'|} + \tfrac{e}{m}E_0(z)\cos\beta\cos\varphi\ ,\end{gathered} \tag{18}$$

$$\begin{gathered}\omega^2 u_3(z) = -\tfrac{i}{2}\omega_p^2\int_0^\infty dz' v_1(z')\tfrac{\partial}{\partial z}e^{-k|z-z'|}+\\ +\tfrac{1}{2k}\omega_p^2\int_0^\infty dz' u_3(z')\tfrac{\partial^2}{\partial z\partial z'}e^{-k|z-z'|} - \tfrac{e}{m}E_0(z)\sin\beta\end{gathered} \tag{19}$$

and

$$\omega^2 v_2(z) = -\frac{e}{m}E_0(z)\cos\beta\sin\varphi\ . \tag{20}$$

Equation (20) is already solved. Equations (18) and (19) can be solved easily by noting that they imply the relationship

$$\frac{\partial v_1}{\partial z} = iku_3 + \frac{e}{m\omega^2}\left(ikE_0\sin\beta + \frac{\partial E_0}{\partial z}\cos\beta\cos\varphi\right)\ . \tag{21}$$

For a plane wave $E_0(z) = E_0e^{i\kappa z}$ we get

$$v_1 = \frac{ie\omega_p^2E_0\sin\beta}{m\omega^2(2\omega^2-\omega_p^2)}e^{-kz} + \frac{eE_0\cos\beta\cos\varphi}{m\omega^2}e^{i\kappa z}\ , \tag{22}$$

$$v_2 = -\frac{eE_0\cos\beta\sin\varphi}{m\omega^2}e^{i\kappa z} \tag{23}$$

and

$$u_3 = -\frac{e\omega_p^2E_0\sin\beta}{m\omega^2(2\omega^2-\omega_p^2)}e^{-kz} - \frac{eE_0\sin\beta}{m\omega^2}e^{i\kappa z}\ . \tag{24}$$

We can see that the displacement field exhibits both a surface term ($\sim e^{-kz}$), with a resonance at the frequency of the surface plasmons, and a bulk term ($\sim e^{i\kappa z}$). The bulk plasmons are absent, in accordance with their longitudinal character. In addition, the bulk contribution to the displacement field is transversal to the propagation wavevector $\mathbf{K}$, as it is produced by the transversal external electric field.

We pass now to computing the radiation field. The displacements given above can be represented as

$$v_1(\mathbf{r}, z; t) = v_{1s}e^{i\mathbf{kr}}e^{-kz}e^{-i\omega t} + v_{1b}e^{i\mathbf{kr}}e^{i\kappa z}e^{-i\omega t} \ , \tag{25}$$

and similarly for v_2 and u_3, where the amplitudes $v_{1s,b}$, $v_{2s,b}$ and $u_{3s,b}$ are given by equations (22)-(24). These fields produce a current density $\mathbf{j} = -en\dot{\mathbf{u}}\theta(z)e^{i\mathbf{kr}}e^{-i\omega t}$ and a charge density $\rho = endiv\mathbf{u} = enu_{30}\delta(z)e^{i\mathbf{kr}}e^{-i\omega t}$ arising from the surface polarization. It is worth noting that the bulk charge $\left(i\mathbf{kv} + \frac{\partial u_3}{\partial z}\right)\theta(z)$ is vanishing, as expected. The current and charge density gives rise to a vector potential

$$\mathbf{A}(\mathbf{r}, z; t) = \frac{1}{c}\int d\mathbf{r}'\int dz'\frac{\mathbf{j}(\mathbf{r}', z'; t - R/c)}{R} \tag{26}$$

and a scalar potential

$$\Phi(\mathbf{r}, z; t) = \int d\mathbf{r}'\int dz'\frac{\rho(\mathbf{r}', z'; t - R/c)}{R} \ , \tag{27}$$

where $R = \sqrt{(\mathbf{r} - \mathbf{r}')^2 + (z - z')^2}$. These integrals can be calculated exactly. They reduce to the known integral[1]

$$\int_{|z|}^{\infty} dx J_0\left(k\sqrt{x^2 - z^2}\right)e^{i\omega x/c} = \frac{i}{\kappa}e^{i\kappa|z|} \ , \tag{28}$$

where J_0 is the zeroth-order Bessel function of the first kind. From $\mathbf{E} = -(1/c)\frac{\partial \mathbf{A}}{\partial t} - grad\Phi$ we get the electric field

