Journal of Theoretical Physics

Founded and Edited by M. Apostol **201** (2011)

ISSN 1453-4428

Electromagnetic field in matter

M. Apostol
Department of Theoretical Physics, Institute of Atomic Physics,
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: apoma@theory.nipne.ro

Abstract

A mechanical representation is introduced for the polarization electromagnetic field in matter in terms of the displacement field of the mobile charges (the polarization field). It is shown that the dynamics of this displacement field, and of the corresponding electromagnetic field, is governed, in general, by three infinite sets of harmonic oscillators, one for the longitudinal component, with the "plasmon" frequency (longitudinal mode), and two for the transverse component, with polaritonic frequencies (coupled transverse and photonic modes). The present method of treating the electromagnetic field in matter is illustrated here for non-magnetic, homogeneous, isotropic matter, by using the well-known Lorentz-Drude model of matter polarization. We give the representations of the electromagnetic field energy, Poynting vector, Lorentz force and charge-field interaction energy in terms of the eigenmodes of the displacement field. It is shown that matter posesses a large amount of electromagnetic energy, originating in the zero-point (vacuum) fluctuations of its polarization field. The electromagnetic coupling between two polarizable bodies is considered within the framework of the present approach, and the corresponding polarization eigenmodes are identified. The Casimir and van der Waals-London forces are derived, as arising from the zero-point fluctuations of the polarization. The nature of the two bodies, represented by their individual longitudinal and transverse polarization modes, is incorporated in the coefficients of these forces. In addition, it is shown that the electromagnetic field (generated by the polarization fluctuations) brings its own contribution to the forces acting between two polarizable bodies. When a dielectric, at least, is present, this contribution is an attractive force which goes like $-1/d^2$, where d is the separation distance between the two bodies. When a conductor, at least, is present, the electromagnetic field brings an additional contribution to the Casimir force ($\sim -1/d^4$). The range of these forces is estimated. The approach is based on Fourier decomposition and, consequently, it is not able to account for the surface (or shape) effects. The particular case of point-like particles is also analyzed.

Key Words: *Matter polarization; Electromagnetic modes; Casimir and van der Waals-London forces*

PACS: 03.50.De; 41.20.Jb; 52.25.Mq; 78.20.Bh

1 Introduction

Recently, there is an extensive, ongoing interest in the Casimir and van der Waals-London forces which act between two polarizable bodies.[1]-[38] The specific points in discussion are the effects of

the nature of the two bodies, the surface (or shape) effects, the temperature dependence, etc. The difficulty resides in the lack of a convenient representation of the electromagnetic (polarization) field in matter, other than the usual semi-phenomenological theory of the dielectric function. We give here such a representation, which incorporates the nature of the electromagnetically interacting bodies. Though not capable of accounting for finite-size effects, it points, at least more specifically, toward the nature of this difficulty and its solution.

The electromagnetic field is generated by electric charges, usually in motion. The motion of the electric charges in usual matter is non-relativistic, since their velocity is much smaller than the light velocity c. For a sufficiently dense condensed matter the motion of the electric charges is also in the quasi-classical limit of the quantum motion, so a quantum-statistical average is appropriate. Therefore, we get a classical (both non-quantum and non-relativistic) motion for the electric charges in matter and, consequently, a classical electromagnetic field. We show here that the electromagnetic field in matter admits a mechanical representation in terms of the matter polarization, which can be decomposed in harmonic oscillators. The polarization is identified as being represented by a displacement field in the position of the mobile charges. The collective motion of the polarization is governed, in general, by three distinct, infinite sets of harmonic oscillators, corresponding to longitudinal, plasmon (one set) and transverse, polariton (two sets) frequencies. This collective motion can be quantized, oferring an example of emergent dynamics. The electromagnetic field generated by the polarization motion has a corresponding mechanical representation in terms of the displacement field. This representation is given here for the electromagnetic field energy, Poynting vector, Lorentz force and charge-field interaction energy.

The electromagnetic coupling between two polarizable bodies is examined in this framework, and the corresponding plasmonic and polaritonic eigenfrequencies are identified. The Casimir and van der Waals-London forces are derived, with coefficients which depend on the nature of the two bodies. It is shown that these forces arise from the vacuum fluctuations of the polarization. It is also shown that the electromagnetic field (generated by these fluctuations) brings its own conribution to the forces acting between two polarizable bodies. When a dielectric, at least, is present, the electromagnetic-field force goes like $-1/d^2$, where d is the separation distance between the two bodies. When a conductor, at least, is present, the electromagnetic-field force is of Casimir type ($\sim -1/d^4$). All these forces are estimated here and their range of validity is given.

The important surface (or shape) effects are not accounted for within the present approach, as a consequence of the use of Fourier decompositions, which do not represent correctly the sharp surfaces (*e.g.*, the well-known Gibbs phenomenon). In order to include such surface effects, we should keep the dependence on the direct-space coordinate transverse to the surface, which is a much more difficult task. In this context, the particular case of point-like, δ-particles is analyzed.

2 Matter polarization

We adopt a generic model of matter polarization consisting of N identical mobile charges q, with mass m and density $n = N/V$, moving in a rigid, neutralizing background of volume V. A small displacement field $\mathbf{u}(\mathbf{r}, t)$ in the position $\mathbf{r}$ of these charges gives, at time t, a local density imbalance $\delta n = -n div \mathbf{u}$ and a polarization charge density $\rho = -nq div \mathbf{u}$. We can see that $\mathbf{P} = nq\mathbf{u}$ is the polarization. Therefore, the displacement field $\mathbf{u}(\mathbf{r}, t)$ is a representation for the polarization field $\mathbf{P}(\mathbf{r}, t)$. The displacement field obeys the Newton law of motion

$$m\ddot{\mathbf{u}} = q\mathbf{E} - m\omega_c^2\mathbf{u} - m\gamma\dot{\mathbf{u}} + q\mathbf{E}_0 \ , \tag{1}$$

where $\mathbf{E}$ is the polarization electric field generated by the polarization charges (and currents), ω_c is a characteristic frequency, γ is a (small) damping factor and $\mathbf{E}_0$ is an external electric field. This is the well-known Lorentz-Drude (plasma) model of polarizable matter,[39]-[43] which assumes a homogeneous, isotropic matter, without spatial dispersion, represented by a field of harmonic oscillators of frequency ω_c. Taking the temporal Fourier transform of equation (1), with $\mathbf{E}_t = \mathbf{E} + \mathbf{E}_0$ the total electric field, we get the electric susceptibility $\chi(\omega) = P/E_t$ and the dielectric function

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega) = \frac{\omega^2 - \omega_c^2 - \omega_p^2}{\omega^2 - \omega_c^2 + i\omega\gamma} = \frac{\omega^2 - \omega_L^2}{\omega^2 - \omega_T^2 + i\omega\gamma} \ , \tag{2}$$

where $\omega_p = \sqrt{4\pi nq^2/m}$ is the plasma frequency. This is also well known as the Lydane-Sachs-Teller dielectric function,[44] with the longitudinal frequency $\omega_L = \sqrt{\omega_c^2 + \omega_p^2}$ and the transverse frequency $\omega_T = \omega_c$. The model can be generalized by including the spatial dispersion, several characteristic frequencies ω_c, or by adding an external magnetic field, etc. It is worth noting the absence of the magnetic part of the Lorentz force in equation (1), according to the non-relativistic motion of the slight displacement $\mathbf{u}$. It is easy to see that, apart from relativistic contributions, it would introduce non-linearities in equation (1), which are beyond our assumption of a small displacement u. Using spatial Fourier transforms, this approximation can be formulated as $\mathbf{ku}(\mathbf{k}) \ll 1$, where $\mathbf{k}$ is the wavevector.

In general, an additional displacement $\mathbf{u}_0$ can be introduced in such a model, originating in external causes, subjected to collisions and obeying a different, averaged equation of motion, $m\dot{\mathbf{u}}_0 = q\mathbf{E}_t\tau$, where τ is a relaxation time; as it is well known, it gives rise to a density of "conduction" current $\mathbf{j}_0 = nq\dot{\mathbf{u}}_0 = (nq^2\tau/m)\mathbf{E}_t$ and the conductivity $\sigma = nq^2\tau/m$. We can see that it implies $\omega_c = 0$ in equation (1), a condition which defines the conductors; for dielectrics, $\omega_c \neq 0$. We leave aside the conduction current $\mathbf{j}_0$.

3 Electromagnetic field

The displacement $\mathbf{u}$ generates also a density of polarization current $\mathbf{j} = nq\dot{\mathbf{u}}$, which satisfies the continuity equation $\dot{\rho} + div\mathbf{j} = 0$. This equation allows an additional current, written as $\mathbf{j}_m = c \cdot curl\mathbf{M}$ (since $div\mathbf{j}_m = 0$); it is easy to see, from Maxwell's equations, that $\mathbf{M}$ is the magnetization. Therefore, the electromagnetic sources are represented both by the displacement $\mathbf{u}$ and magnetization $\mathbf{M}$. These fields ($\mathbf{u}$ and $\mathbf{M}$) are determined, $\mathbf{u}$ by equation (1) and $\mathbf{M}$ by the well-known equation of motion $\dot{\mathbf{M}} = \gamma\mathbf{B} \times \mathbf{M}$ for the magnetization, where $\mathbf{B}$ is the magnetic field and γ is the gyromagnetic factor. With such a representation for the sources, the Maxwell equations in matter are completely soluble. Here, we leave aside the magnetization, and consider only non-magnetic matter. With the polarization charge density ρ and current density $\mathbf{j}$ established above, and with usual notations, the Maxwell equations read

$$\begin{gathered} div\mathbf{E} = 4\pi\rho = -4\pi div\mathbf{P} = -4\pi nqdiv\mathbf{u} \ , \ div\mathbf{H} = 0 \ , \\ curl\mathbf{E} = -\tfrac{1}{c}\tfrac{\partial\mathbf{H}}{\partial t} \ , \ curl\mathbf{H} = \tfrac{1}{c}\tfrac{\partial\mathbf{E}}{\partial t} + \tfrac{4\pi}{c}\mathbf{j} = \tfrac{1}{c}\tfrac{\partial\mathbf{E}}{\partial t} + \tfrac{4\pi}{c}\dot{\mathbf{P}} = \tfrac{1}{c}\tfrac{\partial\mathbf{E}}{\partial t} + \tfrac{4\pi}{c}nq\dot{\mathbf{u}} \ . \end{gathered} \tag{3}$$

We solve these equations, together with equation (1), for the electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$ and the displacement field $\mathbf{u}$ (polarization $\mathbf{P} = nq\mathbf{u}$).

