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Elastic equilibrium of the half-space revisited. Mindlin and Boussinesq problems

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Abstract

The displacement caused in an isotropic elastic half-space by a force localized on or beneath its surface is calculated here by a new method. These classical problems are known as Boussinesq and, respectively, Mindlin problems. The motivation for the present work resides in the fact that the original solutions involve some particular procedures, which may limit their general application. The solutions presented here are obtained by including in a generalized Poisson equation the values of the function and its derivatives on the boundary, and by using in-plane Fourier transforms. This method is general, and it can be extended to other, similar problems.

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Introduction. As it is well known, the elastic displacement caused in an infinite body by a localized force was calculated as early as 1848 by Kelvin.[1]-[3] The displacement caused by a force localized in a point on the plane surface of an elastic half-space is known as the Boussinesq problem.[4]-[6] Various generalizations of such problems have been worked out,[7]-[11] in order to estimate the effects of concentrated forces in elastic bodies bounded by closed surfaces.

The displacement of an isotropic elastic half-space caused by a force localized beneath its surface was tackled as early as 1936 by Mindlin,[12, 13] and reworked in 1953;[14] sometimes it is referred to as Mindlin problem. The displacement caused in an isotropic elastic body by a force localized on or beneath its surface is calculated here by a new method. The motivation of undertaking the present research is derived from some particular devices used in the original solutions, which may limit their application to other, similar problems.. We derive the solution by using Fourier transforms with respect to the coordinates parallel to the plane surface of the half-space, which allow a convenient inclusion of the values of the functions and their derivatives on the surface in a generalized Poisson equation.

General form of the solution. Consider the equilibrium equation

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}grad\cdot div\mathbf{u}=-\frac{2(1+\sigma)}{E}\mathbf{F} \tag{1}$$

for the displacement $\mathbf{u}$ in an isotropic elastic body with Poisson's ratio σ and Young modulus E, subjected to a body force with density $\mathbf{F}$. As it is well known, the solution can be written with Helmholtz potentials φ and $\mathbf{H}$ as

$$\mathbf{u}=grad\varphi+curl\mathbf{H} \tag{2}$$

with $div\mathbf{H}=0$; inserting this solution in equation (1) we get

$$\Delta\mathbf{B}=-\frac{2(1+\sigma)}{E}\mathbf{F}\ , \tag{3}$$

where

$$\mathbf{B}=\frac{2(1-\sigma)}{1-2\sigma}grad\varphi+curl\mathbf{H}\ . \tag{4}$$

Taking the div in equation (4) we are led to

$$div\mathbf{B}=\frac{2(1-\sigma)}{1-2\sigma}\Delta\varphi\ , \tag{5}$$

whose general solution is

$$\varphi=\frac{1-2\sigma}{4(1-\sigma)}(\mathbf{rB}+\beta)\ , \tag{6}$$

where

$$\Delta\beta=\frac{2(1+\sigma)}{E}\mathbf{rF}\ . \tag{7}$$

It follows that we can represent the solution as

$$\mathbf{u}=\mathbf{B}-\frac{1}{4(1-\sigma)}grad(\mathbf{rB}+\beta)\ , \tag{8}$$

in terms of two functions $\mathbf{B}$ and β which satisfy Poisson equations (3) and (7); these functions are sometimes called Grodskii functions.[15]-[17] For a force $\mathbf{F}=\mathbf{f}\delta(\mathbf{r})$ localized at the origin in an infinite body, by solving equations (3) and (7) and using equation (8), we get the well-known Kelvin solution

$$\mathbf{u}=\frac{1+\sigma}{8\pi E(1-\sigma)}\left[\frac{(3-4\sigma)\mathbf{f}}{r}+\frac{[\mathbf{rf}]\mathbf{r}}{r^3}\right]\ . \tag{9}$$

