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A note on the time evolution of the waves

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Abstract

The temporal evolution of delta-pulses is studied by means of the wave equation in one, two and three dimensions. It is shown that waves with a delta-pulse initial shape are different from waves generated by intial delta-pulse sources (Green functions). Special attention is given to the two-dimensional case, where the regularization procedure of the intervening improper integrals should be adopted in agreement with cylindrical waves. The necessity is emphasized of taking into account the small thickness of wires and slabs for wave propagation in purely one and two-dimensional spaces (line or membrane).

The equation of the free (scalar) waves in anisotropic bodies reads

$$\frac{1}{v^2}\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2}-D_{(2)}u(\mathbf{r},t)=0 \ , \tag{1}$$

where v ($v>0$) is a velocity parameter and $D_{(2)}$ is a certain second-order differential operator. We use the Fourier transform

$$\begin{gathered}u(\mathbf{r},t)=\tfrac{1}{(2\pi)^d}\int d\mathbf{k}e^{i\mathbf{kr}}u(\mathbf{k},t) \ , \ u(\mathbf{k},t)=\int d\mathbf{r}e^{-i\mathbf{kr}}u(\mathbf{r},t) \ , \\ \delta(\mathbf{r})=\tfrac{1}{(2\pi)^d}\int d\mathbf{k}e^{i\mathbf{kr}} \ , \ \delta(\mathbf{k})=\tfrac{1}{(2\pi)^d}\int d\mathbf{r}e^{i\mathbf{kr}} \ ,\end{gathered} \tag{2}$$

where d ($d=1,2,3$) is the space dimension; equation (1) becomes

$$\frac{d^2u(\mathbf{k},t)}{dt^2}+v^2F(\mathbf{k})u(\mathbf{k},t)=0 \ , \tag{3}$$

whose solutions are

$$u(\mathbf{k},t)=e^{\pm iv\sqrt{F(\mathbf{k})}t}u(\mathbf{k},0) \ ; \tag{4}$$

their linear combination involves two constants which are determined from the function and its temporal derivative at the initial moment of time (Cauchy's initial value problem); $F(\mathbf{k})$ is given by

$$D_{(2)}u(\mathbf{r},t)=\frac{1}{(2\pi)^d}\int d\mathbf{k}D_{(2)}e^{i\mathbf{kr}}u(\mathbf{k},t)=-\frac{1}{(2\pi)^d}\int d\mathbf{k}F(\mathbf{k})e^{i\mathbf{kr}}u(\mathbf{k},t) \ . \tag{5}$$

Equation (3) can also be written as

$$\left(\frac{d}{dt}+iv\sqrt{F}\right)\left(\frac{d}{dt}-iv\sqrt{F}\right)u=0 \ , \tag{6}$$

which justifies either

$$\frac{du}{dt}=-iv\sqrt{F}u\ ,\ u(\mathbf{k},t)=e^{-iv\sqrt{F(\mathbf{k})}t}u(\mathbf{k},0)\ , \tag{7}$$

or

$$\frac{du}{dt}=iv\sqrt{F}u\ ,\ u(\mathbf{k},t)=e^{iv\sqrt{F(\mathbf{k})}t}u(\mathbf{k},0)\ ; \tag{8}$$

it is easy to see that $\pm\sqrt{F(\mathbf{k})}$ are two frequencies denoted usually by $\pm\omega$; first we choose equation (7), which gives a wave propagating forward,

$$u(\mathbf{r},t)=\frac{1}{(2\pi)^d}\int d\mathbf{k}e^{i\mathbf{kr}-iv\sqrt{F(\mathbf{k})}t}u(\mathbf{k},0)\ ; \tag{9}$$

in addition, $F(\mathbf{k})$ is a positive-defined quadratic form in the components of the wavevector $\mathbf{k}$, $F(\mathbf{k})=\sum_{ij}f_{ij}k_ik_j$, which can be brought to the principal axes. Indeed, we define $k_i=a_{ij}k_j^{'}$, such as $F(\mathbf{k}^{'})=a_{il}f_{ij}a_{jm}k_l^{'}k_m^{'}=f_lk_l^{'2}$, where a_{ij} is the (orthogonal) matrix that diagonalizes the matrix f_{ij} (and summation over repeated labels is included); we change the wavevector to $k_i^{''}=\sqrt{f_i}k_i^{'}$ and the spatial coordinates to $r_i^{''}=r_i^{'}/\sqrt{f_i}$, $r_i^{'}=a_{ji}r_j$ and get $\sqrt{F}=\sqrt{k^{''2}}=k^{''}$ and

