Journal of Theoretical Physics

Founded and Edited by M. Apostol **179** (2009)

ISSN 1453-4428

Diffraction of an electromagnetic plane wave by a metallic sphere. Mie's theory revisited

M. Apostol and G. Vaman
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

The diffraction of a plane electromagnetic wave by an ideal metallic sphere (Mie's theory) is investigated by a new method. The method consists in representing the charge disturbances (polarization) by a displacement field in the positions of the mobile charges (electrons) and using the equation of motion for the polarization together with the electromagnetic potentials. We employ a special set of orthogonal functions, which are combinations of spherical Bessel functions and vector spherical harmonics. This way, we obtain coupled integral equations for the displacement field, which we solve. In the non-retarded limit (Coulomb interaction) we get the branch of "spherical" (surface) plasmons at frequencies $\omega = \omega_p\sqrt{l/(2l+1)}$, where ω_p is the (bulk) plasma frequency and $l = 1, 2, ...$ When retardation is included, for an incident plane wave, we compute the field inside and outside the sphere (the scattered field), the corresponding energy stored by these fields, the Poynting vector and the scattering cross-section. The results agree with the so-called theory of "effective medium permittivity", although we do not start the calculations with the dielectric function. In turn, we recover in our results the well-known dielectric function of metals. We have checked the continuity of the tangential components of the electric field and the continuity of the normal component of the electric displacement at the sphere surface, as well as the conservation of the energy flow and re-derived the "optical theorem". In the limit of small radii (in comparison with the electromagnetic wavelength) the sphere exhibits a series of resonant absorptions at frequencies close to the plasmon frequencies given above. For large radii these resonances disappear.

Keywords: diffraction; metallic sphere; Mie's theory; "spherical" plasmons

OCIS codes: 050.1940; 260.2065; 260.1960; 260.3910

PACS: 42.25.Fx; 42.25.Gy; 78.67.Bf

1 Introduction

The diffraction of an electromagnetic plane wave by metallic spheres has been thoroughly investigated long time ago by Mie[1]. The main result of this investigation is a selective absorption of light by small particles, for some frequencies which, thereafter, were associated with the frequencies of the "spherical" plasmons[2]-[4]. Recently, the subject enjoys a great deal of interest, in connection with plasmons and polaritons in structures with restricted geometry, their role in

the diffraction of the electromagnetic wave and a possible enhancement of the scattered field[5]-[15]. The physics underlying such phenomena is rather obscured in the original Mie's results, due to the mathematical complexity of the problem. Though this degree of complexity is unavoidable, we attempt herein to investigate the problem by a new method, which, we hope, can be more enlightening. We compute the "spherical" plasmons frequencies given by $\omega^2 = \omega_p^2 \frac{l}{2l+1}$ in the non-retarded (Coulomb) limit, where ω_p is the (bulk) plasma frequency and $l = 1, 2, ...$ Including retardation, for an incident plane wave, we compute the fields inside and outside the sphere (the scattered field), the energy stored by these fields, the Poynting vector, the scattering cross-section, and, in general, we try to characterize as completely as possible the interaction of the electromagnetic plane wave with the metallic sphere. We put in evidence both the oscillating regime and the damped regime for the field inside the sphere, identify the polaritonic excitations and make connection with the so-called theory of "effective medium permittivity". We provide compact formulae for such various quantities, which, essentially, are represented as series of partial waves of (total) angular momentum $l = 1, 2, ...$ These formulae can readily be adapted to various particular cases. Such a particular case is the small radius of the sphere (in comparison with the electromagnetic wavelength), where the sphere interacting with the electromagnetic field exhibits a series of resonances for frequencies close to the frequencies of the "spherical" plasmons. For large radii theses resonances disappear.

The method we use herein is based on representing the polarization by a displacement field in the positions of the mobile charges (electrons) and using the equation of motion for this displacement field together with the electromagnetic potentials. The method turns out to be pretty general, and we employed it recently in studying the surface plasmons, the reflected and refracted fields and the reflection coefficient for a semi-infinite metallic plasma[16]. The method does not require the introduction from the beginning of the dielectric function of the medium, but we recover it in our final results. Our procedure leads to coupled integral equations, which seem to have been envisaged long ago in treating the interaction of the electromagnetic field with matter[17]. By using adequate sets of orthogonal functions we are able to solve these equations, and get final, compact results.

We assume a generic model of metals, consisting of mobile charges $-e$ and mass m, moving in a rigid neutralizing background. We can recognize here the well-known jellium-like plasma, which is an adequate representation of an ideal metal in the range of optical frequencies. We assume slight disturbances δn in the density of the charges, given by $\delta n = -n div \mathbf{u}$, where n (constant) is the particles concentration and $\mathbf{u}$ is a displacement field in the particles positions. Such a representation is valid for displacements u much smaller than the wavelengths. These density disturbances give rise to charge and current densities

$$\rho = end iv \mathbf{u} \ , \ \mathbf{j} = -ne\dot{\mathbf{u}} \ . \tag{1}$$

We compute the electric field through $\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - grad\Phi$, where the well-known vector and scalar potentials are given by

$$\mathbf{A} = \frac{1}{c}\int d\mathbf{r}' \frac{\mathbf{j}\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c\right)}{|\mathbf{r} - \mathbf{r}'|} \ , \ \Phi = \int d\mathbf{r}' \frac{\rho\left(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c\right)}{|\mathbf{r} - \mathbf{r}'|} \ . \tag{2}$$

The displacement field $\mathbf{u}$ is subjected to the equation of motion

$$m\ddot{\mathbf{u}} = -e\left(\mathbf{E} + \mathbf{E}_0\right) \ , \tag{3}$$

where $\mathbf{E}_0$ is an external field, or, by using a temporal Fourier transform,

$$\omega^2 \mathbf{u} = \frac{e}{m}\left(\mathbf{E} + \mathbf{E}_0\right) \ . \tag{4}$$

It is easy to see, by making use of the Maxwell equation $div\mathbf{E} = 4\pi\rho$, that equation (3) gives the well-known dielectric function $\varepsilon = 1 - \omega_p^2/\omega^2$ for a bulk plasma, where $\omega_p = \sqrt{4\pi ne^2/m}$ is the plasma frequency. The internal (polarizing) field is given by $\mathbf{E} = 4\pi ne\mathbf{u}$. Similarly, the equation of motion (3) and the Maxwell equation given above lead to the well-known conductivity $\sigma = ine^2/m\omega$. In this treatment we leave aside the magnetization, relativistic effects and dissipation.

The general idea of our procedure can be described as folows. We compute the electric field by equations (2) and get it as an integral containing the displacement field $\mathbf{u}$. Then, we express this electric field through $\mathbf{u}$ by using equation (3) and get an integral equation for $\mathbf{u}$, which we solve. It is such an integral-equation procedure that seems to have been suggested long ago in investigating the electromagnetic field interacting with matter[17], in connection with the so-called Ewald-Oseen extinction theorem. At the same time, we can recognize the elementary theory of classical dispersion in our using of the equation of motion (3) together with Maxwell's equations[18]. Making use of this theory, it is easy to see that the equation of motion (3) can easily be extended to simulate also the behaviour of a simple, classical dielectric, or to include the dissipation. Beside having applied this procedure to a semi-infinite body (half space)[16], we used it also for a slab of finite thickness[19], where we have calculated the dielectric response, the surface plasmons, the refracted, reflected and transmitted waves, surface plasmon-polariton modes, reflection and transmision coefficients, and derived generalized Fresnel relations. For such bodies with finite boundaries, we get coupled integral equations for the components of the displacement vector, which we solve by using suitable sets of orthogonal functions. In addition, we have also calculated the eigenmodes of such equations and derived van der Waals-London and Casimir forces acting between a pair of semi-infinite bodies[20]. We apply herein this treatment to an ideal spherical metallic particle, and compute the field everywhere in space, as was the original aim of Mie theory.