The problem and the solving method. The problem is to solve the equlibrium equation (1) for an elastic half-space which occupies the spatial region $z<0$, bounded by a plane surface $z=0$, and a force localized on or beneath its surface. We consider first a force localized beneath the surface. We introduce the notations $\mathbf{R}=(x,y,z)$ and $\mathbf{r}=(x,y)$ for the position vectors and take a force $\mathbf{F}=(f_x,f_{y,},f_z)\delta(\mathbf{R}-\mathbf{R}_0)$ localized at $\mathbf{R}_0=(0,0,z_0)$, $z_0<0$. The surface $z=0$ is free; consequently, the force $P_i=-n_j\sigma_{ij}$ on the surface $z=0$, where $\mathbf{n}=(0,0,1)$ and σ_{ij} is the stress tensor, is vanishing: $\sigma_{iz}=0$ for $z=0$. The stress tensor is $\sigma_{ij}=\frac{E}{1+\sigma}[u_{ij}+\frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}]$, where $u_{ij}=\frac{1}{2}(\partial_i u_j+\partial_j u_i)$ is the strain tensor; the boundary conditions read

$$u_{xz}=u_{yz}=0\ ,\ (1-\sigma)u_{zz}+\sigma(u_{xx}+u_{yy})=0\ ,\ z=0\ . \tag{10}$$

The usual method of solving this problem is to solve the Poisson equations (3) and (7) for the functions $\mathbf{B}$ and β with the boundary conditions given by equations (10) and to use equation (8) for getting the displacement. Usually, the Poisson equations are solved by using the Green function for the laplacian and the Green theorem. Since the boundary conditions (10) are not simply Dirichlet or von Neumann, their inclusion in the Green theorem requires special elaborations. We give here a different method, which can lead more directly to solution.

Consider the Poisson equation

$$\Delta v=F \tag{11}$$

in the domain D bounded by the closed surface S. We introduce the function $w = vE_D$, where E_D is the support function of the domain D; it is easy to see, by direct calculations, that

$$\Delta w = \Delta v \cdot E_D - \frac{\partial v}{\partial n} \mid_S \delta(n - n_s) - v \mid_S \frac{\partial}{\partial n} \delta(n - n_S) \ , \tag{12}$$

where n is the coordinate along the normal $\mathbf{n}_S$ to the surface and n_S is the value of n on the surface. Making use of equation (11) we get

$$\Delta v = F - \frac{\partial v}{\partial n} \mid_S \delta(n - n_s) - v \mid_S \frac{\partial}{\partial n} \delta(n - n_S) \tag{13}$$

in the closed domain D, where we have re-introduced the notation v for w. Equation (13) provides a generalized form of the original Poisson equation (11). Using the Green function G, $\Delta G = -4\pi\delta(\mathbf{R} - \mathbf{R}')$, we recover the Green theorem

$$v(\mathbf{R}) = -\frac{1}{4\pi} \int_D d\mathbf{R}' G(\mathbf{R} - \mathbf{R}') F(\mathbf{R}') + \frac{1}{4\pi} \int_S dS' \left[G(\mathbf{R} - \mathbf{R}') \frac{\partial v(\mathbf{R}')}{\partial n'} - v(\mathbf{R}') \frac{\partial G(\mathbf{R} - \mathbf{R}')}{\partial n'} \right] \tag{14}$$

from equation (13).

We apply this method to the Poisson equation (11) for the half-space $z < 0$ with the support function $\theta(-z)$ and force $F = f\delta(\mathbf{R} - \mathbf{R}_0)$, $\mathbf{R}_0 = (0, 0, z_0)$, $z_0 < 0$, where $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function. It is convenient to use in-plane Fourier transforms of the type

$$v(\mathbf{r}, z) = \frac{1}{(2\pi)^2} \int d\mathbf{k} v(\mathbf{k}, z) e^{i\mathbf{kr}} \ ; \tag{15}$$

the Poisson equation becomes

$$\Delta v = F - v^{(1)} \delta(z) - v^{(0)} \delta'(z) \ , \tag{16}$$

or

$$\frac{d^2 v}{dz^2} - k^2 v = f\delta(z - z_0) - v^{(1)} \delta(z) - v^{(0)} \delta'(z) \ , \tag{17}$$