$$U(\mathbf{r},t)=\frac{1}{(2\pi)^d}\int d\mathbf{k}e^{i\mathbf{kr}-ivkt}U(\mathbf{k},0)\ , \tag{10}$$

where we have changed the notations $\mathbf{k}^{''}\rightarrow\mathbf{k}$ and $\mathbf{r}^{''}\rightarrow\mathbf{r}$ and denoted by U the function u of the new variables. The procedure described here is well-known as the transition to the ellipsoid of the principal axes of a positive-defined quadratic form. In the new coordinates the orginal equation reads

$$\frac{d^2U(\mathbf{k},t)}{dt^2}+v^2k^2U(\mathbf{k},t)=0\ ,\ \frac{1}{v^2}\frac{\partial^2U(\mathbf{r},t)}{\partial t^2}-\Delta U(\mathbf{r},t)=0\ . \tag{11}$$

In principle, equation (10) solves our problem: using it, we can compute the wave at the moment of time t by knowing the wave at the initial moment of time $t=0$. Usually, we are interested in the time evolution of a wave which initially is a delta-type function localized in the volume V_d, *i.e.* $u(\mathbf{r},0)=V_d\delta(\mathbf{r})$, which, by using equations (2), means $u(\mathbf{k},0)=U(\mathbf{k},0)=V_d$ (delta-pulse).

In three dimensions we have from equation (10)

$$U(\mathbf{r},t)=\frac{V}{(2\pi)^3}\int d\mathbf{k}e^{i\mathbf{kr}-ivkt-\alpha k}=\frac{V}{4\pi^2}\int dk\cdot k^2e^{-ivkt-\alpha k}\int_{-1}^{1}due^{ikru}\ \ , \tag{12}$$

where we have introduced the convergence factor $e^{-\alpha k}$, $\alpha\rightarrow 0^+$. Effecting the integral we get

$$U(\mathbf{r},t)=\frac{V}{4\pi^2 ir}\int dk\cdot k\left[e^{ik(r-vt+i\alpha)}-e^{-ik(r+vt-i\alpha)}\right]\ ; \tag{13}$$

since we are interested in positive moments of time (outgoing wave), we may left aside the term with $r+vt$ (which propagates backwards in time) and write

$$\begin{gathered}U(\mathbf{r},t)=\frac{V}{4\pi^2ir}\int dk\cdot ke^{ik(r-vt+i\alpha)}=\frac{V}{4\pi^2ir}\frac{\partial}{\partial(iR)}\int dke^{ikR}=\\=\frac{iV}{4\pi^2r}\frac{\partial}{\partial r}\frac{1}{r-vt+i\alpha}=\frac{iV}{4\pi^2r}\frac{\partial}{\partial r}\left[P\frac{1}{r-vt}-i\pi\delta(r-vt)\right]\ ;\end{gathered} \tag{14}$$

here we have used

$$\frac{1}{x+i\alpha}\rightarrow P\frac{1}{x}-i\pi\delta(x)\ ,\ \alpha\rightarrow 0^+\ . \tag{15}$$

We retain only the real part of this result, which is equivalent to adding, with equal weight, the conrtibution of the other frequency $-vk$; it corresponds to an initial delta-pulse with zero velocity:

$$U(\mathbf{r},t)=\frac{V}{4\pi r}\delta^{'}(r-vt)\ ; \tag{16}$$

we can see that the original pulse $V\delta(\mathbf{r})$ propagates with velocity v as a spherical surface of radius $r=vt$, while its magnitude decreases as $1/r$. It is worth noting that on this spherical surface the wave is $\delta^{'}$, not δ. The solution given by equation (16) can also be written as

$$U(\mathbf{r},t)=\frac{V}{4\pi r}\delta^{'}(r-vt)=\frac{V}{4\pi r^2}\delta(r-vt)\ \ , \tag{17}$$

which is the temporal evolution of the initial wave $U(\mathbf{r},0)=V\delta(\mathbf{r})=\frac{V}{4\pi r^2}\delta(r)$. In general, if we limit ourselevs only to the the leading terms in the wave equation

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}-\Delta U\simeq\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial U}{\partial r})=0 \tag{18}$$

for $r\rightarrow\infty$, the solution is $U=f(r-vt)/r$ (the outgoing wave); equation (18) indicates the conservation of the energy $E=\frac{1}{2}[\dot{U}^2+v^2(gradU)^2]\simeq V^2f^{'2}/r^2$.