2 Coulomb interaction

We do the calculations in two steps. First, we consider the non-retarded (Coulomb interaction), therafter we include the retardation. In the former case, the equation of motion (4) reads

$$\omega^2\mathbf{u} = -\frac{1}{4\pi}\omega_p^2 grad\int d\mathbf{r}'\frac{div\mathbf{u}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{e}{m}\mathbf{E}_0 \ . \tag{5}$$

For a sphere of radius a, the displacement field $\mathbf{u}$ becomes

$$\mathbf{u} \to \mathbf{u}\theta(a-r) \ , \tag{6}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$ is the step function. We get

$$div\mathbf{u} \to div\mathbf{u}\theta(a-r) + u_r(a)\delta(r-a) \ , \tag{7}$$

where $u_r(a) = u_r(r=a)$ is the radial component of the field $\mathbf{u}$ at $r = a$. We can see the occurrence of the (de)polarizing field associated with $u_r(a)$. We use the well-known decomposition

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\sum_{lm}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) \tag{8}$$

of the Coulomb potential in spherical harmonics, where $r_< = \min(r,r')$ and $r_> = \max(r,r')$. The main ingredient of our calculations is the expansion

$$\mathbf{u}(\mathbf{r}) = \sum_{lm}\left[u_{lm}^0(r)\mathbf{Y}_{llm}(\theta,\varphi) + u_{lm}^-(r)\mathbf{Y}_{ll-1m}(\theta,\varphi) + u_{lm}^+(r)\mathbf{Y}_{ll+1m}(\theta,\varphi)\right] \tag{9}$$

of the displacement field in vector spherical harmonics[21, 22]. We make an extensive use of the properties of these functions, as given in Ref. [21]. Some of the formulae used here are included in **Appendix A**. In particular, we recall that their divergence, involved in equation (5), is related to (scalar) spherical harmonics, so that, by using the expansion (8) and the orthogonality of the spherical harmonics, we express the integral in equation (5) in terms of spherical harmonics. Then we recall that the gradient of the latter functions is related to vector spherical harmonics, so we recover these functions in the *rhs* of equation (5). Doing so, we are left with integral equations which imply integrations only with respect to the radial variable. We get $u^0_{lm} = 0$ and

$$\begin{aligned}
\tfrac{\omega^2}{\omega_p^2} u^-_{lm} &= \tfrac{l}{2l+1} u^-_{lm} - \tfrac{\sqrt{l(l+1)}}{2l+1} u^+_{lm} + \sqrt{l(l+1)} r^{l-1} \textstyle\int_r^a dr' \tfrac{1}{r'^l} u^+_{lm}(r') - \\
&- \tfrac{1}{4\pi ne} \sqrt{\tfrac{l}{2l+1}} \left(\tfrac{d}{dr} \Phi_{lm} + \tfrac{l+1}{r} \Phi_{lm} \right) \ , \\
\tfrac{\omega^2}{\omega_p^2} u^+_{lm} &= \tfrac{l+1}{2l+1} u^+_{lm} - \tfrac{\sqrt{l(l+1)}}{2l+1} u^-_{lm} + \sqrt{l(l+1)} \tfrac{1}{r^{l+2}} \textstyle\int_0^r dr' r'^{l+1} u^-_{lm}(r') + \\
&+ \tfrac{1}{4\pi ne} \sqrt{\tfrac{l+1}{2l+1}} \left(\tfrac{d}{dr} \Phi_{lm} - \tfrac{l}{r} \Phi_{lm} \right) \ ,
\end{aligned} \tag{10}$$

where we have introduced the external potential Φ (expanded in spherical harmonics) through $\mathbf{E}_0 = -grad\Phi$.

The solutions of these coupled equations can be found as series of powers r^n, for $n = 0, 1, 2...$, of the form

$$u^{\pm}_{lm} = \sum_{n=0} u^{\pm}_{lm}(n) r^n \ . \tag{11}$$

We get $u^+_{lm}(n) = 0$, the eigenfrequencies

$$\omega = \omega_p \sqrt{\frac{l}{2l+1}} \tag{12}$$

and

$$u^-_{lm}(n) = -\frac{1}{4\pi ne} \sqrt{l(2l+1)} \frac{\Phi_{lm}(l)}{\frac{\omega^2}{\omega_p^2} - \frac{l}{2l+1}} \delta_{n,l-1} \ , \tag{13}$$

which represents the dielectric response of the sphere. It is worth noting that for a descending series in powers of r^n (negative integers n) in equation (11) we get the eigenfrequencies

$$\omega = \omega_p \sqrt{\frac{l+1}{2l+1}} \ , \tag{14}$$

which correspond to a metallic void of radius a. Both these modes are surface, or "spherical" plasmons.

Making use of equation (13) we find the displacement $\mathbf{u}$ and the internal (depolarizing) field

$$\mathbf{E} = \frac{\omega^2}{\omega^2 - \omega_p^2/3} \mathbf{E}_0 \tag{15}$$

for an external field $\mathbf{E}_0$ oriented along the z-axis. One can see that $1 - \omega_p^2/3\omega^2$ can be viewed as the dielectric function for the $l = 1$- partial wave.

3 External plane wave. The field inside the sphere

We pass now to the retarded case. In equation (4) we consider a plane wave $\mathbf{E}_0 = E_0\mathbf{e}_x e^{ikz}$ for the external field, with frequency $\omega = ck$, propagating along the z-axis; $\mathbf{e}_x$ is the unit vector along the x-axis. We compute the electric field $\mathbf{E}$ from the electromagnetic potentials $\mathbf{A}$ and Φ given by equation (2), by making use of the charge and current densities given by equation (1). We assume the same decomposition given by equation (9) for the displacement field as a series of vector spherical harmonics.

It is very convenient to introduce the functions

$$F_{lmk}(\mathbf{r}) = j_l(kr)Y_{lm}(\theta,\varphi)\ ,\ H_{lmk}(\mathbf{r}) = h_l(kr)Y_{lm}(\theta,\varphi)\ , \tag{16}$$

where $j_l(kr)$ and $h_l(kr)$ are the spherical Bessel functions of the first and, respectively, third rank (the Hankel functions)[23, 24]. Their definition, together with recurrence relations, asymptotic behaviour and other formulae used in our calculations are included in **Appendix B**. The following decomposition holds[25]

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{ik}{4\pi}\sum_{lm} F^*_{lmk}(\mathbf{r}')H_{lmk}(\mathbf{r})\ ,\ r > r' \tag{17}$$

for the "retarded" Coulomb potential appearing in the electromagnetic potentials. We use this decomposition for computing the scalar potential Φ. We define also the vector functions

$$\mathbf{F}^0_{lmk}(\mathbf{r}) = j_l(kr)\mathbf{Y}_{lm}(\theta,\varphi)\ ,$$

$$\mathbf{F}^+_{lmk}(\mathbf{r}) = \frac{1}{\sqrt{2l+1}}\left[\sqrt{l}j_{l+1}(kr)\mathbf{Y}_{ll+1m}(\theta,\varphi) + \sqrt{l+1}j_{l-1}(kr)\mathbf{Y}_{ll-1m}(\theta,\varphi)\right]\ , \tag{18}$$

$$\mathbf{F}^-_{lmk}(\mathbf{r}) = \frac{1}{\sqrt{2l+1}}\left[\sqrt{l}j_{l-1}(kr)\mathbf{Y}_{ll-1m}(\theta,\varphi) - \sqrt{l+1}j_{l+1}(kr)\mathbf{Y}_{ll+1m}(\theta,\varphi)\right]$$

and a similar set of vector functions $\mathbf{H}^q_{lmk}$, $q = 0,\pm$, by replacing j_l by h_l in equations (18). We have the decomposition

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \frac{ik}{4\pi}\sum_{lmq} \mathbf{F}^{q*}_{lmk}(\mathbf{r}')\mathbf{H}^q_{lmk}(\mathbf{r})\ ,\ r > r'\ , \tag{19}$$

which we use in computing the vector potential $\mathbf{A}$. The functions $\mathbf{F}_{lmk}$ are orthogonal, complete and regular in the origin, while $\mathbf{H}_{lmk}$ are not regular in the origin and, together with $\mathbf{F}_{lmk}$, form a complete set for any region excluding the origin.