To this end, as usually, we introduce the electromagnetic potentials $\mathbf{A}$ and Φ, through $\mathbf{E} = -(1/c)\partial\mathbf{A}/\partial t - grad\Phi$ and $\mathbf{H} = curl\mathbf{A}$, subjected to the Lorenz gauge $div\mathbf{A} + (1/c)\partial\Phi/\partial t = 0$,

and use Fourier transforms of the type

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{2\pi\sqrt{N}} \sum_{\mathbf{k}} \int d\omega \mathbf{u}(\mathbf{k},\omega) e^{-i\omega t + i\mathbf{k}\mathbf{r}} \tag{4}$$

(and similar transforms for all the other functions of position and time). We note the well-known symmetry property $\mathbf{u}^*(-k,-\omega) = \mathbf{u}(\mathbf{k},\omega)$, corresponding to the real-valued field $\mathbf{u}(\mathbf{r},t)$. As it is well known, such a Fourier representation does not account for surface effects, *i.e.* effects associated with sharp surfaces. In order to account for finite-size (surface) effects an expansion in eigenfunctions of the laplacian (with outgoing-waves boundary conditions at infinity) is appropiate, preserving the dependence on the direct-space coordinate perpendicular to the surface. It is easy to see that, in general, such a decomposition implies curvilinear coordinates, and leads to serious technical difficulties. The density of these eigenmodes (which appears in summations over states) differs, in general, from the density of the plane waves which appear in the Fourier transformations. Therefore, additional contributions, which depend on the extension and the shape of the bodies, may appear. In addition, the sharpness of the surfaces generates the well-known "depolarizing" factors, especially in the long-wavelength limit, which cannot be taken into account by Fourier transforms.

As it is well known, the Maxwell equations (3) lead to the wave equations with sources

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi = 4\pi\rho \ , \ \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\mathbf{j} \tag{5}$$

for the electromagnetic potentials. Taking the Fourier transforms, we get

$$\mathbf{A}(\mathbf{k},\omega) = 4\pi i\lambda F(\mathbf{k},\lambda)\mathbf{u}(\mathbf{k},\omega) \tag{6}$$

and $\Phi(\mathbf{k},\omega) = \mathbf{k}\mathbf{A}(\mathbf{k},\omega)/\lambda$, where $\lambda = \omega/c$, $F(k,\lambda) = (\lambda^2 - k^2)^{-1}$ is the Green function (for the Helmholtz equation) and the factor nq is left aside (it is restored in the final formulae). The latter relationship is the Lorenz gauge, which expresses the charge conservation, *i.e.* the continuity equation. In order to account for the retardation, the function F must actually be written as $F(k,\lambda) = (\lambda^2 - k^2 + i\lambda 0^+)^{-1}$, as it can be seen by taking the Fourier transforms of the retarded Kirchoff's potentials

$$\begin{aligned} \Phi(\mathbf{r},t) &= \int d\mathbf{r}' \tfrac{\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ , \\ \mathbf{A}(\mathbf{r},t) &= \tfrac{1}{c}\int d\mathbf{r}' \tfrac{\mathbf{j}(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \ , . \end{aligned} \tag{7}$$

which are solutions of equations (5). For simplicity, we often leave aside the arguments $\mathbf{k}$, ω in the Fourier transforms. Making use of equation (6), we get straightforwardly

$$\mathbf{E} = -4\pi F\left[\lambda^2\mathbf{u} - \mathbf{k}(\mathbf{k}\mathbf{u})\right] \ , \ \mathbf{H} = -4\pi\lambda F \mathbf{k}\times\mathbf{u} \ . \tag{8}$$

It is convenient to introduce the longitudinal displacement $u_1 = \mathbf{k}\mathbf{u}/k$ and the transverse displacement $u_2 = \mathbf{k}_\perp\mathbf{u}/k$, where $\mathbf{k}_\perp$ is a vector perpendicular to $\mathbf{k}$ and of the same magnitude as $\mathbf{k}$, and to write $\mathbf{u} = u_1\mathbf{k}/k + u_2\mathbf{k}_\perp/k$. We note the symmetry property $-u^*_{1,2}(-k,-\omega) = u_{1,2}(k,\omega)$. In summations over states we take into account that there are in fact two transverse components u_2 of the displacement, corresponding to two polarizations. Making use of these notations we get

$$E_1 = -4\pi u_1 \ , \ E_2 = -4\pi\lambda^2 F u_2 \ . \tag{9}$$

We can see that the longitudinal component of the internal (polarization) field compensates the longitudinal component of the polarization, as expected, while the transverse polarization introduces a spatial dispersion. Introducing these fields in the equation of motion (1) we get

$$\begin{aligned}(\omega^2-\omega_c^2-\omega_p^2+i\omega\gamma)u_1 &= -\tfrac{q}{m}E_{01}\ ,\\ (\omega^2-\omega_c^2-\omega_p^2\lambda^2F+i\omega\gamma)u_2 &= -\tfrac{q}{m}E_{02}\ ,\end{aligned} \tag{10}$$

whence the longitudinal and transverse polarizabilities

$$\begin{aligned}\alpha_1 = E_{01}/P_1 &= -\tfrac{1}{4\pi}\tfrac{\omega_p^2}{\omega^2-\omega_c^2-\omega_p^2+i\omega\gamma}\ ,\\ \alpha_2 = E_{02}/P_2 &= -\tfrac{1}{4\pi}\tfrac{\omega_p^2}{\omega^2-\omega_c^2-\omega_p^2\lambda^2F+i\omega\gamma}\ .\end{aligned} \tag{11}$$

It is worth noting that for finite-size bodies the plasma frequency in equations (10) is modified by the "depolarizing" factors. For instance, for a half-space it acquires the well-known form $\omega_p/\sqrt{2}$ of the surface plasmon,[45] for a sphere in the dipole approximation it becomes the spherical plasmon $\omega_p/\sqrt{3}$.[46] For uniform fields (in the long-wavelength limit $\mathbf{k}\to 0$) the two polarizabilities coincide, as expected. Making use of equations (10) and (11) we get straightforwardly the electric susceptibility

$$\chi = \frac{\alpha_1}{1-4\pi\alpha_1} = \frac{\alpha_2}{1-4\pi\lambda^2F\alpha_2} = -\frac{1}{4\pi}\frac{\omega_p^2}{\omega^2-\omega_c^2}\ , \tag{12}$$

the same for both longitudinal and transverse field components, as given by equation (2). We can see that the electric susceptibility does not exhibit the spatial dispersion, in contrast with the transverse polarizability.

4 Polarization eigenmodes

The longitudinal polarizability does not depend on the wavevector $\mathbf{k}$. Its singularity (leaving aside the damping parameter γ) gives the longitudinal ("plasmon") mode $\Omega_1=\omega_L=\sqrt{\omega_c^2+\omega_p^2}$, as resulting from two oscillatory motions, one with the local, characteristic frequency ω_c and another collective, driven by the polarization field, with the frequency ω_p. It can also be obtained from the zero of the dielectric function, $\varepsilon=0$. The singularities of the transverse polarizability correspond to propagating polaritons. They are given by the well-known dispersion relation $\varepsilon\omega^2=c^2k^2$. For $\omega_c=0$ (conductors), they acquire the well-known form $\Omega_2(k)=\sqrt{\omega_p^2+c^2k^2}$, which again, exhibits the characteristic pattern of two oscillatory motions (plasmons with frequency ω_p and photons with frequency ck). For $\omega_c\neq 0$, we have two polaritonic branches, $\Omega_{2,3}(k)$, corresponding to the occurrence of a third force, with frequency ω_c. In the long-wavelenght limit ($k\to 0$) they behave like

$$\Omega_2\simeq\sqrt{\omega_L^2+\omega_p^2c^2k^2/\omega_L^2}\ ,\ \Omega_3\simeq\frac{\omega_T}{\omega_L}ck\ , \tag{13}$$

while in the short-wavelenght limit ($k\to\infty$) they go like

$$\Omega_2\simeq\sqrt{c^2k^2+\omega_p^2}\ ,\ \Omega_3\simeq\omega_T=\omega_c\ . \tag{14}$$

We can see that the frequency Ω_3 in the long-wavelength limit corresponds to "photons" with a renormalized phase velocity $v=c\omega_T/\omega_L$ ($\Omega_3=vk$). These two polaritonic branches arise

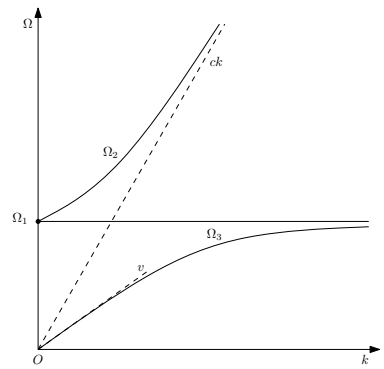

Figure 1: The polarization longitudinal eigenmode Ω_1 and the two transverse eigenmodes $\Omega_{2,3}$, according to equations (10). We note the velocity v of the renormalized "photons" (ck is the photon frequency).

from the transverse mode ω_c and the photonic mode ck, splitted in the region $\omega_c \simeq ck$ by the plasmonic coupling ω_p. If ω_p is a small parameter, we may take approximately ω_c and ck for the two transverse frequencies, as for non-polarizable matter.

For the sake of the generality we preserve three distinct frequencies, Ω_1 for the longitudinal component and $\Omega_{2,3}(k)$ for the transverse components. They are shown schematically in Fig. 1.