In one dimension we have from equation (10)

$$\begin{aligned}U(x,t)&=\tfrac{l}{2\pi}\int_{-\infty}^{+\infty}dke^{ikx-iv|k|t-\alpha|k|}=\\&=\tfrac{il}{2\pi}\left(\tfrac{1}{x-vt+i\alpha}-\tfrac{1}{x+vt-i\alpha}\right)\rightarrow\tfrac{l}{2}\delta(x-vt)+\tfrac{l}{2}\delta(x+vt)\ \ ,\end{aligned} \tag{19}$$

where l is the initial localization length of the pulse. It is easy to see that the term corresponding to the frequency $-v\,|k|$ gives the same contribution, so that the final result is given by equation (19). The original pulse $l\delta(x)$ splits into two pulses each of magnitude $l/2$ wich propagate with velocities v and, respectively, $-v$ in two opposite directions. The wave equation

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2}-\frac{\partial^2 U}{\partial x^2}=0 \tag{20}$$

in one dimension has the general solution $U=f(x-vt)+g(x+vt)$, where f and g are arbitrary functions.

We note that the above meaningful results are obtained only by giving a sense to delta-type functions through the regularization procedure of the α-cutoff. In two dimensions this direct procedure is not possible anymore.

Indeed, in two dimensions we have

$$\begin{aligned}U(\mathbf{r},t)&=\tfrac{S}{(2\pi)^2}\int d\mathbf{k}e^{i\mathbf{kr}-ivkt-\alpha k}=\tfrac{S}{(2\pi)^2}\int d\varphi\int dkke^{ik(r\cos\varphi-vt+i\alpha)}=\\&=-\tfrac{S}{4\pi^2}\int d\varphi\tfrac{1}{(r\cos\varphi-vt)^2}=\tfrac{S}{4\pi^2 r}\tfrac{\partial}{\partial(vt)}\int d\varphi\tfrac{1}{vt/r-\cos\varphi}\ \ ,\end{aligned} \tag{21}$$

where S is the area over which the pulse is originaly localized. The cutoff α is superfluous here, since we know to compute the integral

$$f(x)=\int_{-\pi}^{\pi}d\varphi\frac{1}{x-\cos\varphi}\ ; \tag{22}$$

it can be written as

$$f(x) = 2i \oint dz \frac{1}{z^2 - 2xz + 1} \ , \tag{23}$$

where the integration is carried out over the circle of radius unity. The integrand can be decomposed as $(z - z_1)(z - z_2)$, where $z_{1,2} = x \pm \sqrt{x^2 - 1}$. For $x > 1$ we set $x = \cosh u$, $z_1 = e^u > 1$ and $z_2 = e^{-u} < 1$; the integral in equation (23) becomes

$$f(x) = \frac{2\pi}{\sqrt{x^2 - 1}} \ , \ x > 1; \tag{24}$$

for $x < -1$ we set $x = -\cosh u$ and the integral changes sign ($f(x) = 2\pi sgn(x)/(x^2 - 1)^{1/2}$ for $|x| > 1$). Therefore, we have

$$U(\mathbf{r}, t) = \frac{S}{2\pi} \frac{r}{(r^2 - v^2t^2)^{3/2}} \ , \ r > vt \ . \tag{25}$$

For $|x| < 1$ we set $x = \cos\alpha$ and $z_{1,2} = e^{\pm i\alpha}$; we can see that the integral in equation (23) has two poles on the circle of radius unity. We can give a sense to this integral by pushing the pole $z_1 = e^{i\alpha}$ outside the circle and $z_2 = e^{-i\alpha}$ inside the circle for$x > 0$ and viceversa for $-1 < x < 0$; we get

$$f(x) = -\frac{2\pi i}{\sqrt{1 - x^2}} sgn(x) \ , \ |x| < 1 \ . \tag{26}$$

We can see that $U(\mathbf{r}, t)$ is vanishing for $r < vt$ and has non-zero values outside the circle $r = vt$. This result is not acceptable, because it is not causal. In addition, we can see that $U(\mathbf{r}, 0)$ as given by equation (25) is not the original pulse $S\delta(\mathbf{r})$. This is caused by the manipulation of singular (improper) integrals and by the regularization procedure employed here, which is not acceptable.

