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On some diffraction problems for a scalar wave

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Abstract

An approximate method is devised for estimating scalar waves diffracted on various obstacles. It is applied to a semi-infinite circular pipe, a small aperture in an infinite screen and a semi-infinite plane screen. The method is based upon approximating the effect of the boundary conditions with a surface distribution of sources.

Introduction. We consider first a semi-infinite circular pipe of radius a extending to the half-plane $z<0$ and a scalar field ψ obeying the wave (Helmholtz) equation

$$(\Delta+k^2)\psi=0 \tag{1}$$

where $k=\omega/c$ is the wave number, ω is the frequency and c is the velocity of the wave. The problem is to find the field ψ with suitable boundary conditions at infinity and on the surface of the pipe. We consider the most interesting case $ak<1$. This case was considered by Lord Rayleigh[1] and by Levine and Schwinger[2] for the propagation of sound. The reflection coefficient and the scattering amplitude have been expressed exactly by Levine and Schwinger[2] by quadratures.

Pipe modes. With cylindrical coordinates equation (1) reads

$$(\partial^2/\partial\varrho^2+\partial/\rho\partial\varrho+\partial^2/\varrho^2\partial\varphi^2+\partial^2/\partial z^2+k^2)\psi=0\ . \tag{2}$$

Its solutions can be written as $\psi=e^{\pm iqz}e^{in\varphi}\chi(\varrho)$, where χ satisfies the equation

$$\left\{\varrho^2d^2/d\varrho^2+\varrho d/d\varrho+[(k^2-q^2)\varrho^2-n^2]\right\}\chi=0 \tag{3}$$

The regular solutions of this equation are the Bessel functions $J_n(\kappa\varrho)$ of the first kind, with n integer and $\kappa^2=k^2-q^2$.

These solutions can be used inside the pipe with boundary conditions $J_n'(\kappa a)=0$ (vanishing normal "velocity") or similar ones. κ is then determined by the zeros of the derivatives of the Bessel functions which impose lower bounds to the modes frequencies. For instance, the first mode, corresponding to $J_0(\kappa\varrho)$, starts at $ka=\kappa a=z_1$, where $z_1\simeq 3.85$ is the first zero of $J_0'(z_1)=-J_1(z_1)=0$. Since we are interested in $ak<1$ we choose the plane waves solution

$$\psi=e^{ikz}+Re^{-ikz} \tag{4}$$

which is the dominant (fundamental) solution of equation (2). It satifies the boundary condition. In (4) R is a reflection coefficient.

This solution can be used near the mouth of the pipe ($z \rightarrow -\infty$). Near the open end of the pipe the field is substantially modified by the open-end boundary condition. There, the wave propagates in the free space (ouside the pipe). In general, the inside field near the open end of the pipe is given by a superposition of Bessel functions, both of higher order and higher wavevectors, and imaginary q.

Free space. Outside the pipe it is convenient to use spherical coordinates. The laplacean in spherical coordinates reads $\Delta = \partial(r^2\partial/\partial r)/r^2\partial r - L^2/r^2$, where $L^2 = -\partial(\sin\theta\partial/\partial\theta)/\sin\theta\partial\theta - \partial^2/\sin^2\theta\partial\varphi^2$ is the angular momentum. Its eigenfunctions are the spherical harmonics Y_{lm}, $L^2Y_{lm} = l(l+1)Y_{lm}$. With $\psi = R(r)Y_{lm}(\theta,\varphi)$ the wave equation (1) becomes

$$[d(r^2d/dr)/r^2dr - l(l+1)/r^2 + k^2]R = 0 \ . \tag{5}$$

The regular solutions of this equation are the spherical Bessel functions $j_l(kr)$ of the first kind, $j_l = \sqrt{\pi/2kr}J_{l+1/2}$, where $J_{l+1/2}$ are the Bessel functions of half-integer order satisfying equation (3) .