According to equations (10) the frequencies $\Omega_{1,2,3}$ are the eigenfrequencies of three types of harmonic oscillators, given by

$$\begin{aligned} u_1(\mathbf{k},\omega) &= 2\pi\delta(\omega-\Omega_1)u_1(\mathbf{k}) \ , \\ u_2(\mathbf{k},\omega) &= 2\pi\delta(\omega-\Omega_2)u_2^{(1)}(\mathbf{k}) + 2\pi\delta(\omega-\Omega_3)u_2^{(2)}(\mathbf{k}) \ , \end{aligned} \tag{15}$$

or

$$\begin{aligned} u_1(\mathbf{k},t) &= u_1(\mathbf{k})e^{-i\Omega_1 t} \ , \\ u_2(\mathbf{k},t) &= u_2^{(1)}(\mathbf{k})e^{-i\Omega_2 t} + u_2^{(2)}(\mathbf{k})e^{-i\Omega_3 t} \ . \end{aligned} \tag{16}$$

Equations (15) and (16) include only the positive frequencies. The negative frequencies $-\Omega_{1,2,3}$ are included whenever the case. We emphasize that the functions $u_1(\mathbf{k},\omega)$ and $u_2^{(1,2)}(\mathbf{k},\omega)$ are solutions of equations (10) for a vanishing external field. Making use of equations (9), we get the field generated by the polarization

$$\begin{aligned} E_1(\mathbf{k},t) &= -4\pi u_1(\mathbf{k})e^{-i\Omega_1 t} \ , \\ E_2(\mathbf{k},t) &= -4\pi\tfrac{\Omega_2^2-\omega_c^2}{\omega_p^2}u_2^{(1)}(\mathbf{k})e^{-i\Omega_2 t} - 4\pi\tfrac{\Omega_3^2-\omega_c^2}{\omega_p^2}u_2^{(2)}(\mathbf{k})e^{-i\Omega_3 t} \ . \end{aligned} \tag{17}$$

According to equations (16), the dynamics of the coordinates $u_1(\mathbf{k},t)$, $u_2^{(2,3)}(\mathbf{k},t)$ is governed by a harmonic-oscillator hamiltonian

$$\begin{aligned} H = \textstyle\sum_{\mathbf{k}}\left[\frac{1}{2m}\left|p_1\right|^2 + \frac{1}{2}m\Omega_1^2\left|u_1\right|^2\right] + \sum_{\mathbf{k}}\left[\frac{1}{2m}\left|p_2^{(1)}\right|^2 + \frac{1}{2}m\Omega_2^2\left|u_2^{(1)}\right|^2\right] + \\ + \textstyle\sum_{\mathbf{k}}\left[\frac{1}{2m}\left|p_2^{(2)}\right|^2 + \frac{1}{2}m\Omega_3^2\left|u_2^{(2)}\right|^2\right] \ , \end{aligned} \tag{18}$$

where p_1 and $p_2^{(1,2)}$ are the corresponding momenta (here the negative frequencies are automatically included). By taking the inverse (spatial) Fourier transform we may get a very complicated interaction, involving the displacement field $\mathbf{u}(\mathbf{r},t)$, generated by the polarization field in the direct space. More interesting, we can quantize the motion of the harmonic oscillators described by the hamiltonian given above. For instance, the mean square displacements are given by equations like $\overline{|u_1|^2} = (\hbar/m\Omega_1)(n_1+1/2)$, where n_1 is the quantum number (and $\hbar$ is Planck's constant). These oscillators have also an energy of the form $\hbar\Omega_1(n_1+1/2)$ (including the zero-point energy $\hbar\Omega_1/2$), etc. We can also take a statistical average corresponding to a given temperature, or we can account for the quantum-statistical fluctuations. According to the results given above, the internal, polarization field can be represented entirely in terms of the coordinates u_1 and $u_2^{(1,2)}$, thus acquiring a mechanical representation. It is worth noting that the quantum numbers $n_{1,2,3}$ depend on the wavevector $\mathbf{k}$ (as well as the frequencies $\Omega_{2,3}(k)$, which depend only on the magnitude k of the wavevector). The energy quanta associated with the frequencies $\Omega_{1,2,3}$ are much higher than the usual temperatures, so we may neglect the temperature effects, and limit ourselves to the zero-point contributions (vacuum fluctuations), except for the long-wavelength limit of the frequency $\Omega_3 = vk$, whose contribution, usually, is comparatively small.

By usual procedure, from Maxwell's equations (3) we get the well-known energy conservation

$$\frac{1}{8\pi}\frac{\partial}{\partial t}(E^2+H^2) + \frac{c}{4\pi}div(\mathbf{E}\times\mathbf{H}) = -nq\mathbf{E}\dot{\mathbf{u}} \ , \tag{19}$$

where the *rhs* is the rate of change of the density of the kinetic energy of the mobile charges. We may write equation (19) as

$$\frac{\partial}{\partial t}(u+t) = -div\mathbf{s} \ , \tag{20}$$

where $u = (E^2+H^2)/8\pi$ is the density of the electromagnetic field energy, $t = nm\dot{\mathbf{u}}^2/2$ is the density of kinetic energy and $\mathbf{s} = c(\mathbf{E}\times\mathbf{H})/4\pi$ is the density of the Poynting vector.

We estimate below the time averages of total quantities, like the energy

$$U = \frac{1}{8\pi T}\int dtd\mathbf{r}(E^2+H^2) \ , \tag{21}$$

where T is a time sufficietly long in comparison with any relevant frequency. Making use of the Fourier transforms and of equations (8) and (9) we get straightforwardly

$$U = \frac{nq^2}{T}\sum_{\mathbf{k}}\int d\omega\left[|u_1|^2 + 2\frac{\omega^2}{\omega^2-c^2k^2}|u_2|^2\right] \ , \tag{22}$$

where we have introduced the factor 2 for the two transverse polarizations. We insert here the solutions $u_{1,2}(\mathbf{k},\omega)$ given by equations (15) and take into account that $\delta^2(\omega) = (T/2\pi)\delta(\omega)$. In addition, we note that the negative frequencies give the same contribution as the positive ones, so we account for them by a pre-factor 2. We get

$$\begin{aligned} U = 4\pi nq^2\textstyle\sum_{\mathbf{k}}\left[|u_1(\mathbf{k})|^2 + \frac{2\Omega_2^2}{\Omega_2^2-c^2k^2}\left|u_2^{(1)}(\mathbf{k})\right|^2\right] + \\ +8\pi nq^2\textstyle\sum_{\mathbf{k}}\frac{\Omega_3^2}{\Omega_3^2-c^2k^2}\left|u_2^{(2)}(\mathbf{k})\right|^2 \ . \end{aligned} \tag{23}$$

The mean square fluctuations in equations (23) can be replaced by their zero-point values of the form $|u_1(\mathbf{k})|^2 = \hbar/2m\Omega_1$, etc; we get

$$U = \frac{2\pi\hbar nq^2}{m}\textstyle\sum_{\mathbf{k}}\left[\frac{1}{\Omega_1} + \frac{2\Omega_2}{\Omega_2^2-c^2k^2} + \frac{2\Omega_3}{\Omega_3^2-c^2k^2}\right] \ . \tag{24}$$

The summation (integration) over $\mathbf{k}$ in equation (24) is divergent. It is reasonable to introduce an ultraviolet cutoff $k_c \simeq 1/a$, where a is the mean separation distance between the mobile charges (density $n = 1/a^3$). Under these circumstances we may neglect ck in comparison with the relevant frequencies occurring in equation (24), and replace approximately the integrand by $1/\overline{\Omega} \simeq 1/\Omega_1 + 2/\omega_L + 2/\omega_T$. We get an estimate

$$U \simeq N \frac{q^2}{a} \frac{\hbar^2/ma^2}{\hbar\overline{\Omega}} \tag{25}$$

for the energy. In equation (25) we can identify the Coulomb energy q^2/a per particle, the particle localization energy $\hbar^2/ma^2$ and the energy quanta $\hbar\overline{\Omega}$. We can see that matter stores a large amount of electromagnetic energy, arising from the zero-point motion of its polarization.

Similarly, we can estimate the Poynting vector

$$\begin{gathered} \mathbf{S} = \tfrac{c}{4\pi T} \int dtd\mathbf{r}\mathbf{E} \times \mathbf{H} = \\ = 16\pi n q^2 c^2 \textstyle\sum_{\mathbf{k}} \left[\frac{\Omega_2^3}{(\Omega_2^2 - c^2k^2)^2} \left| u_2^{(1)}(\mathbf{k}) \right|^2 + \frac{\Omega_3^3}{(\Omega_3^2 - c^2k^2)^2} \left| u_2^{(2)}(\mathbf{k}) \right|^2 \right] \mathbf{k} \ . \end{gathered} \tag{26}$$

We can see that it is given only by the transverse components of the displacement field $\mathbf{u}$ and it is directed along the propagation wavevectors $\mathbf{k}$, as expected. The total kinetic energy of the mobile charges is given by

$$\begin{gathered} K = \tfrac{n}{T} \int dtd\mathbf{r} \tfrac{1}{2} m \dot{\mathbf{u}}^2 = \\ = 2 \textstyle\sum_{\mathbf{k}} \left[\frac{1}{2} m\Omega_1^2 \left| u_1(\mathbf{k}) \right|^2 + m\Omega_2^2 \left| u_2^{(1)}(\mathbf{k}) \right|^2 + m\Omega_3^2 \left| u_2^{(2)}(\mathbf{k}) \right|^2 \right] \ . \end{gathered} \tag{27}$$

It can be estimated as $K \simeq N \cdot \hbar\overline{\Omega}$, and we can see that it is comparable with the electromagnetic energy U. Comparing with equation (18), we can see that K includes also the potential energy (therefore it is the total mechanical energy) of the harmonic oscillators, by the pre-factor 2.

It is also well-known that we get the Lorentz force

$$f_i = \rho E_i + \tfrac{1}{c}(\mathbf{j} \times \mathbf{H})_i = \partial_j \sigma_{ij} - \tfrac{\partial}{\partial t} g_i \tag{28}$$

from the Maxwell equations (3), arising from the stress tensor

$$\sigma_{ij} = \frac{1}{4\pi} \left[E_i E_j + H_i H_j - \frac{1}{2} \delta_{ij} (E^2 + H^2) \right] \tag{29}$$

and the electromagnetic momentum $\mathbf{g} = \mathbf{s}/c^2$, where $\rho = -nqdiv\mathbf{u}$ and $\mathbf{j} = nq\dot{\mathbf{u}}$ are the polarization charge and current densities. We give here the representation of the Lorentz force in terms of the displacement (polarization) field

$$\begin{gathered} \mathbf{f} = \tfrac{1}{T} \int dtd\mathbf{r} \left(\rho\mathbf{E} + \tfrac{1}{c}\mathbf{j} \times \mathbf{H} \right) = \\ = -8\pi i n q^2 \textstyle\sum_{\mathbf{k}} \left[\left| u_1(\mathbf{k}) \right|^2 + \frac{2\Omega_2^2}{\Omega_2^2 - c^2k^2} \left| u_2^{(1)}(\mathbf{k}) \right|^2 + \frac{2\Omega_3^2}{\Omega_3^2 - c^2k^2} \left| u_2^{(2)}(\mathbf{k}) \right|^2 \right] \mathbf{k} \ . \end{gathered} \tag{30}$$

Since, usually, u_1 and $u_2^{(1,2)}$ depend only on the magnitude of the wavevector (and not of its direction), the force given by equation (30) is zero. Similarly, we get the charge-field interaction

energy

$$E_{int} = \frac{1}{T}\int dtd\mathbf{r}\left(\rho\Phi - \frac{1}{c}\mathbf{jA}\right) =$$
$$= 8\pi nq^2 \sum_{\mathbf{k}} \left[\left|u_1(\mathbf{k})\right|^2 + \frac{2\Omega_2^2}{\Omega_2^2 - c^2k^2}\left|u_2^{(1)}(\mathbf{k})\right|^2 + \frac{2\Omega_3^2}{\Omega_3^2 - c^2k^2}\left|u_2^{(2)}(\mathbf{k})\right|^2 \right] . \tag{31}$$

Writing $\mathbf{f} = \sum_{\mathbf{k}} \mathbf{f}(\mathbf{k})$ and $E_{int} = \sum_{\mathbf{k}} E_{int}(\mathbf{k})$, we can see that the Fourier components of the force are given by $\mathbf{f}(\mathbf{k}) = -i\mathbf{k}E_{int}(\mathbf{k})$, as expected.