$$\begin{aligned} E_1 &= \frac{\omega_p^2 E_0}{2\omega^2}\frac{\sin\beta}{\sin 2\alpha}\left(\cos 2\alpha + i\frac{\omega_p^2}{2\omega^2}\sin 2\alpha\right)e^{-i\kappa z} \\ E_2 &= -\frac{\omega_p^2 E_0}{4\omega^2}\frac{\cos\beta\sin\varphi}{\cos^2\alpha}e^{-i\kappa z} \\ E_3 &= -\frac{\omega_p^2 E_0}{4\omega^2}\frac{\sin\beta}{\cos^2\alpha}\left(\cos 2\alpha + i\frac{\omega_p^2}{2\omega^2}\sin 2\alpha\right)e^{-i\kappa z} \end{aligned} \tag{29}$$

radiated in the region $z < 0$. We can see that this field represents the reflected plane wave ($\kappa \rightarrow -\kappa$), and we may check easily the orthogonality of the bulk contribution to the wavevector. We give here for convenience the bulk contribution to the radiated field: $E_b = (\omega_p^2/4\omega^2\cos^2\alpha)(\cos\beta\cos\varphi, -\cos\beta\sin\varphi, \sin\beta)e^{-i\kappa z}$. The remaining contribution with respect to equations (29) is the surface contribution (which is not transversal to the propagation vector), arising both from the curent and charge densities. It is worth noting that this contribution goes like $1/\omega^2$ in the high-frequency limit, as does the bulk contribution. The reflection coefficient R in $R^2 = |E|^2/E_0^2$ is given by

$$R^2 = \left(\frac{\omega_p^2}{4\omega^2\cos^2\alpha}\right)^2\left\{\cos^2\beta\sin^2\varphi + \frac{\sin^2\beta}{\sin^2\alpha}\left[\cos^2 2\alpha + \left(\frac{\omega_p^2}{2\omega^2 - \omega_p^2}\right)^2\sin^2 2\alpha\right]\right\} \ . \tag{30}$$

One can see that it exhibits a singularity at the surface plasmons frequency, which, as we see in the next section, is unphysical.

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press (2000), pp. 714-715, 6.677; 1,2.

5 Polaritons

In the presence of an electromagnetic wave we use the equation of motion

$$\omega^2\mathbf{u} = \frac{e}{m}\mathbf{E} + \frac{e}{m}\mathbf{E}_0 e^{i\kappa z} \ , \tag{31}$$

where we have preseved explicitly only the z-dependence ($i.e.$ we leave aside the factors $e^{i\mathbf{kr}}e^{-i\omega t}$). We find it convenient to employ the retarded potentials given by equations (26) and (27), instead of Maxwell's equations, where $\boldsymbol{j}$=-$ne\dot{\mathbf{u}}$ and $\rho = nediv\mathbf{u} = ne\left(i\mathbf{ku} + \frac{\partial u_3}{\partial z}\right)\theta(z) + neu_3(0)\delta(z)$. Obviously, these potentials are equivalent with Maxwell's equations. We preserve the geometry of the incident wave $\mathbf{E}_0$ given in the preceding section, and use the coordinates $v_{1,2}$ given there, as well as the components $E_1 = \mathbf{kE}/k$, $E_2 = \mathbf{k}_\perp\mathbf{E}/k$ and similar ones for the external field $\mathbf{E}_0$. Under these circumstances we get the electric field

$$E_1 = -2\pi ine\kappa \int_0 dz' u_1(z')e^{i\kappa|z-z'|} - 2\pi ne\tfrac{k}{\kappa}\int_0 dz' u_3(z')\tfrac{\partial}{\partial z'}e^{i\kappa|z-z'|}$$

$$E_2 = -2\pi ine\tfrac{\omega^2}{c^2\kappa}\int_0 dz' u_2(z')e^{i\kappa|z-z'|} \tag{32}$$

$$E_3 = 2\pi ne\tfrac{k}{\kappa}\int_0 dz' u_1(z')\tfrac{\partial}{\partial z}e^{i\kappa|z-z'|} - 2\pi ine\tfrac{k^2}{\kappa}\int_0 dz' u_3(z')e^{i\kappa|z-z'|}$$

from the retarded potentials, where we have used equation (28) and $\omega^2 = c^2K^2$. Now, we employ equation of motion (31) and get the integral equations