where $v^{(0)} = v \mid_{z=0}$, $v^{(1)} = \frac{\partial v}{\partial z} \mid_{z=0}$ or their Fourier transforms; for the sake of the simplicity we may omit the arguments $(\mathbf{r}, z)$ or $(\mathbf{k}, z)$, as they can be easily read from the context of the equations. It is known that the Green function for the Helmholtz equation $d^2 g/dz^2 - k^2 g = \delta(z)$ in one dimension is $g = -(1/2k) e^{-k|z|}$, so the solution of equation (17) reads

$$v = -\frac{1}{2k} f e^{-k|z - z_0|} + \frac{1}{2k} v^{(1)} e^{-k|z|} + \frac{1}{2} v^{(0)} e^{-k|z|} \tag{18}$$

for $z < 0$; we eliminate $v^{(1)}$ from this equation and get

$$\begin{gathered} v = -\tfrac{1}{2k} f \left(e^{-k|z - z_0|} - e^{-k|z + z_0|} \right) + v^{(0)} e^{-k|z|} \ , \\ v^{(1)} = f e^{-k|z_0|} + k v^{(0)} \ ; \end{gathered} \tag{19}$$

we recognize here the Green function for the Helmholtz equation in one dimension vanishing on the surface $z = 0$.

We use the representation given by equation (19) for the solutions $\mathbf{B}$ and β of the Poisson equations (3) and (7) and derive the functions and their derivatives (of the form $v^{(0)}$ and $v^{(1)}$ in equation

(19)) from the boundary conditions; in addition, we note that the second-order derivative on the surface is $v^{(2)} = k^2 v^{(0)}$, as it follows immediately from equation (17).

Using the representation given by equation (19) for the solutions of equations (3) and (7) we get

$$\begin{aligned} \mathbf{B} &= \tfrac{(1+\sigma)}{E}\tfrac{\mathbf{f}}{k}\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) + \mathbf{B}^{(0)}e^{-k|z|} \ , \\ \beta &= \tfrac{(1+\sigma)}{E}\tfrac{|z_0| f_z}{k}\left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) + \beta^{(0)}e^{-k|z|} \ ; \end{aligned} \tag{20}$$

in addition, we have the relations

$$\begin{aligned} \mathbf{B}^{(1)} &= -\tfrac{2(1+\sigma)}{E}\mathbf{f}e^{-k|z_0|} + k\mathbf{B}^{(0)} \ , \ \mathbf{B}^{(2)} = k^2\mathbf{B}^{(0)} \ , \\ \beta^{(1)} &= -\tfrac{2(1+\sigma)}{E}|z_0| f_z e^{-k|z_0|} + k\beta^{(0)} \ , \ \beta^{(2)} = k^2\beta^{(0)} \ . \end{aligned} \tag{21}$$

Force perpendicular to the surface. Now we specialize to the case of a force perpendicular to the surface, *i.e.* we take $f_x = f_y = 0$ and $f_z = f$; due to the symmetry of the problem we may also take $B_x = B_y = 0$. Using the Fourier transforms, the boundary conditions from equations (10) are given by

$$(1-2\sigma)B_z^{(0)} - \beta^{(1)} = 0 \ , \ 2(1-\sigma)B_z^{(1)} - k^2\beta^{(0)} = 0 \ , \tag{22}$$

whence, by using relations (21), we get

$$\begin{aligned} B_z^{(0)} &= \tfrac{2(1+\sigma)f}{E}\left[\tfrac{2(1-\sigma)}{k} - z_0\right]e^{-k|z_0|} \ , \\ \beta^{(0)} &= \tfrac{4(1-\sigma)(1+\sigma)f}{E}\left(\tfrac{1-2\sigma}{k^2} - \tfrac{z_0}{k}\right)e^{-k|z_0|} \ . \end{aligned} \tag{23}$$