5 Two electromagnetically coupled bodies

We consider two polarizable bodies, denoted by a and b, one (a) placed at the origin $\mathbf{r} = 0$ and another (b) placed at $\mathbf{r}_0$. The displacement field for the body a is $\mathbf{u}_a(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t)$ and the displacement field for the body b is denoted by $\mathbf{u}_b(\mathbf{r}, t) = \mathbf{v}(\mathbf{r} - \mathbf{r}_0, t)$. We use Fourier transforms of the type given by equation (4), with the pre-factor $N^{-1/2} = (N_a + N_b)^{-1/2}$, where $N_{a,b}$ are the number of mobile charges in body a and, respectively, b, both enclosed in the same volume V. The densities are given by $n_{a,b} = N_{a,b}/V$. The Fourier transform of $\mathbf{u}_b(\mathbf{r}, t)$ acquires an exponential factor $e^{-i\mathbf{kr}_0}$, $\mathbf{u}_b(\mathbf{k}, \omega) = \mathbf{v}(\mathbf{k}, \omega)e^{-i\mathbf{kr}_0}$, which appears in the field $\mathbf{E}_b$ generated by the body b, according to equations (9). Consequently, we can write the equations of motion for $\mathbf{u}_a = \mathbf{u}$ as

$$(\omega^2 - \omega_{ca}^2 - \omega_{pa}^2)u_1 = \omega_{pb}^2 v_1 e^{-i\mathbf{kr_0}} ,$$
$$(\omega^2 - \omega_{ca}^2 - \omega_{pa}^2\lambda^2 F)u_2 = \omega_{pb}^2\lambda^2 F v_2 e^{-i\mathbf{kr_0}} . \tag{32}$$

We can see that the b-field corresponds to the propagating wavevector $-\mathbf{k}$ and the source placed at $\mathbf{r}_0$, *i.e.* it contains the factor $e^{i(-\mathbf{k})\mathbf{r}_0} = e^{-i\mathbf{kr_0}}$, in accordance with the retardation requirements (for positive frequencies). In order to fulfil the retardation requirements, we must limit ouselves only to fields outgoing from their sources. Similar equations of motion can be written for $\mathbf{u}_b$, where the field originates in the source a. In the b-frame this field corresponds to the propagating wavevector $\mathbf{k}$ and source placed at $-\mathbf{r}_0$, *i.e.* it must contains the factor $e^{i\mathbf{k}(-\mathbf{r}_0)} = e^{-i\mathbf{kr_0}}$. In addition we must restrict to $\mathbf{kr}_0 > 0$ for positive frequencies and $\mathbf{kr}_0 < 0$ for negative frequencies. We get

$$(\omega^2 - \omega_{cb}^2 - \omega_{pb}^2)v_1 = \omega_{pa}^2 u_1 e^{-i\mathbf{kr_0}} ,$$
$$(\omega^2 - \omega_{cb}^2 - \omega_{pb}^2\lambda^2 F)v_2 = \omega_{pa}^2\lambda^2 F u_2 e^{-i\mathbf{kr_0}} . \tag{33}$$

We can see that the polarizations of the two bodies are coupled, through their interaction, which depends on their mutual position $\mathbf{r}_0$. The two homogeneous systems of equations (32) and (33) have eigenfrequencies which depend on this interaction, given by the dispersion equations

$$\left[(\omega^2 - \omega_{ca}^2 - \omega_{pa}^2)(\omega^2 - \omega_{cb}^2 - \omega_{pb}^2) - \omega_{pa}^2\omega_{pb}^2 e^{-2i\mathbf{kr_0}}\right] u_1 = 0 ,$$
$$\left[(\omega^2 - \omega_{ca}^2 - \omega_{pa}^2\lambda^2 F)(\omega^2 - \omega_{cb}^2 - \omega_{pb}^2\lambda^2 F) - \omega_{pa}^2\omega_{pb}^2\lambda^4 F^2 e^{-2i\mathbf{kr_0}}\right] u_2 = 0 . \tag{34}$$

They correspond to the equations

$$(4\pi)^2\alpha_{a1,2}\alpha_{b1,2}e^{-2i\mathbf{kr}_0} = 1 \tag{35}$$

for the polarizabilities of the two bodies.

Equations (34), or (35), require $2\mathbf{kr}_0 = \pi n$, where n is a positive integer, $n = 0, 1, 2....$. For realistic values of the parameters $\omega_{ca,b}$ and $\omega_{pa,b}$ we get real solutions of these equations for n an

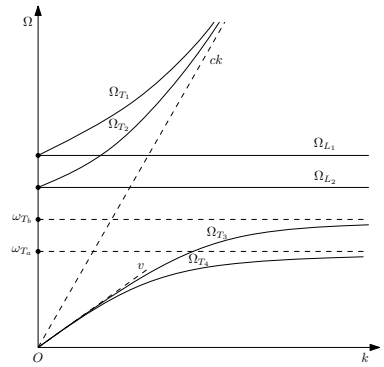

Figure 2: The longitudinal ($\Omega_{L1,2}$) and transverse ($\Omega_{T1,2,3,4}$) eigenmodes for two electromagnetically-coupled bodies a and b, as given by the roots of equations (34). The transverse modes $\omega_{Ta,b} = \omega_{ca,b}$ are also shown, together with the velocity v of the two modes $\Omega_{T3,4}$ (the slope of the curves $\Omega_{T3,4}$ at the origin). ck is the photon frequency.

even integer, so we set $\mathbf{k}\mathbf{r}_0 = \pi n$, $n = 0, 1, 2.....$ The interaction shifts the longitudinal modes $\omega_{La,b}$ of the two bodies, the new longitudinal frequencies being given by

$$\Omega_{L1,2}^2 = \frac{1}{2}\left[\omega_{La}^2 + \omega_{Lb}^2 \pm \sqrt{(\omega_{La}^2 - \omega_{Lb}^2)^2 + 4\omega_{pa}^2\omega_{pb}^2}\right] \tag{36}$$

(we note that they do not depend on the wavevector $\mathbf{k}$); for conductors we have only one longitudinal frequency $\Omega_{L1} = \sqrt{\omega_{pa}^2 + \omega_{pb}^2}$, but we keep both frequncies $\Omega_{L1,2}$ for the sake of the generality. The solution of the first equation (34) is given by

$$u_1(\mathbf{k}, \omega) = 2\pi\delta(\omega - \Omega_{L1})u_1^{(1)}(\mathbf{k}) + 2\pi\delta(\omega - \Omega_{L2})u_1^{(2)}(\mathbf{k}) \tag{37}$$

for $\mathbf{k}\mathbf{r}_0 > 0$ and a similar contribution for negative frequencies and $\mathbf{k}\mathbf{r}_0 < 0$. The coordinate v_1 is obtained directly from equations (34).

Similarly, from the second equation (34) we get four frequencies, in general, for the polaritonic transverse modes, denoted by Ω_{Ti}, $i = 1, 2, 3, 4$. In the long-wavelengh limit they behave like

$$\Omega_{T1,2}^2 \simeq \frac{1}{2}\left[\omega_{La}^2 + \omega_{Lb}^2 \pm \sqrt{(\omega_{La}^2 + \omega_{Lb}^2)^2 - 4\omega_{ca}^2\omega_{cb}^2 - 4\omega_{ca}^2\omega_{pb}^2 - 4\omega_{cb}^2\omega_{pa}^2}\right] \tag{38}$$

(plus a small contribution of the form $const \cdot k^2$) and

$$\Omega_{T3,4} \simeq vk \ , \ v = c\frac{\omega_{ca}\omega_{cb}}{\sqrt{\omega_{ca}^2\omega_{cb}^2 + \omega_{ca}^2\omega_{pb}^2 + \omega_{cb}^2\omega_{pa}^2}} \ ; \tag{39}$$

for $k \to \infty$,

$$\Omega_{T1,2}^2 \simeq c^2k^2 + \omega_{pa}^2 + \omega_{pb}^2 \ , \ \Omega_{T3,4} \simeq \omega_{ca,b} \ . \tag{40}$$

For small couplings ($\omega_{pa,b} \ll \omega_{ca,b}$) these frequencies reduce approximately to the two transverse modes $\omega_{ca,b}$ and two photon mode $\omega \simeq ck$. For conductors ($\omega_{ca} = \omega_{cb} = 0$) we get only one polaritonic branch $\Omega_T = \sqrt{\omega_{pa}^2 + \omega_{pb}^2 + c^2k^2}$. The eigenfrequencies $\Omega_{L1,2}$ and $\Omega_{T1,...4}$ of two electromagnetically coupled bodies a and b are shown schematically in Fig. 2.

The solution of the second equation (34) can be written as

$$u_2(\mathbf{k}, \omega) = 2\pi\sum_{i=1}^{4}\delta(\omega - \Omega_{Ti})u_2^{(i)}(\mathbf{k}) \tag{41}$$

for $\mathbf{k}\mathbf{r}_0 > 0$ and a corresponding decomposition for negative frequencies and $\mathbf{k}\mathbf{r}_0 < 0$. The coordinate v_2 is obtained directly from the second equation (34) as a function of u_2.