Let us introduce the Bessel function J_0. The laplacian in two dimensions applied to a plane wave reads

$$\Delta e^{i\mathbf{kr}} = \left[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{dr}) + \frac{\partial^2}{r^2 \partial\varphi^2} \right] e^{i\mathbf{kr}} = -k^2 e^{i\mathbf{kr}} \ ; \tag{27}$$

we define the Bessel function (of zeroth degree and the first kind)

$$J_0(kr) = \frac{1}{2\pi} \int d\varphi e^{i\mathbf{kr}} = \frac{1}{2\pi} \int d\varphi e^{ikr\cos\varphi} \tag{28}$$

and get

$$\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}) J_0(kr) = -k^2 J_0(kr) \tag{29}$$

or

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + 1 \right) J_0(z) = 0 \ ; \tag{30}$$

this is the Bessel equation for $J_0(z)$ ($z = kr$); we get also

$$\left(\frac{d^2}{dz^2} + \frac{1}{4z^2} + 1 \right) z^{1/2} J_0 = 0 \tag{31}$$

and the asymptotic behaviour

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \pi/4) \ , \ z \to \infty \ . \tag{32}$$

We can see that

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^2}\int d\mathbf{k}e^{i\mathbf{kr}} = \frac{1}{2\pi}\int dkkJ_0(kr) \ , \tag{33}$$

which holds as a convention.

We can use this asymptotic behaviour of the Bessel function J_0 in equation (21); making use of the integral

$$\int_0^\infty dxe^{-\lambda x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \ . \tag{34}$$

we get

$$U(\mathbf{r}, t) \simeq \frac{S}{4\pi\sqrt{2r}}\frac{1}{(vt-r)^{3/2}} \ , \tag{35}$$

which coincides with equation (25) up to the sign of $r - vt$, provided we write the denominator in this latter equation as $(r^2 - v^2t^2)^{3/2} \simeq 2\sqrt{2}r^{3/2}$. Unfortunately, equation (35) is valid for large r, which is not acceptable.

The correct (acceptable) procedure of regularization of the waves in two dimensions is based upon the observation that the wave equation in two dimensions is obtained from the wave equation in three dimensions by an integration over one coordinate, say z; we write the initial condition $V\delta(\mathbf{R})$ as $Sdz\delta(\mathbf{R}) = Sdz\delta(\mathbf{r})\delta(z)$, where we have introduced the notation $\mathbf{R} = (\mathbf{r}, z)$. We have cylindrical waves in this case, and an initial delta-pulse deployed over a line along the z-coordinate, but for the wave equation there is no difference between these waves and purely two-dimensional waves. Making use of equations (12) and (17) we get

$$U(\mathbf{r}, t) = \frac{S}{(2\pi)^3}\int dz\int d\mathbf{K}e^{i\mathbf{KR}-ivKt-\alpha K} = \frac{S}{(2\pi)^2}\int d\mathbf{k}e^{i\mathbf{kr}-ivkt-\alpha K} \ , \ \mathbf{K} = (\mathbf{k}, k_z) \ , \tag{36}$$

which is equation (21) in two dimensions, and

$$\begin{gathered} U(\mathbf{r}, t) = \frac{S}{4\pi}\int dz\frac{\delta'(R-vt)}{R} = \frac{S}{2\pi}\int_r dR\frac{\delta'(R-vt)}{\sqrt{R^2-r^2}} = \\ = \frac{S}{2\pi}\frac{vt}{(v^2t^2-r^2)^{3/2}} \ , \ vt > r \end{gathered} \tag{37}$$