We insert the representation given by equation (17) in the scalar potential Φ, use the orthogonality of the spherical harmonics, perform the integrations by parts for the radial derivatives of the functions $u^{\pm}_{lm}$, reduce the boundary terms, and use equations (77) for the derivatives of the Bessel functions. Thereafter we compute the corresponding electric field, arising from the scalar potential, by using the gradient formula given in equation (65), and recover the vector spherical harmonics. We do similar calculations for the vector poential $\mathbf{A}$, by using equation (19), and get the corresponding electric field. In this calculation equations (77) and (78) are very useful.

Thereafter, we introduce the electric field $\mathbf{E}$, obtained according to the above description, in the equation of motion (4), and use the decomposition of the external field $\mathbf{E}_0$ in vector spherical harmonics as given in **Appendix C**. We identify the coefficients of the vector spherical harmonics

in this equation and get two sets of integral equations for the amplitudes $u_{lm}^{0,\pm}$. The first set consists of one equation

$$\begin{aligned}\tfrac{i(-1)^l 16\pi^2 c^2}{\omega_p^2 k} u_{lm}^0(r) = h_l(kr) \int_0^r dr' r'^2 j_l(kr') u_{lm}^0(r') + \\ + j_l(kr) \int_r^a dr' r'^2 h_l(kr') u_{lm}^0(r') + \tfrac{i(-1)^l \sqrt{\pi}}{nek^3} E_0 \sqrt{2l+1} j_l(kr) \left(\delta_{m,1} + \delta_{m,-1}\right) \ .\end{aligned} \tag{20}$$

The second set consists of two coupled integral equations. The first is

$$\begin{aligned}\tfrac{(-1)^l 16\pi^2 c^2}{\omega_p^2} u_{lm}^-(r) = \tfrac{(-1)^l 16\pi^2}{(2l+1)k^2} \left[l u_{lm}^-(r) - \sqrt{l(l+1)} u_{lm}^+(r) \right] + \\ + ik \tfrac{\sqrt{l+1}}{2l+1} h_{l-1}(kr) \int_0^r dr' r'^2 \left[\sqrt{l+1} j_{l-1}(kr') u_{lm}^-(r') + \sqrt{l} j_{l+1}(kr') u_{lm}^+(r') \right] + \\ + ik \tfrac{\sqrt{l+1}}{2l+1} j_{l-1}(kr) \int_r^a dr' r'^2 \left[\sqrt{l+1} h_{l-1}(kr') u_{lm}^-(r') + \sqrt{l} h_{l+1}(kr') u_{lm}^+(r') \right] + \\ + \tfrac{(-1)^l \sqrt{\pi}}{nek^2} E_0 \sqrt{l+1} j_{l-1}(kr) \left(-\delta_{m,1} + \delta_{m,-1}\right)\end{aligned} \tag{21}$$

and the second equation is given by

$$\begin{aligned}\tfrac{(-1)^l 16\pi^2 c^2}{\omega_p^2} u_{lm}^+(r) = \tfrac{(-1)^l 16\pi^2}{(2l+1)k^2} \left[(l+1) u_{lm}^+(r) - \sqrt{l(l+1)} u_{lm}^-(r) \right] + \\ + ik \tfrac{\sqrt{l}}{2l+1} h_{l+1}(kr) \int_0^r dr' r'^2 \left[\sqrt{l+1} j_{l-1}(kr') u_{lm}^-(r') + \sqrt{l} j_{l+1}(kr') u_{lm}^+(r') \right] + \\ + ik \tfrac{\sqrt{l}}{2l+1} j_{l+1}(kr) \int_r^a dr' r'^2 \left[\sqrt{l+1} h_{l-1}(kr') u_{lm}^-(r') + \sqrt{l} h_{l+1}(kr') u_{lm}^+(r') \right] + \\ + \tfrac{(-1)^l \sqrt{\pi}}{nek^2} E_0 \sqrt{l} j_{l+1}(kr) \left(-\delta_{m,1} + \delta_{m,-1}\right) \ .\end{aligned} \tag{22}$$

We pass now to solving these equations. We take the second derivative in equation (20) with respect to r and eliminate the intervening integrals by using equation (20) and its first derivative with respect to r. Then, we use equations (77) and (78) to get

$$r^2 \frac{d^2}{dr^2} u_{lm}^0 + 2r \frac{d}{dr} u_{lm}^0 + \left[k_1^2 r^2 - l(l+1) \right] u_{lm}^0 = 0 \ , \tag{23}$$

which is the Bessel equation for $j_l(k_1 r)$, where

$$k_1 = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2} \ . \tag{24}$$

We have therefore $u_{lm}^0 \sim j_l(k_1 r)$, where the coefficient is determined from equation (20). Making use of equation (73) we get

$$u_{lm}^0 = A_{lm} j_l(k_1 r) \ , \tag{25}$$

where

$$A_{lm} = \frac{(-1)^{l+1} \sqrt{\pi(2l+1)} c \omega_p^2 E_0}{nea^2 \omega^3} \cdot \frac{\left(\delta_{m,1} + \delta_{m,-1}\right)}{k_1 h_l(ka) j_{l+1}(k_1 a) - k h_{l+1}(ka) j_l(k_1 a)} \ . \tag{26}$$

In order to solve the system of equations (21) and (22) we introduce two new functions defined by

$$\begin{aligned}U_{lm} = \sqrt{l} u_{lm}^- - \sqrt{l+1} u_{lm}^+ \ , \\ r V_{lm} = \sqrt{l}(l-1) u_{lm}^- + \sqrt{l+1}(l+2) u_{lm}^+ \ .\end{aligned} \tag{27}$$

These combinations of amplitudes appear in $div\mathbf{u}\theta(a-r)$, equation (7), and we find easily, by equations (21) and (22), the relation $\frac{\partial U_{lm}}{\partial r} = V_{lm}$. This relation expresses the vanishing of the volume charge, as expected for a transverse field. Making use of this relation, we reduce the system of equations (21) and (22) to only one equation for U_{lm}, which we solve following the same method as the one described above for the function u^0_{lm}. We get

$$r^2\frac{d^2}{dr^2}U_{lm} + 4r\frac{d}{dr}U_{lm} + \left[k_1^2r^2 - l(l+1) + 2\right]U_{lm} = 0 \ , \tag{28}$$

whose solution is $U_{lm} \sim j_l(k_1r)/r$. We determine the amplitude of this solution from the integral equation for U_{lm} and then, making use of equations (27), we get the fields $u^{\pm}_{lm}$. These fields are given by

$$u^+_{lm}(r) = \frac{B_{lm}}{\sqrt{l+1}}j_{l+1}(k_1r) \ , \ u^-_{lm}(r) = \frac{B_{lm}}{\sqrt{l}}j_{l-1}(k_1r) \ , \tag{29}$$

where

$$B_{lm} = \frac{(-1)^{l+1}\sqrt{\pi l(l+1)}c^3k_1E_0}{nea^2\omega^3}\cdot$$
$$\cdot\frac{(-\delta_{m,1}+\delta_{m,-1})}{\left[\left(-\frac{\omega^2}{\omega_p^2}+\frac{l}{2l+1}\right)h_{l+1}(ka)+\frac{l+1}{2l+1}h_{l-1}(ka)\right]j_l(k_1a)+\frac{\omega^2k_1}{\omega_p^2k}h_l(ka)j_{l+1}(k_1a)} \ . \tag{30}$$

We put these results in a more compact and symmetrical form by introducing the notations

$$A_{lm} = \frac{e}{m\omega^2}E_0A_la_{lm} \ , \ B_{lm} = \frac{e}{m\omega^2}E_0\sqrt{\frac{l(l+1)}{2l+1}}B_lb_{lm} \ , \tag{31}$$

where a_{lm} and b_{lm} are the amplitudes of the plane wave given by equation (82),

$$A_l = \frac{16\pi^2(-1)^{l+1}c}{\omega a^2}\cdot\frac{1}{k_1h_l(ka)j_{l+1}(k_1a) - kh_{l+1}(ka)j_l(k_1a)} \tag{32}$$

and

$$B_l = \frac{16\pi^2(-1)^{l+1}c^3k_1}{\omega_p^2\omega a^2}\cdot$$
$$\cdot\frac{1}{\left[\left(-\frac{\omega^2}{\omega_p^2}+\frac{l}{2l+1}\right)h_{l+1}(ka)+\frac{l+1}{2l+1}h_{l-1}(ka)\right]j_l(k_1a)+\frac{\omega^2k_1}{\omega_p^2k}h_l(ka)j_{l+1}(k_1a)} \ . \tag{33}$$