It is worth computing the Lorentz force acting on one body on behalf of the other; for instance, the force by which body a acts upon body b. We follow the same procedure as that which led to equation (30) and use the coupled equations (32) and (33) in order to eliminate the coordinates $v_{1,2}$ in favour of the coordinates $u_{1,2}$. We get

$$\begin{aligned}
\mathbf{f}_{ab} &= \tfrac{1}{T}\int dt d\mathbf{r}\left(\rho_b \mathbf{E}_a + \tfrac{1}{c}\mathbf{j}_b \times \mathbf{H}_a\right) = \\
&== -\tfrac{2in_a n_b q^2 V}{T\omega_{pb}^2 N}\sum_{\mathbf{k}}\int d\omega(\omega^2 - \omega_{ca}^2 - \omega_{pb}^2)\left|u_1(\mathbf{k},\omega)\right|^2 \mathbf{k} - \\
&- \tfrac{4in_a n_b q^2 V}{T\omega_{pb}^2 N}\sum_{\mathbf{k}}\int d\omega(\omega^2 - \omega_{ca}^2 - \omega_{pa}\lambda^2 F)\left|u_2(\mathbf{k},\omega)\right|^2 \mathbf{k} = \\
&= \tfrac{in_a n_b q^2 \omega_{pa}^2 V}{2\pi T\omega_{pb}^2 N}\sum_{\mathbf{k}}\int d\omega\left[\tfrac{1}{\alpha_{a1}}\left|u_1(\mathbf{k},\omega)\right|^2 + \tfrac{2}{\alpha_{a2}}\left|u_2(\mathbf{k},\omega)\right|^2\right]\mathbf{k} \ ,
\end{aligned} \tag{42}$$

where $\alpha_{a1,2}$ are the polarizabilities of the body a, as given by equations (11). If we eliminate the coordinates $u_{1,2}$ in favour of the coordinates $v_{1,2}$ (by using equations (32) and (33), we get a similar expression for the force $\mathbf{f}_{ab}$, with a and b interchanged; this corresponds to the force $\mathbf{f}_{ba}$ and, since the sign does not change, we conclude that $\mathbf{f}_{ab} = \mathbf{f}_{ba} = 0$. Indeed, when introducing the decompositions given by equations (37) and (41), we note that for positive frequencies $\mathbf{k}\mathbf{r}_0 > 0$, while $\mathbf{k}\mathbf{r}_0 < 0$ for negative frequencies. Since both contributions are equal, we can write

$$\begin{aligned}
\mathbf{f}_{ab} &= -\tfrac{4\pi i n_a n_b q^2 V}{\omega_{pb}^2 N}\sum_{\mathbf{k}}\sum_{i=1}^{2}(\Omega_{Li}^2 - \omega_{La}^2)\left|u_1^{(i)}(\mathbf{k})\right|^2 \mathbf{k} \\
&- \tfrac{8\pi i n_a n_b q^2 V}{\omega_{pb}^2 N}\sum_{\mathbf{k}}\sum_{i=1}^{4}(\Omega_{Ti}^2 - \omega_{ca}^2 - \tfrac{\omega_{pa}^2\Omega_{Ti}^2}{\Omega_{Ti}^2 - c^2k^2})\left|u_2^{(i)}(\mathbf{k})\right|^2 \mathbf{k} \ ,
\end{aligned} \tag{43}$$

where the summation extends over the whole $\mathbf{k}$-space. We can see, indeed, that $\mathbf{f}_{ab} = 0$, as long as the amplitudes $u_{1,2}^{(i)}$ do not depend on the direcion of the wavevector $\mathbf{k}$.

6 Casimir force

Equations (32) and (33) describe two pairs of coupled harmonic oscillators. One pair, say $v_{1,2}$ is completely determined by the motion of the pair $u_{1,2}$; its energy is taken up in the modified frequencies given by the dispersion equations (34). Therefore, we have, in fact, only one pair of harmonic oscillators for the two coupled bodies a and b, of coordinates $u_{1,2}$ and eigenfrequencies Ω_{Li}, $i = 1, 2$ and Ω_{Ti}, $i = 1, 2, 3, 4$, governed by a harmonic-oscillator hamiltonian of the type given in equation (18). Its ground-state (zero-point) energy reads

$$E = \sum_{\mathbf{k}}\left(\sum_{i=1}^{2}\frac{1}{2}\hbar\Omega_{Li} + \sum_{i=1}^{4}\hbar\Omega_{Ti}(k)\right) \ , \tag{44}$$

where we have introduced a factor 2 in order to account for the two transverse polarizations. We may choose to eliminate $u_{1,2}$ in favour of the coordinates $v_{1,2}$. The energy given by equation (44) remains unchanged. Both situations are equally valid, though they introduce an asymmetry, in the sense that in each case one pair of coordinates are completely determined by the other pair.

These frequencies are obtained from equations (34) by requiring $\mathbf{k}\mathbf{r}_0 = \pi n$. It is convenient to introduce the component κ of the wavevector $\mathbf{k}$ along the position vector $\mathbf{r}_0$, whose magnitude is denoted by d ($r_0 = d$). Therefore, we have $\kappa_n = \pi n/d$ and $k_n^2 = k_\perp^2 + \kappa_n^2$, where $\mathbf{k}_\perp$ is the transverse wavevector, *i.e.* the component of the wavevector $\mathbf{k}$ perpendicular to $\mathbf{r}_0$. The energy given by equation (44) can then be written as

$$E = \frac{S}{2\pi} \sum_{n=0} \int dk_\perp k_\perp \left(\sum_{i=1}^{2} \frac{1}{2} \hbar \Omega_{Li} + \sum_{i=1}^{4} \hbar \Omega_{Ti}(k_n) \right) \quad , \tag{45}$$

where S is the transverse area. We estimate the change brought about by the finite distance d in the energy E by using the Euler-Maclaurin formula:[47]

$$\Delta E = \sum_{m=1} \frac{(-1)^m B_m (\pi/d)^{2m-1}}{(2m)!} f^{(2m-1)}(0) \quad , \tag{46}$$

where B_m are the Bernoulli's numbers and

$$f(\kappa) = \frac{S}{2\pi} \int dk_\perp k_\perp \left(\sum_{i=1}^{2} \frac{1}{2} \hbar \Omega_{Li} + \sum_{i=1}^{4} \hbar \Omega_{Ti}(\sqrt{k_\perp^2 + \kappa^2}) \right) \quad . \tag{47}$$

It is easy to see that the longitudinal frequencies Ω_{Li} do not contribute to equation (46). Similarly, the transverse frequencies $\Omega_{T1,2}$, which, according to equation (38), go like $\sim\sqrt{const + \kappa^2}$ in the long-wavelength limit, do not contribute to equation (46), since all their odd-order derivatives are vanishing for $\kappa = 0$. We are left with the contributions arising from $\Omega_{T3,4}$, which behave like vk in the long-wavelength limit, according to equation (39). Equation (46) becomes

$$\Delta E = \frac{\hbar v S}{2\pi} \sum_{m=1} \frac{(-1)^m B_m (\pi/d)^{2m-1}}{(2m)!} \left(\int_{\kappa^2} du \sqrt{u} \right)_0^{(2m-1)} \quad , \tag{48}$$

where we have introduced $u = k_\perp^2 + \kappa^2$. The only contribution to equation (48) comes from the third-order derivative. We get ($B_2 = 1/30$)

$$\Delta E = -\frac{\pi^2 \hbar v S}{360} \cdot \frac{1}{d^3} \tag{49}$$

and an attractive force

$$F = -\frac{\pi^2 \hbar v S}{120} \cdot \frac{1}{d^4} \tag{50}$$

acting between the two bodies. This is the Casimir force, arising between two polarizable bodies from the polarization vacuum fluctuations. It differs from the classical formula $F = -\pi^2 \hbar c S/240 d^4$, derived for two conducting half-spaces,[2, 8] by a factor 2, arising from the two branches $\Omega_{T3,4}$ (which behave identically in the long-wavelength limit), as well as by the presence of the renormalized velocity v instead of c. The nature of the bodies is incorporated in the polariton velocity

$$v = c \frac{\omega_{ca}\omega_{cb}}{\sqrt{\omega_{ca}^2\omega_{cb}^2 + \omega_{ca}^2\omega_{pb}^2 + \omega_{cb}^2\omega_{pa}^2}} \tag{51}$$

as given by equation (39). It is worth noting that this velocity v differs from the corresponding pre-factor in the formula given in Ref. 8 for two dielectrics. For identical bodies there remains only one vk-branch, with velocity v given by equation (51) for $\omega_{c,pa} = \omega_{c,pb}$. The formula (50) for the force should then be modified accordingly.

The effect of the temperature $T = 1/\beta$ can be incorporated in equation (48) by the change

$$\int_{\kappa^2} du\sqrt{u} \to \int_{\kappa^2} du\sqrt{u}\coth\left[\frac{1}{2}\beta\hbar v\sqrt{u}\right] \ . \tag{52}$$

For realistic values of the parameters we have $\beta\hbar v/d \gg 1$, so we get a temperature correction factor $\simeq \coth(\beta\hbar v/d)$ in the expression of the force. In the opposite limit (very high temperature, $\beta\hbar v/d \ll 1$) the force is vanishing.

The situation described above corresponds to two dielectrics ($\omega_{ca,b} \neq 0$). Let us assume that we have a conductor a ($\omega_{ca} = 0$) and a dielectric b ($\omega_{cb} \neq 0$). The two transverse modes $\Omega_{T3,4}$ disappear in this case, and the two longitudinal modes $\Omega_{L1,2}$ do not contribute to the energy. We are left with the two transverse modes $\Omega_{T1,2}$, which, in the long-wavelength limit go like

$$\Omega_{T1,2}^2 \simeq \Omega_{L1,2}^2 + v_{1,2}^2 k^2 \ , \tag{53}$$

where

$$v_{1,2}^2 = \frac{1}{2}c^2\left[1 \pm \frac{\omega_{pa}^2 + \omega_{pb}^2 - \omega_{cb}^2}{\sqrt{(\omega_{pa}^2 + \omega_{pb}^2 + \omega_{cb}^2)^2 - 4\omega_{pa}^2\omega_{cb}^2}}\right] \ . \tag{54}$$

It is worth noting that, while Ω_{T1} goes like $\sim ck$ for $k \to \infty$, the mode Ω_{T2} approaches ω_{cb} in the same limit. The frequencies $\Omega_{T1,2}$ given by equation (53) can be written as

$$\Omega_{T1,2} = v_{1,2}\sqrt{k_\perp^2 + \Omega_{L1,2}^2/v_{1,2}^2 + \kappa^2} \tag{55}$$

and we can see that $k_\perp$ may acquire imaginary values, such that $k_\perp^2 + \Omega_{L1,2}^2/v_{1,2}^2 = k_\perp^{'2}$ behaves as the square of a new transverse wavevector $k_\perp^{'}$ ranging from 0 to $\Omega_{L1,2}/v_{1,2}$ (a similar situation for two dielectrics would imply imaginary frequencies $\Omega_{3,4}$ which are not physically acceptable). The transverse $\mathbf{k}_\perp$-waves are damped waves in this case. It is easy to see that the Casimir force will have the same expression as given by equation (50) above, with the proper replacement of the velocity v by $(v_1 + v_2)/2$. The same situation holds for two conductors, where we have only one polaritonic branch $\Omega_{T1} = \sqrt{\omega_{pa}^2 + \omega_{pb}^2 + c^2k^2}$. In this case, we get the classical result $F = -\pi^2\hbar cS/240d^4$ (corresponding to two half-spaces).