and 0 otherwise. We can see that these cylindrical pulses propagate with a sharp front on the circle $r = vt$, without warning for $vt < r$ and with a wake behind for $vt > r$. In contrast with one or three dimensions, the two dimensional (cylindrical) waves have a tail (a rear, a wake). Similarly, we can obtain the one-dimensional waves given by equation (19) by integrating over a surface the three-dimensional waves

$$U(x, t) = \frac{l}{(2\pi)^3}\int dy\int dz\int d\mathbf{K}e^{i\mathbf{KR}-ivKt-\alpha K} = \frac{l}{2\pi}\int dke^{ikx-iv|k|t-\alpha|k|} \ , \ \mathbf{K} = (k, k_y, k_z) \ , \tag{38}$$

where $R = (x, y, z)$; here, the direct integration in equation (38) leads immediately to the result given by equation (19) for one-dimensional wave, rather than integrating the three-dimensional solution $\frac{V}{4\pi r}\delta'(r - vt)$ given by equation (17). The result corresponds to pulses concentrated at the initial moment over a surface and propagating in one dimension along the direction perpendicular to the original surface, rather than pulses restricted to a line; but, the wave equation does not make a difference between these two situations. For realistic situations concerning waves confined to a plane surface or to a line we should include the (small) finite thickness of the surface (the slab) or the line (thread, wire, rod, etc), when the corresponding wave equation in three dimensions becomes the wave equation in two or one dimensions with an additional term of the form $const \cdot U$ (Klein-Gordon equation, as for waveguides).

It is worth noting that in connection with wave delta-pulses a different problem is also well known, namely the wave equation (in three dimensions)

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = \delta(t)\delta(\mathbf{R}) \ , \tag{39}$$

which assumes a delta-pulse source localized at the initial moment and at the origin of the space; this is different from an initial delta-pulse wave. The solution of equation (39) is of the form of a Fourier transform

$$U(\mathbf{R},t) = \frac{1}{2\pi}\int d\omega U(\mathbf{R},\omega)e^{-i\omega t} \ , \tag{40}$$

which leads to the equation

$$\Delta U(\mathbf{R},\omega) + \frac{\omega^2}{v^2}U(\mathbf{R},\omega) = -\delta(\mathbf{R}) \ ; \tag{41}$$

we can check easily

$$(\Delta+\lambda^2)\frac{e^{i\lambda R}}{R} = 0 \ , \ \ R \neq 0 \tag{42}$$

and

$$\int d\mathbf{R}\frac{e^{i\lambda R}}{R} = \int d\mathbf{S} grad\frac{e^{i\lambda R}}{R} = -4\pi \tag{43}$$

for the infinitesimal volume; therefore,

$$U(\mathbf{R},\omega) = \frac{1}{4\pi}\frac{e^{i\frac{\omega}{v}R}}{R} \tag{44}$$

which is the Green function of the Helmholtz equation (41) (spherical wave); and $1/4\pi R$ is the Green function of the Laplace equation $\lambda = \omega/v = 0$). It follows

$$U(\mathbf{R},t) = \frac{1}{2\pi}\int d\omega U(\mathbf{R},\omega)e^{-i\omega t} = \frac{1}{2\pi}\int d\omega\frac{e^{i\frac{\omega}{v}R}}{4\pi R}e^{-i\omega t} = \frac{\delta(R/v-t)}{4\pi R} \ , \tag{45}$$

which is the Green function of the full wave equation in three dimesnions; we can see that it describe a delta-pulse propagating as a spherical surface with velocity v and with an amplitude decreasing with the distance like $1/R$; it differs from the pulse given by equation (17). As it is well known, the usefulnesss of the Green functions resides in that it gives the (particular) solution

$$U(\mathbf{R},t) = \int d\mathbf{R}' dt' f(\mathbf{R}',t')\frac{\delta(\left|\mathbf{R}-\mathbf{R}'\right|/v-t)}{4\pi\left|\mathbf{R}-\mathbf{R}'\right|} \tag{46}$$

of the wave equation

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = f(\mathbf{R},t) \ . \tag{47}$$