With these notations the displacement field can be written as

$$\mathbf{u}(\mathbf{r}) = \frac{e}{m\omega^2}E_0\sum_{l=1m}^{\infty}\left[A_la_{lm}\mathbf{F}^0_{lmk_1}(\mathbf{r}) + B_lb_{lm}\mathbf{F}^+_{lmk_1}(\mathbf{r})\right] \tag{34}$$

and the field inside the sphere is given by

$$\mathbf{E}_i(\mathbf{r}) = \frac{m\omega^2}{e}u(\mathbf{r}) = E_0\sum_{l=1m}^{\infty}\left[A_la_{lm}\mathbf{F}^0_{lmk_1}(\mathbf{r}) + B_lb_{lm}\mathbf{F}^+_{lmk_1}(\mathbf{r})\right] \ . \tag{35}$$

We can see from the above equations that the external field $\mathbf{E}_0$ is modified inside the sphere, by the coefficients A_l and B_l, and replaced by the total field $\mathbf{E}_i = \mathbf{E} + \mathbf{E}_0$, which goes like the spherical Bessel functions $j_{l,l\pm1}(k_1r)$, where the "wave number" $k_1 = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$ is different than $k = \omega/c$. This is an illustration of the so-called Ewald-Oseen extinction theorem[17]. In addition, the field inside the sphere is either oscillating, for k_1 real ($\omega > \omega_p$), or damped, for k_1 purely imaginary ($\omega < \omega_p$); in the latter case there will appear the modified Bessel functions in the above

formulae. We note also that equation (24) which gives the "wave number" k_1, can also be written as $c^2k_1^2 = \varepsilon\omega^2$, where $\varepsilon = 1-\omega_p^2/\omega^2$ is the dielectric function of a metal. We get $\omega^2 = c^2k_1^2+\omega_p^2$ from this equation, which is the dispersion relation of polaritons in metals. The dispersion relationship $c^2k_1^2 = \varepsilon\omega^2$ is well-known in the theory of "effective medium permittivity".

Making use of $\mathbf{H}_i = (-i/k)curl\mathbf{E}_i$ and equations (66) we get the magnetic field inside the sphere

$$\mathbf{H}_i(\mathbf{r}) = iE_0 \sum_{l=1m}^{\infty} \left[A_l a_{lm}\mathbf{F}^+_{lmk_1}(\mathbf{r}) + B_l b_{lm}\mathbf{F}^0_{lmk_1}(\mathbf{r})\right] \quad , \tag{36}$$

which allows the calculation of the energy

$$W_i = \frac{1}{16\pi}\int_0^a dr\cdot r^2 \int d\Omega \left(|\mathbf{E}_i|^2 + |\mathbf{H}_i|^2\right) \tag{37}$$

stored inside the sphere. Using the orthogonality of the vector spherical harmonics this energy can be written as

$$\begin{aligned} W_i = \tfrac{1}{16\pi}E_0^2 \textstyle\sum_{l=1m}^{\infty}\left(|A_l a_{lm}|^2 + |B_l b_{lm}|^2\right)\cdot \\ \cdot\left[\int_0^a dr\cdot r^2\left(|j_l(k_1r)|^2 + \tfrac{l}{2l+1}\,|j_{l+1}(k_1r)|^2 + \tfrac{l+1}{2l+1}\,|j_{l-1}(k_1r)|^2\right)\right] \quad , \end{aligned} \tag{38}$$

where the integrals can be computed with the aid of formula (74).

We can compute also the Poynting vector defined as the real part of $\mathbf{S}_i = (c/8\pi)\,(\mathbf{E}_i \times \mathbf{H}_i^*)$. We limit ourselves to the radial component $(\mathbf{S}_i)_r$, which gives the radial flow

$$Q_i = \int d\Omega\,(\mathbf{S}_i)_r \quad , \tag{39}$$

where the integration is performed over the solid angle Ω. It can be calculated easily by using equations (70) and (71) and the orthogonality of the vector spherical harmonics. We get

$$\begin{aligned} Q_i = \tfrac{c}{8\pi}E_0^2 \textstyle\sum_{l=1m}^{\infty}\left(|A_l a_{lm}|^2 + |B_l b_{lm}|^2\right)\cdot \\ \cdot j_l(k_1r)\left[\tfrac{l}{2l+1}j^*_{l+1}(k_1r) + \tfrac{l+1}{2l+1}j^*_{l-1}(k_1r)\right] \quad . \end{aligned} \tag{40}$$

It is easy to see that this expression is purely imaginary, *i.e.* the net radial flow through the sphere is vanishing, as expected for such an ideal (non-dissipative) plasma. It is easy to compare equation (40) with the radial flow Q_0 of the plane wave (obtained by putting formally $A_l = B_l = 1$ in equation (40)), which is also vanishing.

4 The scattered field

Having known the displacement field $\mathbf{u}$ given by equation (34) we can compute the scattered electric field, *i.e.* the field created outside the sphere by charges and currents, via equations (1) and (2). In the electromagnetic potentials we use again the decompositions given by equations (17) and (19), employ the orthogonality of the spherical harmonics and integrals given by equation (73) for the Bessel functions, together with recurrence relations of the type (77) and (78). We get the scattered field

$$\mathbf{E}_s(\mathbf{r}) = \frac{ka^2}{16\pi^2}E_0 \sum_{l=1m}^{\infty}\left[A_l a_{lm} f_l^0\mathbf{H}^0_{lmk}(\mathbf{r}) + B_l b_{lm} f_l^+\mathbf{H}^+_{lmk}(\mathbf{r})\right] \quad , \tag{41}$$

where

$$f_l^0 = (-1)^l \left[k_1 j_l(ka) j_{l+1}(k_1 a) - k j_{l+1}(ka) j_l(k_1 a)\right] \tag{42}$$

and

$$f_l^+ = (-1)^l \left[k j_l(ka) j_{l+1}(k_1 a) - k_1 j_{l+1}(ka) j_l(k_1 a) - \frac{i\omega_p^2 (l+1)}{c^2 k k_1 a} j_l(ka) j_l(k_1 a)\right] \ . \tag{43}$$

We note again the modification of the incident plane wave in the scattered field, through the coefficients A_l, B_l and $f_l^{0,+}$. The scattered field contains the propagating functions $h_{l,l\pm1}$ (Hankel functions), with the wavevector $k = \omega/c$.

We have checked that the scattered field obtained above is the same as the scattered field from Mie's theory, as given in Ref. [17]. The coefficients ${}^eB_l$ and ${}^mB_l$ in Mie's theory are related to our coefficients A_l, B_l and $f_l^{0,+}$ through

$$\begin{aligned} {}^mB_l &= -\tfrac{i^{l+1}(2l+1)}{16\pi^2 l(l+1)} k a^2 A_l f_l^0 \ , \\ {}^eB_l &= -\tfrac{i^{l+1}(2l+1)}{16\pi^2 l(l+1)} k a^2 B_l f_l^+ \ . \end{aligned} \tag{44}$$

Making use of the relations (70) and (71) it is easy to prove the continuity of the tangential spherical components of the electric field at the sphere surface, *i.e.*

$$(\mathbf{E}_i)_{\theta,\varphi}\,|_{r=a} = (\mathbf{E}_0 + \mathbf{E}_s)_{\theta,\varphi}\,|_{r=a} \ . \tag{45}$$

Similarly, using relations (68) and (69) it is easy to get the continuity of the electric displacement

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)(\mathbf{E}_i)_r\,|_{r=a} = (\mathbf{E}_0 + \mathbf{E}_s)_r\,|_{r=a} \ , \tag{46}$$

where we recognize the dielectric function $\varepsilon = 1 - \omega_p^2/\omega^2$. This result emphasizes again the validity of the theory of "effective medium permittivity".