7 van der Waals-London force

In the non-retarded regime $\lambda = 0$, which corresponds to $k = 0$ for free waves. It follows that $\kappa = -ik_\perp$ acquires purely imaginary values (the waves are damped along the $\mathbf{r}_0$). The second equations (34) give the transverse modes $\omega_{ca,b}$, which do not depend on $k_\perp$ and, consequently do not contribute to the change in energy. We are left with the longitudinal modes, given by the first equation (34) which reads now

$$(\omega^2 - \omega_{La}^2)(\omega^2 - \omega_{Lb}^2) - \omega_{pa}^2\omega_{pb}^2 e^{-2k_\perp d} = 0 \ . \tag{56}$$

For realistic situations we may take $\omega_{La} \simeq \omega_{Lb}$ and neglect $(\omega_{La}^2 - \omega_{Lb}^2)^2$ in comparison with $\omega_{pa}^2\omega_{pb}^2$. Further, we may expand the solutions of equation (56) in powers of $\omega_{pa}\omega_{pb}e^{-k_\perp d}$, and subtract the energy corresponding to $d \to \infty$. We get the approximate change in the ground-state (zero-point) energy

$$\Delta E \simeq -\frac{\hbar S}{4\pi\sqrt{2}}\frac{\omega_{pa}^2\omega_{pb}^2}{(\omega_{La}^2 + \omega_{Lb}^2)^{3/2}}\int_0 dk_\perp k_\perp e^{-2k_\perp d} \tag{57}$$

and the van der Waals-London force

$$F \simeq -\frac{\hbar S}{8\pi\sqrt{2}} \frac{\omega_{pa}^2 \omega_{pb}^2}{(\omega_{La}^2 + \omega_{Lb}^2)^{3/2}} \cdot \frac{1}{d^3} \ . \tag{58}$$

We note that the pre-factor in equation (58) differs from the result given in Ref. 8.

Comparing equations (50) and (58), we get a crossover distance of the order $d_1 \simeq c/\omega_p$, which separates the ranges of the Casimir and van der Waals-London forces (ω_p being a frequency of the order of the plasma frequencies of the two bodies).

It is worth noting the solutions of equation (56) for two identical conductors: $\Omega^2 = \omega_p^2(1 \pm e^{-kd})$. They differ from the well-known solutions $\Omega^2 = \omega_p^2(1 \pm e^{-kd})/2$ corresponding to two half-spaces separated by distance d, which exhibit the surface plasmons (for instance, for $d \to \infty$ we get the well-known surface plasmon $\omega_p/\sqrt{2}$). This is a particular example that our approach does not account for surface effects, as we discussed before. The surface effects may bring significant changes both in numerical coefficients, as well as in the d-dependence of the forces, especially through the depolarizing fields the surfaces generate, with contributions that may differ for identical, or distinct bodies. Such effects are more significant for the non-retarded regime.

In order to make this situation more clear, we resort here to another well-known decomposition of the spherical wave. Indeed, we preserve the coordinate z along the direction between the two bodies, and use only transverse Fourier transforms. The decomposition reads[45, 48]

$$\frac{e^{i\lambda R}}{R} = \frac{i}{2\pi} \int d\mathbf{k} \frac{1}{\kappa} e^{i\mathbf{kr}} e^{i\kappa|z|} \ . \tag{59}$$

If we further take the Fourier transform with respect to the z-coordinate we get the function $F(k, \lambda)$ used here. Now, the derivatives of the function $e^{i\kappa|z|}$ are given by

$$\frac{\partial}{\partial z} e^{i\kappa|z|} = i\kappa sgn(z) e^{i\kappa|z|} \ , \ \frac{\partial^2}{\partial z^2} e^{i\kappa|z|} = 2i\kappa\delta(z) - \kappa^2 e^{i\kappa|z|} \ . \tag{60}$$

We can see easily that, while the Fourier transform reproduces the first derivative of the function $e^{i\kappa|z|}$, it does not reproduce the second derivative (as a consequence of the Gibbs phenomenon in the Fourier transform of the function $sgn(z)$). Such second-derivative-terms are responsible for the surface charges.

8 Electromagnetic field energy

It is worth examining the electromagnetic field energy, as given by equation (21), for the total field generated by the two bodies. The fields are given by $\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b$ and $\mathbf{H} = \mathbf{H}_a + \mathbf{H}_b$, where $\mathbf{E}_{a,b}$ and $\mathbf{H}_{a,b}$ are given by equation (8) and (9). We get straightforwardly

$$U = \frac{nq^2}{T} \sum_{\mathbf{k}} \int d\omega \left(|w_1|^2 + 2\lambda^2 F |w_2|^2 \right) \ , \tag{61}$$

where $\mathbf{w} = (n_a/n)\mathbf{u} + (n_b/n)\mathbf{v}$. We choose first to eliminate the coordinates $v_{1,2}$ in favour of the coordinates $u_{1,2}$, by using equations (32). Introducing the zero-point mean values for $|u_{1,2}|^2$, we get

$$\begin{aligned} U = & \frac{\pi\hbar n_a^2 q^2}{nm} \sum_{\mathbf{k}} \sum_{i=1}^{2} \left(\frac{\Omega_{Li}^2 - \omega_{ca}^2}{\omega_{pa}^2} \right)^2 \frac{1}{\Omega_{Li}} + \\ & + \frac{2\pi\hbar n_a^2 q^2}{nm} \sum_{\mathbf{k}} \sum_{i=1}^{4} \left(\frac{\Omega_{Ti}^2 - \omega_{ca}^2}{\omega_{pa}^2} \right)^2 \frac{\Omega_{Ti}^2 - c^2 k^2}{\Omega_{Ti}^3} \ . \end{aligned} \tag{62}$$

If we eliminate $u_{1,2}$ in favur of the coordinates $v_{1,2}$, and use the zero-point mean square fluctuations for $|v_{1,2}|^2$, we get a similar equation for U, with a and b interchanged. Since both situations are equally valid we take the mean value, corresponding to a symmetrized contribution arising from the two bodies.

Let us assume that we have two dielectrics. It is easy to see that the change in energy brought by the finite distance d comes from the two branches $\Omega_{T3,4} = vk$ of "renormalized" photons. The corresponding energy can be written as

$$U_{3,4} \simeq -\frac{2\pi\hbar q^2}{nm}\left[\left(\frac{n_a\omega_{ca}^2}{\omega_{pa}^2}\right)^2 + \left(\frac{n_b\omega_{cb}^2}{\omega_{pb}^2}\right)^2\right]\frac{c^2-v^2}{v^3}\sum_{n=0}\sum_{\mathbf{k}_\perp}\frac{1}{\sqrt{k_\perp^2+\kappa_n^2}} \ . \tag{63}$$

We apply the usual procedure of extracting the finite change in energy given by the Euler-Maclaurin formula (46). The contribution comes from the first-order derivative in equation (46). We get

$$\Delta U = -\frac{\pi\hbar q^2 S}{12nm}\left[\left(\frac{n_a\omega_{ca}^2}{\omega_{pa}^2}\right)^2 + \left(\frac{n_b\omega_{cb}^2}{\omega_{pb}^2}\right)^2\right]\frac{c^2-v^2}{v^3}\cdot\frac{1}{d} \ . \tag{64}$$

We can see that the electromagnetic field generates an attractive force ($v < c$) which goes like $-1/d^2$; it acts at long distance, beyond $d_2 \simeq a\sqrt{\frac{mc^2}{q^2/a}}$, where mc^2 is the rest energy of the mobile charges and q^2/a is their Coulomb energy (a being the mean separation distance between the charges). It is easy to see that $d_2 = c\sqrt{a^3/(q^2/m)} \simeq c/\omega_p \simeq d_1$, *i.e.* the $-1/d^4$-Casimir force and the $-1/d^2$-electromagnetic field force are competitive for moderate distances. According to equation (52), the effect of the temperature on this d^2-force could more visible, as the force acts at a longer distance.

If body a is a conductor ($\omega_{ca} = 0$), then its contribution to the change in the electromagnetic field energy goes like $-1/d^3$ (as for a Casimir force). The ratio of this force to the Casimir force is of the order of $(\hbar^2/ma^2)(q^2/a)/(\hbar\omega_p)^2$. The contribution of the dielectric ($\omega_{cb} \neq 0$) to the energy goes like $-1/d$; it is given by the corresponding term (ω_{cb}) in equation (64). Both contributions arise from the damped $\Omega_{T1,2}$-modes. Finally, for two conductors, the electromagnetic field energy contributes a Casimir force, given by $F = -\hbar cS/480\pi d^4$. It is much smaller (by factor $1/2\pi^3$) than the Casimir force $F = -\pi^2\hbar cS/240d^4$ arising from polarization.

9 Point-like particles

We specialize the above calculations to point-like, δ-particles, since such localized particles exhibit certain important particularities. Let us consider a displacement field given by

$$\mathbf{u}(\mathbf{r},t) = \mathbf{u}(t)a^3\delta(\mathbf{r}) \tag{65}$$

for a δ-particle of volume a^3 placed at $\mathbf{r} = 0$. According to equation (4), its Fourier transform is

$$\mathbf{u}(\mathbf{k},\omega) = \frac{a^3\sqrt{N}}{V}\mathbf{u}(\omega) \ , \tag{66}$$

where N is the number of mobile charges in the particle (density $n = N/a^3$). The charge and current densities are given by

$$\rho = -i\frac{nqa^3\sqrt{N}}{V}\mathbf{ku} \ , \ j = -i\omega\frac{nqa^3\sqrt{N}}{V}\mathbf{u} \ . \tag{67}$$

We get the electric and magnetic fields

$$\mathbf{E} = -4\pi F \frac{nqa^3\sqrt{N}}{V}\left[\lambda^2\mathbf{u} - \mathbf{k}(\mathbf{ku})\right] \ , \ \mathbf{H} = -4\pi\lambda F \frac{nqa^3\sqrt{N}}{V}\mathbf{k}\times\mathbf{u} \tag{68}$$

and the components

$$E_1 = -4\pi\frac{nqa^3\sqrt{N}}{V}u_1 \ , \ E_2 = -4\pi\lambda^2 F\frac{nqa^3\sqrt{N}}{V}u_2 \ . \tag{69}$$

The internal, polarization field $\mathbf{E}$ in the equation of motion (1) does not appear anymore in this case, in view of the lack of spatial extension of the δ-particle. We can use the external field $\mathbf{E}_0$ at the location $\mathbf{r} = 0$ of the particle, and the equation of motion reads (with $\gamma = 0$)

$$(\omega^2 - \omega_c^2)\mathbf{u}(\omega) = -\frac{q}{m}\mathbf{E}_0(\mathbf{r} = 0, \omega) \ . \tag{70}$$

We can see that there is only one polarizability, which coincides with the electric susceptibiity

$$\alpha(\omega) = \chi(\omega) = -\frac{1}{4\pi}\frac{\omega_p^2}{\omega^2 - \omega_c^2} \ . \tag{71}$$

In the absence of an external field equation (70) has harmonic-oscillator solutions

$$\mathbf{u}(\omega) = 2\pi\delta(\omega - \omega_c)\mathbf{u} \ , \tag{72}$$

where $\mathbf{u}$ are now constant amplitudes, corresponding to $\mathbf{u}(t) = \mathbf{u}e^{-i\omega_c t}$. Their dynamics is governed by the hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{1}{2}m\omega_c^2\mathbf{u}^2 \ , \tag{73}$$

where $\mathbf{p}$ is the momentum. The zero-point mean square fluctuations are $\hbar/2m\omega_c$ for each component.