Denoting $\mathbf{R} = (\mathbf{r}, z)$ and integrating over z in equation (39) we get the wave equation in two dimensions

$$\frac{1}{v^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = \delta(t)\delta(\mathbf{r}) \tag{48}$$

with a delta-source; its Green function is obtained immediately from equation (45) as

$$U(\mathbf{r},t) = \int dz\frac{\delta(R/v-t)}{4\pi R} = \frac{1}{2\pi}\int_r^\infty dR\frac{\delta(R/v-t)}{\sqrt{R^2-r^2}} = \frac{v}{2\pi}\frac{1}{\sqrt{v^2t^2-r^2}} \ , \ r < vt \tag{49}$$

and zero otherwise; we note again that this pulse differs from the pulse given by equation (37). It is interesting to compute the temporal Fourier transform of the Green function given by equation (49),

$$\begin{gathered}U(\mathbf{r},\omega)=\frac{v}{2\pi}\int_{r/v}^{\infty}dt\frac{1}{\sqrt{v^2t^2-r^2}}e^{i\omega t}=\\=\frac{1}{2\pi}\int_r dR\frac{1}{\sqrt{R^2-r^2}}e^{ikR}=\frac{1}{2\pi}\int_0^{\infty}due^{ikr\cosh u}\ ,\ k=\omega/v\end{gathered}\tag{50}$$

which is the integral over z of the three-dimensional Green function given by equation (44),

$$U(\mathbf{r},\omega)=\frac{1}{2\pi}\int_r^{\infty}dR\frac{1}{\sqrt{R^2-r^2}}e^{ikR}=\frac{1}{2\pi}\int_0^{\infty}dz\frac{1}{R}e^{ikR}\ .\tag{51}$$

This function is the solution of the equation

$$\Delta U(\mathbf{r},\omega)+k^2U(\mathbf{r},\omega)=-\delta(\mathbf{r})\ ,\tag{52}$$

which is obtained from equation (41) by integrating over z. We are interested in solutions which do not depend on the direction of $\mathbf{r}$, *i.e.* solutions of the equation

$$\Delta_rU(\mathbf{r},\omega)+k^2U(\mathbf{r},\omega)=0\ ,\ \frac{1}{r}\frac{d}{dr}(r\frac{dU}{dr})+k^2U=0\ ;\tag{53}$$

this is the Bessel equation (29), which, beside the solution J_0, has another, independent solution, which goes like $\ln r$ for $r\rightarrow 0$; we can check

$$\int d\mathbf{r}\Delta_r(\ln r)=2\pi\int dr\frac{d}{dr}(r\frac{d}{dr}\ln r)=2\pi\tag{54}$$

for an infinitesimal circle, so that $U\sim-(1/2\pi)\ln r$ for $r\rightarrow 0$. The solution which behaves like $\ln r$ for $r\rightarrow 0$ and like an outgoing wave at the infinity is the Hankel function (of zeroth degree and the first kind)

$$H_0^{(1)}(kr)\sim\left\{\begin{array}{c}\frac{2i}{\pi}\ln(kr)\ ,\ kr\rightarrow 0\ ,\\ \\ \sqrt{\frac{2}{\pi kr}}e^{i(kr-\pi/4)}\ ,\ kr\rightarrow\infty\ .\end{array}\right.\tag{55}$$

Therefore,

$$U(\mathbf{r},\omega)=\frac{i}{4}H_0^{(1)}(kr)\ ,\ k=\omega/v\ .\tag{56}$$

This is the Green function of the wave equation (Helmholtz equation) in two dimensions; it gives also the integrals in equations (50).

Similarly, we can integrate equation (44) over $\mathbf{r}$ in $\mathbf{R}=(x,\mathbf{r})$, and get the Green function

$$U(x,\omega)=\frac{1}{4\pi}\int d\mathbf{r}\frac{e^{ikR}}{R}=\frac{1}{2}\int_{|x|}^{\infty}dRe^{ikR}=-\frac{1}{2}\frac{1}{ik-\alpha}e^{ik|x|}\ ,\ \alpha\rightarrow 0\ ,\tag{57}$$

of the wave (Helmholtz) equation

$$\Delta U(x,\omega)+\frac{\omega^2}{v^2}U(x,\omega)=-\delta(x)\ ,\ \frac{d^2U}{dx^2}+k^2U=-\delta(x)\ ,\tag{58}$$

where $k=\omega/v$. The temporal Fourier transform of the Green function given by equation (57) leads to a step-function, a result which can be obtained directly by integrating over $\mathbf{r}$ the three-dimensional Green function given by equation (45).