In view of the cylindrical symmetry we can limit ourselves to $Y_{l0} = i^l\sqrt{(2l+1)/4\pi}P_l(\cos\theta)$, where P_l are the Legendre polyonomials. We can write therefore the wave outside the pipe as a general superposition

$$\psi = 2ik\sum_{l=0}^{\infty} i^l(2l+1)f_lP_l(\cos\theta)j_l(kr) \ . \tag{6}$$

Having in mind the decompostion of the plane wave in spherical Bessel functions we can see that the above field is a superposition of plane waves. The asymptotic behaviour of this function for $r \rightarrow \infty$ is

$$\psi \sim 2ik\sum_{l=0}^{\infty} i^l(2l+1)f_lP_l(\cos\theta)\cdot\frac{1}{kr}\sin(kr - l\pi/2) \ , \tag{7}$$

or

$$\psi \sim \sum_{l=0}^{\infty}(2l+1)f_lP_l(\cos\theta)\left[\frac{e^{ikr}}{r} - (-1)^l\frac{e^{-ikr}}{r}\right] \ . \tag{8}$$

The outgoing part of this wave is the scattering wave

$$\psi \sim f(\theta)\cdot\frac{e^{ikr}}{r} \tag{9}$$

with the scattering amplitude

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)f_lP_l(\cos\theta) \tag{10}$$

where f_l are the partial scattering amplitudes. Such a field can be used outside the pipe, providing the boundary conditions (vanishing normal derivative on the surface of the pipe) are satisfied. Equations (9) and (10) together with (4) provide the framework of the radiation (diffraction) problem for a scalar wave (fundamental mode) propagating in a semi-infinite circular pipe.

Approximations. In view of the great difficulties related to satisfying the boundary conditions it is hopeless (and even useless) to try to get the field directly by analytical methods. However, some reasonable approximations can be made for the radiation problem.

The wave equation with a δ-source

$$(\Delta + k^2)\psi = -\delta(\mathbf{r} - \mathbf{r}') \tag{11}$$

has the Green function

$$G(\mathbf{r}-\mathbf{r}') = e^{ik|\mathbf{r}-\mathbf{r}'|}/4\pi\,|\mathbf{r}-\mathbf{r}'| \tag{12}$$

as a solution (outgoing part). We consider a distribution of sources $g(\mathbf{r}-\mathbf{r}')$ on the surface S of the pipe and write the field generated by them as

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\int_S d\mathbf{r}' \cdot g(\mathbf{r}')G(\mathbf{r}-\mathbf{r}') \ . \tag{13}$$

This idea was probably first used by Bethe,[3] related of course to the Green theorem. In view of (11) the field and its derivatives may have discontinuities on crossing the surface S. Since the normal derivative is vanishing on both sides of the surface S the field has a discontinuity in this case (the field is higher inside than outside, as expected). The method used here is to identify the asymptotic field given by (13) for $z \to -\infty$ with the asymptotic wave given by (4) at the mouth of the pipe (outgoing part). This matching condition gives an integral equation for g, which, together with the boundary conditions, would solve the problem. This illustrates the principle that the field is entirely determined by its values on the boundaries. The principle of the method is essentially due to Kirchoff. The source function $g(\mathbf{r}')$ may be taken as depending only on z.

There is however a great difficulty in having a useful representation of the integral in (13), due to the presence of sources near the mouth of the pipe. Since $\mathbf{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ and $\mathbf{r}' = (a\cos\varphi', a\sin\varphi', z)$ we have

$$|\mathbf{r}-\mathbf{r}'| = \left[r^2 + a^2 + z^2 - 2ar\sin\theta\cos(\varphi'-\varphi) - 2zr\cos\theta\right]^{1/2} \ ; \tag{14}$$