Following the calculations which led to equation (23), the energy of the electromagnetic field can be written as

$$U = 4\pi\frac{(nqa^3)^2}{V}\sum_{\mathbf{k}}\left(|u_1|^2 + \frac{2\omega_c^2}{\omega_c^2 - c^2k^2}|u_2|^2\right) \ , \tag{74}$$

where, for zero-point fluctuations $|u_{1,2}|^2$ are replaced by $\hbar/2m\omega_c$. Similarly, we can have a representation in terms of the $\mathbf{u}$-coordinates for the Poynting vector, the "self-Lorentz force" and the charge-field "self-interaction", though, the latter two quantities are in fact meaningless, since the field acts only outside the particle.

We consider another point-like particle of volume b^3 placed at $\mathbf{r}_0$, with the displacement field $\mathbf{v}(t)b^3\delta(\mathbf{r} - \mathbf{r}_0)$, and follow the treatment given in the preceding sections. The coupled equations (32) and (33) become

$$\begin{aligned}(\omega^2 - \omega_{ca}^2)a^3u_1 &= \omega_{pb}^2 b^3 v_1 e^{-i\mathbf{kr_0}} \ , \\ (\omega^2 - \omega_{ca}^2)a^3u_2 &= \omega_{pb}^2 b^3\lambda^2 F v_2 e^{-i\mathbf{kr_0}}\end{aligned} \tag{75}$$

and

$$\begin{aligned}(\omega^2 - \omega_{cb}^2)b^3v_1 &= \omega_{pa}^2 a^3 u_1 e^{-i\mathbf{kr_0}} \ , \\ (\omega^2 - \omega_{cb}^2)b^3v_2 &= \omega_{pa}^2 a^3\lambda^2 F u_2 e^{-i\mathbf{kr_0}} \ .\end{aligned} \tag{76}$$

From the dispersion equations we get two longitudinal modes

$$\Omega_{L1,2}^2 = \frac{1}{2}\left[\omega_{ca}^2 + \omega_{cb}^2 \pm \sqrt{(\omega_{ca}^2 - \omega_{cb}^2)^2 + 4\omega_{pa}^2\omega_{pb}^2}\right] \tag{77}$$

corresponding to the oscillator coordinates

$$u_1(\omega) = 2\pi\delta(\omega - \Omega_{L1})u_1^{(1)} + 2\pi\delta(\omega - \Omega_{L2})u_1^{(2)} \tag{78}$$

and, in general, four branches of transverse modes Ω_{Ti}, $i = 1, ...4$. In the long-wavelength limit they go like

$$\Omega_{T1,2}^2 \simeq \Omega_{L1,2}^2 + const \cdot k^2 \ , \ \Omega_{T3,4} \simeq vk \ , \tag{79}$$

where the velocity v is given by

$$v = c\sqrt{\frac{\omega_{ca}\omega_{cb}}{\omega_{ca}\omega_{cb} + \omega_{pa}\omega_{pb}}} \ . \tag{80}$$

We can see that all these transverse modes depend on the magnitude of the wavevector $\mathbf{k}$, while our model, given by equation (65), for δ-particles does not allow for a $\mathbf{k}$-dependence of the displacement field. Therefore, all these "transverse" solutions of the dispersion equations are not acceptable, and we are left with the longitudinal modes only. We conclude that there is not a Casimir force (nor an electromagnetic field force) for two δ-particles (the longitudinal modes do not give such forces). This is an expected result, because the transverse modes are spatially-dispersive and the δ-particles, in view of their lack of spatial extension, cannot accomodate them. The same conclusion holds for the interaction of a δ-particle and a spatially-extended body.

Similarly, in the non-retarded regime, the dispersion equation (56) (with $\omega_{La,b} \to \omega_{ca,b}$) gives damped modes which cannot be sustained by the δ-localized particles. Therefore, we conclude that our approach gives not a van der Waals-London force for two δ-particles.

The δ-particles offer the opportunity of a direct-space approach. Indeed, the displacement field given by equation (65) gives a current density $\mathbf{j}(\mathbf{r},\omega) = -i\omega nqa^3\mathbf{u}(\omega)\delta(\mathbf{r})$ (and a charge density $\rho(\mathbf{r},\omega) = -nqa^3[\mathbf{u}(\omega)grad]\delta(\mathbf{r})$). By equations (7) we get immediately the vector potential

$$\mathbf{A} = -i\lambda nqa^3\mathbf{u}f(r) \ , \tag{81}$$

where $f(r) = e^{i\lambda r}/r$ is the (outgoing) spherical wave. From the Lorenz gauge we get the scalar potential

$$\Phi = -nqa^3\frac{\mathbf{ur}}{r}f^{'}(r) \ , \tag{82}$$

so we have straightforwardly the electric field

$$\mathbf{E} = nqa^3\left[\lambda^2 f\mathbf{u} + \frac{f^{'}}{r}\mathbf{u} - \frac{(\mathbf{ru})f^{'}}{r^3}\mathbf{r} + \frac{(\mathbf{ru})f^{''}}{r^2}\mathbf{r}\right] \ . \tag{83}$$

We estimate this field, produced by particle a, at the location $\mathbf{r} = \mathbf{r}_0 = (0, 0, d)$ of the particle b. It is convenient to use the longitudinal projection $\|$ on the z-direction and the transverse projection $\perp$ on the (x, y)-plane. We get

$$E_{a\|} = n_a qa^3(\lambda^2 f + f^{''})u_\| \ , \ \mathbf{E}_{a\perp} = n_a qa^3(\lambda^2 f + \frac{1}{d}f^{'})\mathbf{u}_\perp \ , \tag{84}$$

where $f = e^{i\lambda d}/d$. The equations of motion for the displacement field $\mathbf{v}(t)b^3\delta(\mathbf{r}-\mathbf{r}_0)$ of the particle b reads

$$\begin{aligned}(\omega^2-\omega_{cb}^2)v_\parallel &= -\tfrac{\omega_{pa}^2}{4\pi}a^3(\lambda^2 f+f'')u_\parallel \ , \\ (\omega^2-\omega_{cb}^2)v_\perp &= -\tfrac{\omega_{pa}^2}{4\pi}a^3(\lambda^2 f+\tfrac{1}{d}f')u_\perp \ .\end{aligned} \tag{85}$$

Similarly, we compute the field produced by particle b at the location $\mathbf{r}=0$ of the particle a (in equation (83) $\mathbf{r}$ is replaced by $\mathbf{r}-\mathbf{r}_0$), and get the equations of motion

$$\begin{aligned}(\omega^2-\omega_{ca}^2)u_\parallel &= -\tfrac{\omega_{pb}^2}{4\pi}b^3(\lambda^2 f+f'')v_\parallel \ , \\ (\omega^2-\omega_{ca}^2)u_\perp &= -\tfrac{\omega_{pb}^2}{4\pi}b^3(\lambda^2 f+\tfrac{1}{d}f')v_\perp \ .\end{aligned} \tag{86}$$

From equations (85) and (86) we get the dispersion equations

$$\begin{aligned}(\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) &= \tfrac{\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}a^3b^3(\lambda^2 f+f'')^2 \ , \\ (\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) &= \tfrac{\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}a^3b^3(\lambda^2 f+\tfrac{1}{d}f')^2 \ .\end{aligned} \tag{87}$$

It is easy to see that the two equations (87) ae not compatible with one another, so we set either $u_\parallel = v_\parallel = 0$, or $u_\perp = v_\perp = 0$. Let us take $u_\perp = v_\perp = 0$, so we have the first equation (87), which can be cast in the form

$$(\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) = 4\frac{\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}a^3b^3\frac{e^{2i(\lambda d+\varphi)}}{d^6} \ , \tag{88}$$

where $\tan\varphi = \lambda d$. This equation implies $\lambda d+\varphi = \pi n$, where n is any integer, and

$$(\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) = 4\frac{\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}\frac{a^3b^3}{d^6} \ . \tag{89}$$

It is easy to see that these two conditions cannot be satisfied simultaneously. The same result holds for the other dispersion equation, corresponding to $u_\parallel = v_\parallel = 0$. We conclude that the dispersion equations have not solutions, and, therefore, the energy is not changed by the presence of the finite distance d, *i.e.* there is not a Casimir force.

In the non-retarded regime $\lambda = 0$ and the dispersion equations (87) become

$$(\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) = \frac{4\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}\frac{a^3b^3}{d^6} \tag{90}$$

for the longitudinal components and

$$(\omega^2-\omega_{ca}^2)(\omega^2-\omega_{cb}^2) = \frac{\omega_{pa}^2\omega_{pb}^2}{(4\pi)^2}\frac{a^3b^3}{d^6} \tag{91}$$

for the transverse components. The two equations (90) and (91) differ by a factor 4. For realistic situations we may take $\omega_{ca} \simeq \omega_{cb} = \omega_c$, as for identical particles; we take also $\omega_{pa} = \omega_{pb} = \omega_p$. We may consider the *rhs* term of these equations as a small perturbation. The first equation (90) gives

$$\omega = \omega_c\left[1 \pm \frac{\omega_p^2}{4\pi\omega_c^2}\left(\frac{a^3b^3}{d^6}\right)^{1/2} - \frac{\omega_p^4}{32\pi\omega_c^4}\frac{a^3b^3}{d^6} + ...\right] \tag{92}$$

and a change in the zero-point energy

$$\Delta E = -\frac{\hbar\omega_p^4}{32\pi^2\omega_c^3} \cdot \frac{a^3 b^3}{d^6} \,. \tag{93}$$

This energy gives a $-1/d^7$-force, which is the classical van der Waals-London force acting between two point-like particles. A similar result is obtined from the second dispersion equation, corresponding to the transverse modes. Statistically, the latter force appears with a weight factor 2 in comparison with the former, due to the two transverse polarizations of the displacement field. It is easy to see that the average energy contains a factor 1/2 which multplies equation (93).