Making use of the same equations (68)-(71) and the asymptotic behaviour of the function $h_l(kr)$ given by equation (76), it is easy to show that the radial component of the scattered field goes like $\sim 1/r^2$ at large distances, while the tangential components go like $\sim 1/r$. Indeed, at large distances the scattered field is practically a transverse field. It is also easy to check that $(\mathbf{E}_s)_\varphi \sim \sin\varphi$ and $(\mathbf{E}_s)_\theta \sim \cos\varphi$, which give the degree of polarization for an arbitrary azimuthal angle φ.

In the same manner as for the field inside the sphere we can compute the scattered magnetic field, the energy stored by this field and the Poynting vector. The scattered magnetic field is given by

$$\mathbf{H}_s(\mathbf{r}) = \frac{ika^2}{16\pi^2} E_0 \sum_{l=1m}^{\infty} \left[A_l a_{lm} f_l^0 \mathbf{H}_{lmk}^+(\mathbf{r}) + B_l b_{lm} f_l^+ \mathbf{H}_{lmk}^0(\mathbf{r})\right] \tag{47}$$

and the stored energy can be written as

$$\begin{aligned} W_s = \tfrac{1}{16\pi}\left(\tfrac{ka^2}{16\pi^2}\right)^2 E_0^2 \textstyle\sum_{l=1m}^{\infty} \left(\left|A_l a_{lm} f_l^0\right|^2 + \left|B_l b_{lm} f_l^+\right|^2\right) \cdot \\ \cdot \left[\int_a^\infty dr \cdot r^2 \left(|h_l(kr)|^2 + \tfrac{l}{2l+1}\,|h_{l+1}(kr)|^2 + \tfrac{l+1}{2l+1}\,|h_{l-1}(kr)|^2\right)\right] \ . \end{aligned} \tag{48}$$

The radial flow of energy is given by the real part of

$$\begin{aligned} Q_s = \textstyle\int d\Omega\,(\mathbf{S}_s)_r = \tfrac{c}{8\pi}\left(\tfrac{ka^2}{16\pi^2}\right)^2 E_0^2 \textstyle\sum_{l=1m}^{\infty} \left(\left|A_l a_{lm} f_l^0\right|^2 + \left|B_l b_{lm} f_l^+\right|^2\right) \cdot \\ \cdot h_l(kr) \left[\tfrac{l}{2l+1} h_{l+1}^*(kr) + \tfrac{l+1}{2l+1} h_{l-1}^*(kr)\right] \ . \end{aligned} \tag{49}$$

This quantity is not vanishing anymore. In fact it defines the scatterring cross-section (or the relative scatterred intensity)

$$\sigma = Re\left[r^2 \frac{Q_s}{|\mathbf{S}_0|}\,|_{r\to\infty}\right] = \frac{a^4}{16\pi^2}\sum_{l=1m}^{\infty}\left(\left|A_l a_{lm} f_l^0\right|^2 + \left|B_l b_{lm} f_l^+\right|^2\right) \ , \tag{50}$$

where $|\mathbf{S}_0| = \frac{1}{8\pi}E_0^2$ is the modulus of the Poynting vector of the incident wave. It can be checked straightforwardly, by direct calculations, that (the real part of) the cross-product contribution to the flow

$$Q_s^{'} = \int d\Omega\,(\mathbf{E}_0 \times \mathbf{H}_s^* + \mathbf{E}_s \times \mathbf{H}_0^*)_r =$$

$$= \tfrac{c}{8\pi}\left(\tfrac{ka^2}{8\pi^2}\right)\left(\tfrac{E_0^2}{16\pi}\right)\textstyle\sum_{l=1}^{\infty}\{\left(A_l f_l^0 + B_l f_l^+\right) h_l(kr)\left[l j_{l+1}^*(kr) + (l+1) j_{l-1}^*(kr)\right] + \tag{51}$$

$$+\left(A_l^* f_l^{0*} + B_l^* f_l^{+*}\right) j_l(kr)\left[l h_{l+1}^*(kr) + (l+1) h_{l-1}^*(kr)\right]\}$$

cancels out exactly the contribution ReQ_s given above for the scattering field, *i.e.* $ReQ_s^{'} + ReQ_s = 0$, leading thereby to a vanishing total net flow outside the sphere, in agreement with the vanishing flow of the external field and the field inside the sphere.

Moreover, for large distances the flow $Q_s^{'}$ given above can be written as

$$Q_s^{'} \sim_{r\to\infty} \frac{c}{8\pi}\frac{E_0^2 a^2}{8\pi k r^2}\sum_{l=1}^{\infty}(2l+1)\left[Re\left(A_l f_l^0 + B_l f_l^+\right) + iIm\left(A_l f_l^0 + B_l f_l^+\right) e^{2i(kr - l\pi/2)}\right] \ , \tag{52}$$

while the forward scattered field becomes

$$[\mathbf{E}_s(\theta = 0)]_x \sim_{r\to\infty} -\frac{iE_0 a^2}{32\pi^2}\frac{e^{ikr}}{r}\sum_{l=1}^{\infty}(2l+1)\left(A_l f_l^0 + B_l f_l^+\right) \ ; \tag{53}$$

if we denote by E_f the amplitude of the spherical wave in this equation, and compare the two equations (52) and (53), we get

$$r^2 ReQ_s^{'} = -E_0\frac{c}{2k}ImE_f \ , \tag{54}$$

or, making use of equation (50) and the flow balance $ReQ_s^{'} + ReQ_s = 0$,

$$\sigma = \frac{4\pi}{kE_0}ImE_f \ , \tag{55}$$

which is the well-known "optical theorem"[17].

5 Discussion and conclusions

The results presented in this paper are cast in the form of well-known series of partial waves, with respect to the angular momentum number $l = 1, 2, ...$ We emphasize that this is the total angular momentum, arising from the coupling of the orbital momentum to the angular momentum 1. The latter reflects the vectorial character of the electromagnetic field, and the effect of the coupling can be seen, for instance, from the occurrence of the $l \pm 1$-partial waves and from the lowest value $l = 1$ acquired by this label. The partial-waves expansions (which essentially are multipole expansions)

were attained by means of the orthonormal functions $\mathbf{F}^{0,\pm}_{lmk}$, supplemented by $\mathbf{H}^{0,\pm}_{lmk}$, which form together a complete set. The field inside the sphere is regular in the origin, and is represented by functions $\mathbf{F}^{0,+}_{lmk}$, while the scattered field is propagating and is represented by functions $\mathbf{H}^{0,+}_{lmk}$. The functions $\mathbf{F}^{-}_{lmk}$ and $\mathbf{H}^{-}_{lmk}$ do not appear, because they are associated with a net charge. The terms containing $\mathbf{F}^{0}_{lmk}$ ($\mathbf{H}^{0}_{lmk}$) correspond to magnetic multipoles (TE), while those containing $\mathbf{F}^{+}_{lmk}$ ($\mathbf{H}^{+}_{lmk}$) correspond to electric multipoles (TM).

We have computed these fields for a plane wave incident on an ideal metallic sphere and recover, in a different form, the original results of Mie's theory. We have also computed the energy stored by such fields, their Poynting vector, the scattering cross-section (scattered intensity), and checked the well-known continuity conditions at the surface of the sphere and the balance of the energy flow. We re-derived also the well-known "optical theorem". We do not introduce the dielectric function from the beginning in our calculations, but we recover the results of the so-called theory of "effective medium permittivity" in our final results. This is possible due to our different method of calculation, which employs the equation of motion for the polarization and the electromagnetic potentials. The characteristic feature of this method consists in the fact that it leads to coupled integral equations, which we solved.