we can see that a power expansion in (13) involves powers of $\cos\theta$ and $\sin^2\theta$ (since the φ'-integration leaves only even powers of $\sin\theta$). Therefore, we may suggest an expansion of the field given by (13) in Legendre polynomials. Since we want to use this field near the open end of the pipe it is advisable to limit ourselves to a few terms in such an expansion, of the lowest order in $\cos\theta$ (in accordance also with $ak < 1$). We choose therefore a representation of the form

$$\psi = (f_0 + 3f_1\cos\theta)\cdot\frac{e^{ikr}}{r} \tag{15}$$

for the asymptotic field given by (13) and identify it with the corresponding part of the asymptotic wave given by (4). The coefficients f_0, f_1 are related to the source function g. The field given by (15) satisfies the boundary condition near the mouth of the pipe, as it and its derivatives are vanishing in the limit $z \to -\infty$.

Matching condition. The decomposition of the plane wave in spherical Bessel functions is

$$e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)P_l(\cos\theta)j_l(kr) \ . \tag{16}$$

Its asymptotic form is given by

$$e^{ikz} \sim \sum_{l=0}^{\infty} i^l(2l+1)P_l(\cos\theta)\cdot\frac{1}{kr}\sin(kr - l\pi/2) \tag{17}$$

or

$$e^{ikz} \sim \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\left[\frac{e^{ikr}}{r} - (-1)^l\frac{e^{-ikr}}{r}\right] \ . \tag{18}$$

From the matching condition

$$\left[e^{ikz}+Re^{-ikz}\right]_{out}=(f_0+3f_1\cos\theta)\cdot\frac{e^{ikr}}{r} \tag{19}$$

we get the equations

$$1+R=2ikf_0\ ,\ 1-R=2ikf_1\ . \tag{20}$$

It follows that the reflection coefficient can be represented as

$$R=-1+2ikf_0 \tag{21}$$

or $R=|R|\,e^{i\alpha}$, where

$$\begin{aligned}|R|&=\left(1+4kImf_0+4k^2\left|f_0^2\right|\right)^{1/2}\ ,\\ \alpha&=-\arctan\left[2kRef_0/(1+2kImf_0)\right]\ .\end{aligned} \tag{22}$$

In addition we have $f_1=1/ik-f_0$.

Conservation of the current. The total amount of incident field is given by $N_{inc}=\pi a^2\cdot 2D$, where $D\rightarrow\infty$. Similarly, the amount of reflected field is $N_r=\pi a^2\left|R\right|^2\cdot 2D$. We identify the scattered field with the field given by (15). Its main defect is that it is not valid very close to the open end of the pipe. The amount of this scattered field is $N_{sc}=2\pi D\left(2\left|f_0\right|^2+6\left|f_1\right|^2\right)$. The conservation of the total amount of field reads

$$a^2k^2\left(1-|R|\right)^2=2k^2\left|f_0\right|^2+6\left|1-ikf_0\right|^2+C \tag{23}$$

where the constant C is necessary to correct for the approximations made. It will be determined from the condition to have conservation in the limit $ak\rightarrow 0$. Equation (23) can also be written as

$$a^2k^2\left(1-|R|\right)^2=12kImf_0+8k^2\left|f_0\right|^2+6+C\ . \tag{24}$$

It is more convenient to write this equation in terms of R, by making use of (21); we get

$$(2+a^2k^2)\left|R\right|^2-2\left|R\right|\cos\alpha+(2-a^2k^2)+C=0\ . \tag{25}$$

In the limit $ak\rightarrow 0$ we must have $|R|=1$ and $\alpha=0$; these conditions set the constant $C=-2$. Equation (25) is solved for

$$\alpha=A\cdot ak\ ,\ |R|=1-\frac{1}{2}A^2\cdot a^2k^2\ . \tag{26}$$

The coefficient A remains undetermined. We get also $kRef_0=Aak/2$ and $kImf_0=-1+A^2a^2k^2$. α is usually written as $\alpha=2ak(l/a)$, where l is a fictitious additional length of the pipe (introduced by Lord Rayleigh). We get $l/a=A/2$. The exact value is $\simeq 0.6$,[2] so $A\sim 1$. Indeed, for large values of k the reflection coefficient is vanishing and the continuity of the plane wave e^{ikz} and the forward-scattered wave $(Aa/2)(-2)(1/a)$ at the open end gives $A=-1$. The phase goes to π with respect to our conventions and we get $A=1$. The results given by (26) agree qualitatively with the exact results given by Levine and Schwinger.[2]