Making use of equations (20), (23) and

$$\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k}e^{-k|z|} = \frac{1}{(r^2+z^2)^{1/2}} \ , \tag{24}$$

we get, by inverse Fourier transformation,

$$B_z = \frac{(1+\sigma)f}{2\pi E}\left[\frac{1}{R} + \frac{3-4\sigma}{\overline{R}} + \frac{2z_0(z+z_0)}{\overline{R}^3}\right] \ , \tag{25}$$

where

$$R = \left[r^2 + (z-z_0)^2\right]^{1/2} \ , \ \overline{R} = \left[r^2 + (z+z_0)^2\right]^{1/2} \ ; \tag{26}$$

we can see the contribution of the "image" solution corresponding to $z_0 \to -z_0$. Similarly, we get from equations (20) and (23)

$$\beta = \frac{(1+\sigma)f}{2\pi E}\left[\frac{|z_0|}{R} + \frac{(3-4\sigma)|z_0|}{\overline{R}} + 4(1-\sigma)(1-2\sigma)G\right] \ , \tag{27}$$

where

$$G = \frac{1}{2\pi}\int d\mathbf{k}\frac{1}{k^2}e^{i\mathbf{kr}}e^{-k|z+z_0|} \ . \tag{28}$$

In the original solution[14] the function G is replaced by $\ln(\overline{R} + |z+z_0|)$, which can be obtained by integration of the derivative $\partial G/\partial|z+z_0|$ (a minus sign should be included for the half-space

$z<0$ in comparison with the half-space $z>0$). For the displacement functions $u_{x,y,z}$ given by equation (8) we need $grad\beta$ and, therefore, $gradG$. The derivatives of the function G can be calculated from equation (28) by using Bessel functions. For example, it is easy to get

$$\frac{\partial G}{\partial \mathbf{r}} = -\left(1-\frac{|z+z_0|}{\overline{R}}\right)\frac{\mathbf{r}}{r^2} = -\frac{\mathbf{r}}{\overline{R}\left(\overline{R}+|z+z_0|\right)} \ . \tag{29}$$

We give here the displacement of the surface $z=0$, calculated from equation (8) by using B_z given by equation (25) and the derivatives of the function β and function G :

$$\begin{aligned} u_r &= \tfrac{(1+\sigma)f}{2\pi E}\left(\tfrac{|z_0|}{R_0^2}+\tfrac{1-2\sigma}{R_0+|z_0|}\right)\tfrac{r}{R_0} \ , \\ u_z &= \tfrac{(1+\sigma)f}{2\pi E}\left[2(1-\sigma)+\tfrac{z_0^2}{R_0^2}\right]\tfrac{1}{R_0} \ , \end{aligned} \tag{30}$$

where u_r is the radial component of the displacement (along $\mathbf{r}$) and $R_0 = \left[r^2+z_0^2\right]^{1/2}$. We can see that the radial component of the displacement u_r has a maximum of the order $\simeq f/E\,|z_0|$ for distances of the order $r \simeq |z_0|$, while the z-component u_z of the displacement attains its maximum value $\simeq f/E\,|z_0|$ for $r=0$.

Force parallel to the surface. We consider now a force parallel to the x-axis $f_x=f$, $f_y=f_z=0$; due to the symmetry of the problem we take also $B_y=0$. We introduce the function $C = xB_x+\beta$ and the boundary conditions (10) become

$$\begin{aligned} 2(1-\sigma)B_x^{(1)}+i(1-2\sigma)k_xB_z^{(0)}-ik_xC^{(1)} &= 0 \ , \\ (1-2\sigma)B_z^{(0)}-C^{(1)} &= 0 \ , \\ 2(1-\sigma)(1-2\sigma)B_z^{(1)}-(1-\sigma)C^{(2)}+4i\sigma(1-\sigma)k_xB_x^{(0)}+\sigma k^2C^{(0)} &= 0 \ . \end{aligned} \tag{31}$$