We have presented the results, more or less, in a compact form. Various particular cases can be derived from our results, by using well-known definitions and properties of, essentially, the spherical harmonics, the vector spherical harmonics and the spherical Bessel functions. It is worth noting that the fields given by us exhibit the corresponding modifications of the incident plane wave (decomposed in partial waves) through our coefficients A_l,B_l given by equations (32) and (33), and the coefficients $f_l^{0,+}$ given by equations (42) and (43).

A case of interest is the limit of small radii, *i.e.* $ka \ll 1$. For realistic values of the bulk plasma frequency ω_p, the "wave number" k_1 inside the sphere acquires purely imaginary values, *i.e.* $k_1 = i\alpha_1$, where $\alpha_1 = \frac{1}{c}\sqrt{\omega_p^2 - \omega^2} \simeq \omega_p/c$. The argument ak_1 can then be written as $ak_1 = ia\alpha_1 \simeq ia\omega_p/c$, which, in general, may not be small. Making use of the asymptotic behaviour of the spherical Bessel functions given by equations (75), we give here the leading contributions to the coefficients A_l, B_l, $f_l^{0,+}$ in the limit $ka \ll 1$ and for any value of the k_1a:

$$
\begin{aligned}
A_l &= \frac{4\pi(ika)^l}{(2l+1)!!j_l(k_1a)} \ , \\
B_l &= 4\pi\frac{ck_1\omega}{\omega_p^2}\cdot\frac{(ika)^l}{\left(\frac{\omega^2}{\omega_p^2}-\frac{l}{2l+1}\right)(2l+1)!!j_l(k_1a)} \ , \\
f_l^0 &= 4\pi k_1\frac{(-ika)^l}{(2l+1)!!}j_{l+1}(k_1a) \ , \\
f_l^+ &= -4\pi\frac{\omega_p^2}{c^2k_1}\frac{(-ika)^{l-1}(l+1)}{(2l+1)!!}j_l(k_1a)
\end{aligned}
\qquad (56)
$$

(where we have assumed $j_l(k_1a) \neq 0$). We can see that, in this limit, the leading contributions correspond to $l = 1$ (so-called dipolar contributions).

It is worth noting the resonances occurring in the denominator of the coefficient B_l for frequencies $\omega = \omega_p\sqrt{l/(2l+1)}$, which are the frequencies of the "spherical" plasmons. They appear as singularities in the fields, but, if the dissipation is included, these singularities are smoothed out, the fields exhibit an enhancement, and the absorption is very high. The dissipation can easily be included in the above formulae by changing ω^2 into $\omega(\omega + i\gamma)$, where the dissipation parameter γ is, in general, much smaller than the relevant values of the frequencies ω. The absorption is then

related to the imaginary parts of the singularities given above, and one can see easily that it is very high ($\sim 1/\gamma$) at resonance.

It is worth investigating the dependence on radius of the resonances found above. This can be achieved by taking the next-to-leading contributions to the denominator of the coefficient B_l. We get

$$\omega^2 = \omega_p^2 \frac{l}{2l+1} \cdot \frac{1}{1 + iak_1 \frac{j_{l+1}(k_1 a)}{(2l+1) j_l(k_1 a)}} \ . \tag{57}$$

The resonance frequencies $\omega = \omega_p \sqrt{l/(2l+1)}$ are of the same order of magnitude as ω_p. It follows that $a\omega_p/c$ can also be taken as being much lesser than unity, *i.e.* $a\alpha_1 \ll 1$ in $ak_1 = ia\alpha_1 \simeq ia\omega_p/c$. Then, equation (57) yields

$$\omega^2 \simeq \omega_p^2 \frac{l}{2l+1} \left[1 - \frac{1}{(2l+1)(2l+3)} (a\omega_p/c)^2 \right] \ , \tag{58}$$

which, within these limiting case, gives the radius dependence of the resonance frequencies. However, a word of caution must be inserted here, regarding such series expansions. For practical situations the "small radius" conditon $ka = a\omega/c \ll 1$ might not be fulfilled for resonance frequencies $\omega \sim \omega_p \sqrt{l/(2l+1)}$, so, actually, the series expansions are not valid, and the resonances remain to be estimated numerically. In the opposite limit $ka \gg 1$ these resonances disappear.

Finally, we give an illustration of the scattered intensity at large distances ($kr \gg 1$) for the most interesting case $ka \ll 1$. To this end, we use the scattered field $\mathbf{E}_s$ given by equation (41), where the coefficients A_l, B_l as well as the coefficients $f_l^{0,+}$ are given by equations (56). We can see easily that the leading contributions correspond to $l = 1$ in this case (dipolar approximation). For $\omega < \omega_p/\sqrt{2}$ we have the resonance regime, while for $\omega < \omega_p$ we have the damped regime (k_1 purely imaginary). In order to simplify the calculations we avoid these frequency regions. Moreover, we assume $\omega \gg \omega_p$, such that $k_1 a \simeq ka \ll 1$. Although this is a rather unrealistic situation, corresponding to a small plasma frequency ω_p, we choose it for the purpose of illustrating the angular dependence of the scattered radiation intensity, which, in spite of all these simplifications, still exhibits a sufficiently complex behaviour. Under these circumstances we find out the angular components of the scattered field

$$\begin{aligned} E_{s\theta} &= \tfrac{1}{30} E_0 k^4 a^5 \tfrac{e^{ikr}}{r} (1 + \varepsilon \cos\theta) \cos\varphi \ , \\ E_{s\varphi} &= \tfrac{1}{30} E_0 k^4 a^5 \tfrac{e^{ikr}}{r} (3\varepsilon - \cos\theta) \sin\varphi \end{aligned} \tag{59}$$

where $\varepsilon = 10\omega_p^2/3\omega^2(ka)^2$, so that we get the intensities

$$I_{\parallel} = |E_{s\theta}|^2 = I_0 (1 + \varepsilon \cos\theta)^2 \ , \ I_{\perp} = |E_{s\varphi}|^2 = I_0 (3\varepsilon - \cos\theta)^2 \tag{60}$$

with customary notations, for naturally polarized incident radiation ($\bar{\cos^2}\varphi = \bar{\sin^2}\varphi = 1/2$), where $I_0 = \frac{1}{2} (E_0 k^4 a^5/30r)^2$. A plot of these intensities is shown in the polar diagram given in Fig. 1. We can see that these intensities are very sensitive to the parameter ε, which is the ratio of two small parameters (ω_p^2/ω^2 to $(ka)^2$), thus illustrating the complexity of the angular dependence of the scatterring pattern.

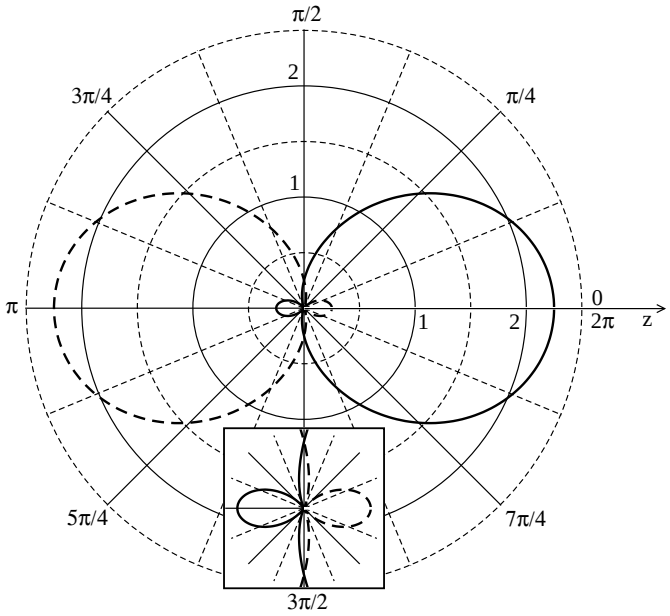

Figure 1: Polar diagram for the reduced intensities $I_{\parallel}/\varepsilon^2 I_0$ for$\varepsilon = 2$ (solid line) and $I_{\perp}/I_0$ for $\varepsilon = 0.17$ (dashed line), as given by equation (60). The inset shows the central region enlarged.