Cross-section. Making use of $f_0=Aa/2-i/k$ and $f_1=-Aa/2$ derived above the scattering amplitude given by (15) becomes

$$f(\theta)=\frac{Aa}{2}(1-3\cos\theta)-i/k\ . \tag{27}$$

The imaginary term in (27) coming from f_0 accounts for the reflected wave $R = 1$ in the first equation (20). Therefore, it must be omitted from the scattering wave. The cross-section reads then

$$|f(\theta)|^2 = (Aa/2)^2(1 - 3\cos\theta)^2 \ . \qquad (28)$$

It has an extinction point at $\cos\theta = 1/3$ and an inflexion point near $\theta \simeq \pi/4$. It is large for large angles $\theta > \pi/2)$, where the approximation is not valid.

As we said above, higher-order Legendre polynomials in the scattering amplitude are more suitable for larger values of ak. In the limit $ak \to 0$ the cross-section is practically independent of θ, and we can limit ourselves to f_0 in the expansion of the scattering amplitude.

A small circular aperture in an infinite screen. The related problem of diffraction of a scalar wave through a small aperture in an infinite screen was considered by Bethe[3] and also by Levine and Schwinger,[4] especially for resonant cavities of microwaves. An exact representation of the solution has been given by Bouwkamp.[5] An approximate solution is also known as the Kirchoff solution and another as the Rayleigh solution.

We consider a small circular aperture of radius a in an infinite screen and an incident plane wave perpendicular to the screen. The boundary conditions consist in the vanishing of the field on the screen. Due to these boundary conditions the reflection coefficient has a phase near π (corresponding to nodes on the screen), so we use the asymptotioc wave (4) in the form

$$\psi = e^{ikz} - Re^{-ikz} \ . \qquad (29)$$

We consider the screen thin, but still of a finite length, so a $\cos\theta$-term is present in (14). The scattered wave is therefore assumed to be of the form given by (15). The matching condition gives the same equations (20) with R changed into $-R$:

$$1 - R = 2ikf_0 \ , \ 1 + R = 2ikf_1 \ . \qquad (30)$$

The conservation of the current (equation (25)) gives the equation

$$(2 + a^2k^2)\,|R|^2 + 2\,|R|\cos\alpha + (2 - a^2k^2) + C = 0 \qquad (31)$$

with $C = -6$ for $|R| = 1$ and $\alpha = 0$ in the limit $ak \to 0$, *i.e.*

$$(2 + a^2k^2)\,|R|^2 + 2\,|R|\cos\alpha - (4 + a^2k^2) = 0 \ . \qquad (32)$$

Since the tickness of the screen is assumed to be very small the shift l is vanishing; therefore we put $\alpha = A \cdot (ak)^2$. We get from (32) $|R| = 1 + A^2(ak)^4/6$ and $f_0 = -f_1 = -A(ak)^2/2k$. The scattering amplitude is therefore

$$f(\theta) = -(A/2k)(ak)^2(1 - 3\cos\theta) \qquad (33)$$

and the cross-section goes like $(ak)^4$ in agreement with Rayleigh scattering. The reflection coefficient $|R|$ is increased in order to compensate for the transmitted wave when the aperture is open. It follows that the transmission coefficient is $t^2 = A^2(ak)^4/6$, in agreement with previous results.