$$\omega^2 v_1 = -\tfrac{i\omega_p^2\kappa}{2}\int_0 dz' v_1(z')e^{i\kappa|z-z'|} - \tfrac{\omega_p^2 k}{2\kappa}\int_0 dz' u_3(z')\tfrac{\partial}{\partial z'}e^{i\kappa|z-z'|} + \tfrac{e}{m}E_0\cos\beta\cos\varphi e^{i\kappa z}$$

$$\omega^2 v_2 = -\tfrac{i\omega_p^2\omega^2}{2c^2\kappa}\int_0 dz' v_2(z')e^{i\kappa|z-z'|} - \tfrac{e}{m}E_0\cos\beta\sin\varphi e^{i\kappa z} \tag{33}$$

$$\omega^2 u_3 = \tfrac{\omega_p^2 k}{2\kappa}\int_0 dz' v_1(z')\tfrac{\partial}{\partial z}e^{i\kappa|z-z'|} - \tfrac{i\omega_p^2 k^2}{2\kappa}\int_0 dz' u_3(z')e^{i\kappa|z-z'|} - \tfrac{e}{m}E_0\sin\beta e^{i\kappa z}$$

for the coordinates $v_{1,2}$ and u_3 in the region $z > 0$.

The second equation (33) can be solved by noticing that

$$\frac{\partial^2}{\partial z^2}\int_0 dz' v_2(z')e^{i\kappa|z-z'|} = -\kappa^2\int_0 dz' v_2(z')e^{i\kappa|z-z'|} + 2i\kappa v_2 \ . \tag{34}$$

We get

$$\frac{\partial^2 v_2}{\partial z^2} + (\kappa^2 - \omega_p^2/c^2)v_2 = 0 \tag{35}$$

The solution of this equation is

$$v_2 = \frac{eE_0\cos\beta\sin\varphi}{m\omega_p^2}\cdot\frac{2c^2\kappa\left(\kappa_2' - \kappa\right)}{\omega^2}e^{i\kappa_2' z} \ , \tag{36}$$

where

$$\kappa_2' = \sqrt{\kappa^2 - \omega_p^2/c^2} \ . \tag{37}$$

For $\kappa^2 < \omega_p^2/c^2$ this wave does not propagate. For $\kappa^2 > \omega_p^2/c^2$ it represents a refracted wave with the refraction angle α_2' given by

$$\frac{\sin\alpha_2'}{\sin\alpha} = \frac{1}{\sqrt{1-\omega_p^2/\omega^2}} = 1/\sqrt{\varepsilon} \ . \tag{38}$$

The polariton frequency corresponding to this mode is given by

$$\omega_2^2 = c^2K^2 = \omega_p^2 + c^2K^{'2} \ , \tag{39}$$

as it is well known, where $K^{'2} = \kappa_2^{'2} + k^2$. In the limit $c \to \infty$ equation (36) reduces to equation (23), as expected.

The first and the third equations (33) can be solved by using the same equation (34) and by noticing that they imply $\kappa^2 u_3 = ik\frac{\partial v_1}{\partial z}$, which is the transversality condition. We get

$$v_1 = -\frac{eE_0\cos\beta\cos\varphi}{m\omega_p^2}\cdot\frac{2c^2\kappa^2\left(\kappa_1^{'}-\kappa\right)}{\omega^2\kappa + c^2k^2\left(\kappa_1^{'}-\kappa\right)}e^{i\kappa_1^{'}z} \tag{40}$$

and

$$u_3 = \frac{eE_0\cos\beta\cos\varphi}{m\omega_p^2}\cdot\frac{2c^2k\kappa_1^{'}\left(\kappa_1^{'}-\kappa\right)}{\omega^2\kappa + c^2k^2\left(\kappa_1^{'}-\kappa\right)}e^{i\kappa_1^{'}z} \ , \tag{41}$$

where

$$\kappa_1^{'} = \kappa^2\sqrt{\frac{\omega^2-\omega_p^2}{\omega^2\kappa^2+\omega_p^2k^2}} \ . \tag{42}$$

For $\omega < \omega_p$ these waves do not propagate. For $\omega > \omega_p$ these displacement fields represent another refracted wave, with the refraction angle $\alpha_1^{'}$ given by

$$\frac{\sin\alpha_1^{'}}{\sin\alpha} = \left(1-\frac{\omega_p^2}{\omega^2+\omega_p^2\tan^2\alpha}\right)^{-1/2} \ . \tag{43}$$