In conclusion, we may say that we computed the diffraction of an electromagnetic plane wave by an ideal metallic sphere and recovered the results of Mie's theory. We did these calculations by a method different than Mie's method, which uses the boundary conditions at the sphere surface and the dielectric function of the metal. By our method we recover these "effective medium theory" results. We have characterized as fully as posible, in compact formulae, the interaction of the plane electromagnetic wave with the metallic sphere, and have given additional results, like the field inside the sphere, the energy stored by these fields, their Poynting vector, scattering cross-section (scattered intensity) and have also checked the balance of the energy flow. In addition, we have computed by our method the "spherical" plasmons and put in evidence the diffraction resonances occurring at frequencies close to these plasmons frequencies in the limit of small radii.

The method used in this paper is not restricted to metallic spheres. For dielectrics we replace ω^2 in our formulae (including k_1 given by equation (24)) by $\omega_p^2/(1-\varepsilon)$ and use the dielectric function ε for dielectrics. Within our method (without loss) it is represented as $\varepsilon = 1 + \omega_p^2/\omega_0^2$, where both ω_p and ω_0 are parameters. Under these circumstances, the resonances found above for a metallic sphere do not appear anymore. Instead, for a dielectric sphere, there exist small oscillations in the relevant coefficients A_l, B_l, $f_l^{0,\pm}$, arising by a different mechanism which originates in the oscillatory behaviour of the Bessel functions.

Acknowledgments. The authors are indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for enlightening discussions and to Dr. L. C. Cune for many useful comments.

Appendix A. Vector spherical harmonics

The vector spherical harmonics are defined by[21, 22]

$$\mathbf{Y}_{JlM} = \sum_{mq} C_{l1}(JM; mq) Y_{lm} \mathbf{e}_q \ , \tag{61}$$

where $C_{l1}(JM;mq)$ are the Clebsch-Gordan coefficients for the coupling of the angular momenta lm and $1q$ to the angular momentum JM, Y_{lm} are (scalar) spherical harmonics and $\mathbf{e}_q$, $q=0,\pm1$, are defined by $\mathbf{e}_\pm = \mp\frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y)$, $\mathbf{e}_0 = \mathbf{e}_z$; $\mathbf{e}_{x,y,z}$ are unit vectors of the reference frame. Obviously, $J = l, l\pm1$. The scalar spherical harmonics are defined as in Ref.[21]

$$Y_{lm}(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\varphi} \ , \tag{62}$$

where the associated Legendre polynomials are given by

$$P_l^m(x) = \frac{(1-x^2)^{m/2}}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l \ . \tag{63}$$

The vector spherical harmonics are orthogonal functions over the sphere. We give here a few useful formulae used in the main text[21]:

$$div\,(f(r)\mathbf{Y}_{llm}) = 0 \ ,$$

$$div\,(f(r)\mathbf{Y}_{ll-1m}) = \sqrt{\tfrac{l}{2l+1}}\left(\tfrac{d}{dr} - \tfrac{l-1}{r}\right) f(r)Y_{lm} \ , \tag{64}$$

$$div\,(f(r)\mathbf{Y}_{ll+1m}) = -\sqrt{\tfrac{l+1}{2l+1}}\left(\tfrac{d}{dr} + \tfrac{l+2}{r}\right) f(r)Y_{lm}$$

and

$$grad\,(f(r)Y_{lm}) = -\sqrt{\frac{l+1}{2l+1}}\left(\frac{d}{dr} - \frac{l}{r}\right) f(r)\mathbf{Y}_{ll+1m} + \sqrt{\frac{l}{2l+1}}\left(\frac{d}{dr} + \frac{l+1}{r}\right) f(r)\mathbf{Y}_{ll-1m} \ . \tag{65}$$

for any arbitrary function $f(r)$. We have also

$$curl\mathbf{F}^0_{lmk} = -k\mathbf{F}^+_{lmk} \ , \ curl\mathbf{F}^+_{lmk} = -k\mathbf{F}^0_{lmk} \ , \ curl\mathbf{F}^-_{lmk} = 0 \tag{66}$$

and similar relations for $\mathbf{H}^{0,\pm}_{lmk}$, where $\mathbf{F}^{0,\pm}_{lmk}$ and $\mathbf{H}^{0,\pm}_{lmk}$ are defined in the main text. Another useful formula is

$$\mathbf{e}_r Y_{lm} = \frac{1}{\sqrt{2l+1}}\left(\sqrt{l}\mathbf{Y}_{ll-1m} - \sqrt{l+1}\mathbf{Y}_{ll+1m}\right) \ ,. \tag{67}$$

where $\mathbf{e}_r$ is the radial unit vector.

Making use of their definiton (61) and of the Clebsch-Gordan coefficients we can compute the spherical components the vector spherical harmonics[21, 22]. We give here a few useful formulae for these spherical components. First we have

$$(\mathbf{Y}_{llm})_r = 0 \ , \ (\mathbf{Y}_{ll-1m})_r = \sqrt{\frac{l}{2l+1}}Y_{lm} \ , \ (\mathbf{Y}_{ll+1m})_r = -\sqrt{\frac{l+1}{2l+1}}Y_{lm} \tag{68}$$

for the radial components, whence

$$\left(\mathbf{F}^0_{lmk}\right)_r = 0 \ , \ \left(\mathbf{F}^+_{lmk}\right)_r = \frac{\sqrt{l(l+1)}}{ikr} j_l(kr)Y_{lm} \tag{69}$$

and similar formulae for the functions $H^{0,+}_{lmk}$, replacing j_l by h_l. We have also

$$(\mathbf{Y}_{ll+1m})_\varphi = -i\sqrt{\frac{l}{2l+1}}\,(\mathbf{Y}_{llm})_\theta \ , \ (\mathbf{Y}_{ll-1m})_\varphi = -i\sqrt{\frac{l+1}{2l+1}}\,(\mathbf{Y}_{llm})_\theta \tag{70}$$

and

$$(\mathbf{Y}_{ll+1m})_\theta = i\sqrt{\frac{l}{2l+1}}\,(\mathbf{Y}_{llm})_\varphi \ , \ (\mathbf{Y}_{ll-1m})_\theta = i\sqrt{\frac{l+1}{2l+1}}\,(\mathbf{Y}_{llm})_\varphi \ , \tag{71}$$

which we use to get the tangential spherical components of the functions $\mathbf{F}_{lmk}$ and $\mathbf{H}_{lmk}$ in terms of the corresponding components of $\mathbf{Y}_{llm}$.

Appendix B. Spherical Bessel functions

We use the following definitions for the spherical Bessel functions

$$j_l(z) = (2\pi)^{3/2} i^l \frac{J_{l+1/2}(z)}{\sqrt{z}} \ ,$$

$$n_l(z) = (2\pi)^{3/2} i^l \frac{N_{l+1/2}(z)}{\sqrt{z}} \ , \tag{72}$$

$$h_l(z) = j_l(z) + i n_l(z) = (2\pi)^{3/2} i^l \frac{H_{l+1/2}(z)}{\sqrt{z}} \ ,$$

where $J_{l+1/2}$, $N_{l+1/2}$ and $H_{l+1/2}$ are (cylindrical) Bessel functions of half-integer order and of the first, second and, respectively, third rank ($H_{l+1/2}$ known also as the Hankel function)[23, 24]. We note the "orthogonality" property

$$\int dz \cdot z w_\nu(\alpha z) W_\nu(\beta z) = \frac{z}{\beta^2 - \alpha^2} \left[\beta W_{\nu+1}(\beta z) w_\nu(\alpha z) - \alpha W_\nu(\beta z) w_{\nu+1}(\alpha z)\right] \tag{73}$$

for any pair $w_\nu \, , W_\nu$ of (cylindrical) Bessel functions of the same rank. We have also the useful integral

$$\int dz \cdot z^2 j_l^2(\alpha z) = \frac{z^3}{2} \left[j_l^2(\alpha z) - j_{l+1}^2(\alpha z) - \frac{2l+1}{i\alpha z} j_l(\alpha z) j_{l+1}(\alpha z) \right] \tag{74}$$

and a similar integral for $h_l(z)$.