A semi-infinite plane screen. We consider a semi-infinite plane screen with the edge along the x-axis and an incident plane wave propagating perpendicular to the screen along the z-axis. The problem does not depend on the coordinate x, so it is convenient to use cylindrical coordinates $z = \varrho\cos\theta$ and $y = \rho\sin\theta$. The boundary conditions are either a vanishing field (nodes) or a vanishing normal derivative on the screen. This is a famous problem, because it has been solved exactly, in the sense that an integral representation has been given for the field (in the more general case of a wedge), by Sommerfeld;[6] additional asymptotic terms has been computed by Pauli.[7]

The asymptotic form of the wave for $z \to -\infty$ is

$$\psi = e^{ikz} + Re^{-ikz} \tag{34}$$

where R equals 1 or -1 depending on the boundary conditions. We choose $R = 1$. The diffracted wave, created by sources on the screen, is a superposition of adequate solutions of the Bessel equation (3) (with $q = 0$). We choose the outgoing waves which are given by Hankel's functions $H_n(k\rho)$ of the first kind. Their asymptotic behaviour is given by

$$H_n(k\rho) \sim \sqrt{2/\pi k\rho} e^{i(k\rho - \pi n/2 - \pi/4)} \tag{35}$$

(for $n > 0$; $H_{-n} = (-1)^n H_n$; the factor $(-1)^n$ is incorporated in f_n below). It is important to stress upon the fact that this behaviour is valid for fixed n and $k\rho \to \infty$. In addition, these functions are divergent at the origin. We take therefore for the diffracted field

$$\psi = \sum_n f_n H_n(k\rho) e^{in\theta} \tag{36}$$

where f_n are partial scattering amplitudes.

We use the expansion of the plane wave

$$e^{ikz} = e^{ik\rho\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho) e^{in\theta} \tag{37}$$

and its asymptotic behaviour

$$e^{ikz} \sim \sqrt{2/\pi k\rho} \sum_{n=-\infty}^{\infty} i^n \cos(k\rho - \pi n/2 - \pi/4) e^{in\theta} \ . \tag{38}$$

If we extend formally this asymptotic behaviour to $n \to \pm\infty$ we would get

$$e^{ikz} \sim \sqrt{2\pi/k\rho} \left[\delta(\theta) e^{i(k\rho - \pi/4)} + \delta(\pi - \theta) e^{-i(k\rho - \pi/4)} \right] \ . \tag{39}$$

According to the matching principle we identify the asymptotic forms (34) and (36) for $z \to -\infty$ ($\theta \sim \pi$) and get $f_n = (-i)^n/2$. The scattering wave is given therefore by

$$\psi = \frac{1}{2} \sum_n (-i)^n H_n(k\rho) e^{in\theta} \sim \frac{1}{\sqrt{2\pi k\rho}} \sum_n (-1)^n e^{i(k\rho - \pi/4)} e^{in\theta} \ . \tag{40}$$

The scattering amplitude $f(\theta)$ normalized by $|\psi|^2 \rho d\rho d\theta = |f(\theta)|^2 d\rho d\theta$ can be represented as

$$f(\theta) = \frac{1}{\sqrt{2\pi k}} e^{-i\pi/4} \sum_n (-1)^n e^{in\theta} \ . \tag{41}$$

As said above, this representation is valid for a limited number of indices n for any fixed ρ, say $n < k\rho$. For fixed ρ and large k many n enter the scattering amplitude, and we have a rapid succession of extinctions (diffraction fringes); it corresponds to particle scattering. For low k we have a few diffraction fringes. For instance, the first-order contributons to the scattering amplitude give the angular dependence $1 - 2\cos\theta$; it has an extinction point at $\theta = \pi/6$. It is a Fraunhofer diffraction. For large $k\rho$ we may integrate over n in (41) up to $n \sim k\rho$; we get $f \sim [\sin k\rho(\pi - \theta)] / (\pi - \theta)$ which gives Fresnel fringes near the classical path.

Acknowledgments. All the mathematical formulae used herein are taken from Abramowitz and Stegun[8] and from Gradshteyn and Ryzhik.[9]

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