The difference between the result obtained here regarding the van der Waals-London force for δ-particles and the (null) result obtained by using the Fourier-transform approach is an extreme instance of the inadequacy of the Fourier-transform approach to problems concerning the matter polarization and the interaction of electromagnetically-coupled bodies. In general, the Fourier-transform approach to such problems is valid for bodies of a sufficient spatial extension, such that the surface effects can be neglected in comparison with the bulk contribution. However, for finite-size bodies, especially in the long-wavelength limit, this requirement is never achieved, due to the long-range character of the Coulomb interaction. Therefore, we expect deviations from our results presented here, whenever finite-size bodies are involved.

10 Concluding remarks

The matter polarization can be represented by a displacement field of the mobile charges, within the well-known Lorentz-Drude model. By Maxwell's equations, the electromagnetic field generated by the polarization acquires a corresponding mechanical representation in terms of the displacement field. The collective motion of the polarization (reflected in the polarization electromagnetic field) is an example of emergent dynamics. In general, it consists of three infinite sets of harmonic oscillators, corresponding to longitudinal (plasmons, one set) and transverse (polaritons, two sets), which can be quantized. The electromagnetic field energy, Poynting vector, Lorentz force and charge-field interaction energy are represented in terms of these three types of eigenmodes. It is shown that the polarizable matter stores a large amount of electromagnetic energy, arising from the zero-point (vacuum) fluctuations of the displacement (polarization) field.

Two electromagnetically-coupled bodies are treated within this framework, and their polarization eigenmodes are identified. The Casimir and van der Waals-London forces are derived, as arising from the zero-point fluctuations of the displacement (polarization) field. These forces incorporate the nature of the two bodies (by their individual longitudinal and transverse frequencies). The electromagnetic field (generated by the polarization fluctuations) brings its own contribution to the forces acting bewteen two polarizable bodies. If a dielectric, at least, is present, the electromagnetic-field force goes like $\sim -1/d^2$, where d is the separation distance between the two bodies (it is a long-range force). If a conductor, at least, is present, the electromagnetic field adds its own contribution to the Casimir force ($\sim -1/d^4$). The magnitude and their range of action have been estimated here for all these forces.

The Fourier decomposition employed throughout as a general tool does not allow for including surface (or shape) effects. In general, the Fourier decomposition can be replaced by expansions in eigenmodes of the laplacian for specific, finite-size bodies (with outgoing-waves boundary condition at infinity), preserving the direct-space coordinae perpendicular to the surface. Such expansions

may change, in general, the density of modes, which enters summations over states. In the long-wavelength limit, which gives usually the relevant contributions, this change is not significant; but the sharpness of the surfaces may bring important effects, through the depolarizing fields. Such effects are more evident for the van der Waals–London forces. The particular case of point-like, δ-particles was analyzed in this context, as a limiting case of inadequacy of the Fourier-decomposition approach.

Acknowledgments. The author is indebted to the members of the Laboraratory of Theoretical Physics at Magurele-Bucharest for many enlightening discussions, and to the members of the Institute for Atomic Physics Seminar, Magurele-Bucharest, for a careful analysis of this work. The support of the Institute for Physics and Nuclear Engineering, Magurele-Bucharest, is also gratefully acknowledged.

References

[1] H. B. G. Casimir and D. Polder, "The influence of retardation on the London-van der Waals forces," Phys. Rev. **73** 360-372 (1948).

[2] H. Casimir, "On the atttraction between two perfectly conducting plates," Proc. Kon. Ned. Ak. Wet. **51** 793 (1948).

[3] R. Eisenschitz and F. London, "Uber das Verhaltnis der van der Waalsschen Krafte zu den homoopolaren Bindungskraften", Z. Physik **60** 491-527 (1930).

[4] F. London, "Zur Theorie und Systematik der Molekularkrafte", Z. Physik **63** 245-279 (1930).

[5] F. London, "The general theory of molecular forces", Trans. Faraday Soc. **33** 8-26 (1937).

[6] E. Lifshitz, ZhETF **29** 94-105 (1956) (Sov. Phys. JETP **2** 73-83 (1956)).

[7] I. E. Dzyaloshinskii, E. M. Lifshitz and L. P. Pitaevskii, "The general theory of van der Waals forces," Adv. Phys. **10** 165-209 (1961).

[8] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 5 (*Statistical Physics*), part. 2, Butterworth-Heinemann, Oxford (2003).

[9] J. Schwinger, "Casimir effect in source theory", Lett. Math. Phys. **1** 43-47 (1975).

[10] J. Schwinger, L. L. deRaad, Jr., and K. A. Milton, "Casimir effect in dielectrics", Ann. Phys. **115** 1-23 (1978).

[11] N. G. Van Kampen, B. R. A. Nijboer and K. Schram, "On the macroscopic theory of van der Waals forces," Phys. Lett. **A26** 307-308 (1968).

[12] T. H. Boyer, "Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle", Phys. Rev. **174** 1764-1776 (1968).

[13] T. H. Boyer, "Recalculations of long-range van der Waals potentials", Phys. Rev. **180** 19-24 (1969).

[14] E. Gerlach, "Equivalence of van der Waals forces between solids and the surface-plasmon interaction," Phys. Rev. **B4** 393-396 (1971).

[15] K. Schram, "On the macroscopic theory of retarded van der Waals forces," Phys. Lett. **A43** 282-284 (1973).

[16] J. Heinrichs, "Theory of van der Waals interaction between metal surfaces," Phys. Rev. **B11** 3625-3636 (1975).

[17] P. W. Milloni, *The Quantum Vacuum* (Academic Press, San Diego, 1994).

[18] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon, Oxford, 1997).

[19] S. K. Lamoreaux, "Demonstration of the Casimir force in the 0.6 to 6μm range," Phys. Rev. Lett. **78** 5-8 (1997).

[20] L. H. Ford, "Casimir sphere between a dielectric sphere and a wall: a model for amplification of vacuum fluctuations, Phys. Rev. **A58** 4279-4286 (1998).

[21] U. Mohideen and A. Roy, "Precision measurement of the Casimir force from 0.1 to 0.9μm", Phys. Rev. Lett. **81** 4549-4552 (1998).

[22] S. K. Lamoreaux, "Calculation of the Casimir force between imperfectly conductiong plates," Phys. Rev. **A59** R3149-R3153 (1999).

[23] A. Lambrecht and S. Reynard, "Comment on "Demonstration of the Casimir force in the 0.6 to 6μm range"," Phys. Rev. Lett. **84** 5672-5672 (2000).

[24] A. Lambrecht and S. Reynaud, "Casimir force between metallic mirrors", Eur. Phys. J. **D8** 309-318 (2000).

[25] M. Bordag, U. Mohideen and V. M. Mostepanenko, "New developments in the Casimir effect," Phys. Reps. **353** 1-205 (2001).

[26] K. A. Milton, *The Casimir Effect* (World Scientific, Singapore, 2001).

[27] G. Bressi, G. Carugno, R. Onofrio and G. Ruoso, "Measurement of the Casimir force between parallel metallic surfaces", Phys. Rev. Lett. **88** 041804 (2002) (1-4).

[28] C. Genet, A. Lambrecht and S. Reynaud, "Casimir force and the quantum theory of lossy optical cavities," Phys. Rev. **A67** 043811 (1-18) (2003).

[29] K. A. Milton, "The Casimir effect: recent controversies and progress", J. Phys. A: Math. Gen. **37** R209-R277 (2004).

[30] F. Chen, U. Mohideen, G. L. Klimchitskaya and V. M. Mostepanenko, "Investigation of the Casimir force between metal and semiconductor test bodies," Phys. Rev. **A72** 020101(R1-4) (2005).

[31] S. K. Lamoreaux, "The Casimir force: background, experiments and applications," Reps. Progr. Phys. **65** 201-236 (2005).

[32] F. Intravaia, "Effet Casimir et interaction entre plasmons de surface," These de Doctorat de l'Universite Paris VI, 1-177, June 2005.

[33] J. M. Obrecht, R. J. Wild, M. Antezza, L. P. Pitaevskii, S. Stringari and E. A. Cornell, "Measurement of the temperature dependence of the Casimir-Polder force," Phys. Rev. Lett. **98** 063201 (1-4) (2007).

[34] F. Intravaia, C. Henkel and A. Lambrecht, "Role of surface plasmons in the Casimir effect," Phys. Rev. **A76** 033820 (1-11) (2007).

[35] T. Emig, N. Graham, R. L. Jaffe and M. Kardar, "Casimir forces between arbitrary compact objects", Phys. Rev. Lett. **99** 170403 (2007) (1-4).

[36] M. Kruger, T. Emig, G. Bimonte and M. Kardar, "Non-equilibrium Casimir forces:: spheres and sphere-plate", Eur. Phys. Lett. **95** 21002 (2011) (1-6).

[37] M. Apostol and G. Vaman, "Electromagnetic eigenmodes in matter. van der Waals-London and Casimir forces", Pogr. Electrom. Res. PIER **B19** 115-131 (2010).

[38] M. Apostol and G. Vaman, "Attraction force between a polarizable point-like particle and a semi-infinite solid", Roum. J. Phys. **55** 764-771 (2010).

[39] P. Drude, "Zur Elektronentheorie der Metalle", Ann. Phys. **306** 566-613 (1900).

[40] P. Drude, "Zur Elektronentheorie der Metalle, 2. Teile. Galvanomagnetische und thermomagnetische Effecte, **308** 369-402 (1900).

[41] H. A. Lorentz, *The Theory of Electrons*, Teubner, Leipzig (1916).

[42] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Saunders College,Philadelphia (1976).

[43] G. D. Mahan, *Many-Particle Physics*, Plenum, NY (1980).

[44] C. Kittel, *Introduction to Solid State Physics*, Wiley, NJ (2005).

[45] M. Apostol and G. Vaman, "Electromagnetic field interacting with a semi-infinite plasma", J. Opt. Soc. Am. **A26** 1747-1753 (2009).

[46] M. Apostol and G. Vaman, "Plasmons and difraction of an electromagnetic plane wave by a metallic sphere", Progr. Electrom. Res. PIER **98** 97-118 (2009).

[47] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Mathematical Library, Cambridge (2004), p. 128.

[48] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, eds. A. Jeffrey and D. Zwillinger, 6th edition, Academic Press, San Diego (2000), pp. 714-715, 6.677.1,2.