Similarly, the polariton frequency corresponding to this mode is obtained from $\omega_1^2 = c^2(\kappa^2+k^2)$, where equation (42) is used to get κ as a function of $\kappa_1^{'}$. This is a third-order equation

$$\kappa^6 - \left(K^{'2}\cos 2\alpha_1^{'} + \omega_p^2/c^2\right)\kappa^4 - \frac{1}{4}K^{'4}\sin^2 2\alpha_1^{'}\left(\kappa^2+1\right) = 0 \tag{44}$$

in κ^2, which has only one solution. In the long-wavelength limit the polariton frequency is given by $\omega_1^2(K^{'}\to 0) \simeq \omega_p^2 + c^2\kappa_1^{'2}$, while in the short-wavelength limit it reaches the photon frequency, $\omega_1(K^{'}\to\infty) = cK^{'}$, as expected (and $\kappa_1^{'}\to\kappa$).

The electric field propagating in the semi-infnite plasma ($z > 0$) can be derived straightforwardly from equation (31) and the displacement field given above. It is easy to see that it is not anymore transversal to its wavevector, due to the change $\kappa \to \kappa_1^{'}$.

In order to get the reflected wave (the region $z < 0$) we turn to equation (32) and use therein the solutions given above for $v_{1,2}$ and u_3. We get straightforwardly the fields

$$E_1 = E_0\cos\beta\cos\varphi\cdot\frac{1-f\tan^2\alpha}{1+f\tan^2\alpha}\cdot\frac{1-f}{1+f}e^{-i\kappa z} \ , \tag{45}$$

where

$$f = \sqrt{\frac{\omega^2-\omega_p^2}{\omega^2+\omega_p^2\tan^2\alpha}} \ , \tag{46}$$

$$E_2 = E_0\cos\beta\sin\varphi\cdot\frac{\sqrt{\omega^2\cos^2\alpha-\omega_p^2}-\omega\cos\alpha}{\sqrt{\omega^2\cos^2\alpha-\omega_p^2}+\omega\cos\alpha}e^{-i\kappa z} \tag{47}$$

and $E_3 = kE_1/\kappa$. We can see that these fields represent the reflected wave ($\kappa \to -\kappa$), and we can check its transversality to the propagation wavevector. The reflection coefficient can be written as

$$R^2 = \cos^2\beta\sin^2\varphi\left|\frac{\sqrt{\omega^2\cos^2\alpha-\omega_p^2}-\omega\cos\alpha}{\sqrt{\omega^2\cos^2\alpha-\omega_p^2}+\omega\cos\alpha}\right|^2 + \frac{\sin^2\beta}{\sin^2\alpha}\left|\frac{1-f\tan^2\alpha}{1+f\tan^2\alpha}\cdot\frac{1-f}{1+f}\right|^2 . \tag{48}$$

We can see that there is no singularity in this function, but there are two cusps in its behaviour, associated with the transition from the propagating regime to the damping regime. They occurr at $\omega^2 = \omega_p^2$ and $\omega^2 = \omega_p^2/\cos^2\alpha$ where the reflection coefficient exhibits a sudden enhancement on passing from the propagating regime to the damping one, as expected. For $\omega^2 \leq \omega_p^2$ the reflection coefficient is given by

$$R^2 = \cos^2\beta\sin^2\varphi + \frac{\sin^2\beta}{\sin^2\alpha} , \tag{49}$$

while

$$R^2 = \cos^2\beta\sin^2\varphi + \frac{\sin^2\beta}{\sin^2\alpha}\cdot\frac{1}{\left(\sqrt{1+\sin^2\alpha}+\sin\alpha\right)^4} \tag{50}$$

for $\omega^2 = \omega_p^2/\cos^2\alpha$. Slightly above ω_p^2 and $\omega_p^2/\cos^2\alpha$ the slope of the function $R^2(\omega^2)$ is $-\infty$.

6 Conclusions

The approach presented here is a quasi-classical one, valid for wavelengths much longer than the amplitude of the Fourier components of the displacement field. This is not a particularly restrictive condition. When this condition is violated, as, for instance, for wavelengths much shorter than the mean separation distance between electrons, there appear both higher-order terms in the equations of motion and the coupling to the individual motion of the electrons. These couplings affect in general the dispersion relations and introduce a finite lifetime (damping) for the plasmon and polaritons modes. Some of these questions are left for a forthcoming investigation. Several other particular issues, as the derivation of the van der Waals force from the fluctuations of the surface plasmons, surface impedance, surface plasmons radiative decay, response to point-like charges, etc, may get a new, more accurate aspect by means of the results presented here.

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