The asymtotic behaviour of the spherical Bessel functions is given by

$$\begin{gathered} j_l(z) \sim_{z\to 0} 4\pi i^l \tfrac{z^l}{(2l+1)!!} \ , \ j_l(z) \sim_{z\to\infty} 4\pi i^l \tfrac{\sin(z - l\pi/2)}{z} \ , \\ n_l(z) \sim_{z\to 0} -4\pi i^l \tfrac{(2l-1)!!}{z^{l+1}} \ , \ n_l(z) \sim_{z\to\infty} -4\pi i^l \tfrac{\cos(z - l\pi/2)}{z} \end{gathered} \tag{75}$$

and

$$h_l(z) \sim_{z\to\infty} -4\pi i^{l+1} \frac{e^{i(z - l\pi/2)}}{z} \ . \tag{76}$$

The following recurrence relations are used in the main text:

$$\begin{gathered} \tfrac{d}{dz} j_l(z) = \tfrac{i}{2l+1} \left[l j_{l-1}(z) + (l+1) j_{l+1}(z) \right] \ , \\ j_l(z) = \tfrac{iz}{2l+1} \left[j_{l-1}(z) - j_{l+1}(z) \right] \end{gathered} \tag{77}$$

and

$$\begin{gathered} J_\nu(z) H_\nu'(z) - J_\nu'(z) H_\nu(z) = \tfrac{2i}{\pi z} \ , \\ J_{\nu-1}(z) H_\nu(z) - J_\nu(z) H_{\nu-1}(z) = \tfrac{2}{\pi i z} \ , \\ J_\nu'(z) = J_{\nu-1}(z) - \tfrac{\nu}{z} J_\nu(z) = -J_{\nu+1}(z) + \tfrac{\nu}{z} J_\nu(z) \ . \end{gathered} \tag{78}$$

Appendix C. Plane wave

Beside the well-known decomposition

$$\mathbf{E}_0(\mathbf{r}) = E_0 \mathbf{e}_x \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} j_l(kr) Y_{l0}(\theta, \varphi) \tag{79}$$

for the plane wave $\mathbf{E}_0 = E_0 \mathbf{e}_x e^{ikz}$, we have also the decomposition

$$\mathbf{E}_0(\mathbf{r}) = \tfrac{1}{2\sqrt{4\pi}} E_0 \sum_{l=1}^{\infty} \{\sqrt{2l+1} j_l(kr)[\mathbf{Y}_{ll1}(\theta,\varphi) + \mathbf{Y}_{ll-1}(\theta,\varphi)] +$$

$$+\sqrt{l+1} j_{l-1}(kr)[\mathbf{Y}_{ll-1-1}(\theta,\varphi) - \mathbf{Y}_{ll-11}(\theta,\varphi)] + \tag{80}$$

$$+\sqrt{l} j_{l+1}(kr)[\mathbf{Y}_{ll+1-1}(\theta,\varphi) - \mathbf{Y}_{ll+11}(\theta,\varphi)]\}$$

as a series of vector spherical harmonics. It is very convenient to write this equation in the more compact form

$$\mathbf{E}_0(\mathbf{r}) = E_0 \sum_{l=1m}^{\infty} \left(a_{lm} \mathbf{F}^0_{lmk}(\mathbf{r}) + b_{lm} \mathbf{F}^+_{lmk}(\mathbf{r}) \right) \ , \tag{81}$$

where

$$a_{lm} = \frac{1}{4}\sqrt{\frac{2l+1}{\pi}} \left(\delta_{m1} + \delta_{m,-1}\right) \ , \ b_{lm} = \frac{1}{4}\sqrt{\frac{2l+1}{\pi}} \left(-\delta_{m1} + \delta_{m,-1}\right) \tag{82}$$

and the functions $\mathbf{F}^{0,+}_{lmk}$ are given by equation (18) in the main text. The magnetic field is calculated from $\mathbf{H}_0 = (-i/k) curl \mathbf{E}_0$, and, making use of equations (64), we get

$$\mathbf{H}_0 = i E_0 \sum_{l=1m}^{\infty} \left[a_{lm} \mathbf{F}^+_{lmk}(\mathbf{r}) + b_{lm} \mathbf{F}^0_{lmk}(\mathbf{r}) \right] \ . \tag{83}$$

References

[1] G. Mie, "Beitrage zur optik truber Medien, speziell kolloidaler Metallosungen," Ann. Physik **25** 377-445 (1908).

[2] H. C. van de Hulst, "*Light Scattering by Small Particles,*" (Wiley, New York, 1957).

[3] W. T. Doyle and A. Agarwal, "Optical extinction of metal spheres," J. Opt. Soc. Am. **55** 305-309 (1965).

[4] J. Crowell and R. H. Ritchie, "Radiation decay of Coulomb-stimulated plasmons in spheres," Phys. Rev. **172** 436-440 (1968).

[5] A. Ashkin and J. M. Dziedzic, "Observation of resonances in the radiation pressure on dielectric spheres," Phy. Rev. Lett. **38** 1351-1355 (1977).

[6] P. Chylek, J. T. Kiehl and M. K. W. Ko, "Narrow resonance structures in the Mie scattering characteristics," Appl. Optics, **17** 3019-3021 (1978).

[7] P. R. Conwell, P. W. Barber and C. K. Rushforth, "Resonant spectra of dielectric spheres," J. Opt. Soc. Am. **A1** 62-67 (1984).

[8] P. L. Marston and J. H. Crichton, "Radiation torque on a sphere caused by a circularly-polarized electromagnetic wave," Phys. Rev. **A30** 2508- 2516 (1984).

[9] S. Chang, J. T. Kim, J. H. Jo and S. S. Lee, "Optical force on a sphere caused by the evanescent field of a Gaussian beam; effects of multiple scattering," Optics Commun. **139** 252-261 (1997).

[10] R. Ruppin, "Optical properties of small metallic spheres," Phys. Rev. **B11** 2871-2876 (1975).

[11] B. J. Messinger, K. U. von Raben, R. K. Chang and P. W. Barber, "Local fields at the surface of noble-metal microspheres," Phys. Rev. **B24** 649-657 (1981).

[12] S. Arnold, A. B. Pluchino and K. M. Leung, "Influence of surface-mode-enhanced local fields on photophoresis," Phys. Rev. **A29** 654-660 (1984).

[13] C. Brechignac, Ph. Cahuzac, J. Leygnier and A Sarfati, "Optical response of large lithium clusters: evolution toward the bulk," Phys. Rev. Lett. **70** 2036-2039 (1993).

[14] P. Markowicz, K. Kolwas and M. Kolwas, "Experimental determination of free-electron plasma damping rate in large sodium clusters," Phys. Lett. **A236** 543-547 (1997).

[15] C. Brechignac, Ph. Cahuzac, N. Kebaili, J. Leygnier and H. Yoshida, "Interband effect in the optical response of strontium clusters," Phys. Rev. **B61** 7280-7283 (2000).

[16] M. Apostol and G. Vaman, "Electromagnetic field interacting with a semi-infinite plasma," J. Opt. Soc. Am. A, in print.

[17] M. Born and E. Wolf, *Principles of Optics* (Pergamon, London, 1959) and references therein.

[18] See, for instance, H. A. Lorentz, *The Theory of Electrons* (Leipzig, Teubner, 1916).

[19] M. Apostol and G. Vaman, "Plasmons and polaritons in a semi-infinite plasma and a plasma slab," Physica B, in print.

[20] M. Apostol and G. Vaman, "Electromagnetic eigenmodes in matter. van der Waals-London and Casimir forces," J. Theor. Phys. **177** 1-12 (2009).

[21] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton, New Jersey, 1957).

[22] J. M. Blatt and V. E. Weisskopf, *Theoretical Nuclear Physics* (Dover, New York, 1991).

[23] A. Erdelyi, ed., "*Higher Transcendental Functions*," vol. 2, Bateman Project (McGraw-Hill, New York, 1957).

[24] M. Abramowitz and I. A Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, 1972).

[25] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 2000), pp. 930, 8.533.

© **J. Theor. Phys.** 2009, apoma@theor1.theory.nipne.ro