Making use of the Fourier transform $C = i\partial B_x/\partial k_x+\beta$ and equations (21) we get the solutions of the system of equations (31) The solutions of the boundary conditions (10) are

$$\begin{aligned} B_x^{(0)} &= \tfrac{2(1+\sigma)f}{E}\tfrac{1}{k}e^{-k|z_0|} \ , \\ B_z^{(0)} &= \tfrac{2(1+\sigma)f}{E}\left(1-2\sigma-k\,|z_0|\right)\tfrac{ik_x}{k^2}e^{-k|z_0|} \ , \\ \beta^{(0)} &= \tfrac{2(1+\sigma)(1-2\sigma)f}{E}\left(1-2\sigma-k\,|z_0|\right)\tfrac{ik_x}{k^3}e^{-k|z_0|} \ . \end{aligned} \tag{32}$$

Hence, by using equations (20), we get

$$\begin{aligned} B_x &= \tfrac{(1+\sigma)f}{E}\tfrac{1}{k}\left(e^{-k|z-z_0|}+e^{-k|z+z_0|}\right) \ , \\ B_z &= \tfrac{2(1+\sigma)f}{E}\left(1-2\sigma-k\,|z_0|\right)\tfrac{ik_x}{k^2}e^{-k|z+z_0|} \ , \\ \beta &= \tfrac{2(1+\sigma)(1-2\sigma)f}{E}\left(1-2\sigma-k\,|z_0|\right)\tfrac{ik_x}{k^3}e^{-k|z+z_0|} \ , \end{aligned} \tag{33}$$

and

$$\begin{aligned} B_x &= \tfrac{(1+\sigma)f}{2\pi E}\left(\tfrac{1}{R}+\tfrac{1}{\overline{R}}\right) \ , \\ B_z &= \tfrac{(1+\sigma)f}{\pi E}\left(\tfrac{|z_0|}{\overline{R}^2}-\tfrac{1-2\sigma}{\overline{R}+|z+z_0|}\right)\tfrac{x}{\overline{R}} \ , \\ \beta &= \tfrac{(1+\sigma)(1-2\sigma)f}{\pi E}\left[\tfrac{|z_0|}{\overline{R}}-(1-2\sigma)\right]\tfrac{x}{\overline{R}+|z+z_0|} \ ; \end{aligned} \tag{34}$$

in equations (34) the function

$$H = \frac{1}{2\pi}\int d\mathbf{k}\frac{ik_x}{k^3}e^{i\mathbf{kr}}e^{-k|z+z_0|} \tag{35}$$

has been calculated by integrating the derivative $\partial H/\partial\left|z+z_0\right|$ $(H=-x/\left(\overline{R}+\left|z+z_0\right|\right))$. The results given in equations (34) coincide with the original Mindlin's results[14], (except the sign of B_z).

Having known the functions $B_{x.z}$ and β, we can calculate the displacement by using equation (8). We give here the displacement u_z on the surface $z=0$

$$u_z = \frac{(1+\sigma)f}{2\pi E}\left(\frac{|z_0|}{R_0^2}-\frac{1-2\sigma}{R_0+|z_0|}\right)\frac{x}{R_0} \tag{36}$$

and the asymptotic behaviour of $u_{x,y}$ on the surface

$$\begin{gathered} u_x \simeq \tfrac{(1+\sigma)(3-2\sigma)f}{4\pi E}\tfrac{1}{|z_0|}\ ,\ u_y \simeq \tfrac{(1+\sigma)(3+2\sigma)f}{8\pi E}\tfrac{xy}{|z_0|^3}\ ,\ r \ll |z_0|\ , \\ u_x \simeq \tfrac{(1+\sigma)(1-\sigma)f}{\pi E}\tfrac{1}{r}\ ,\ u_y \simeq \tfrac{\sigma(1+\sigma)f}{\pi E}\tfrac{xy}{r^3}\ ,\ r \gg |z_0|\ ; \end{gathered} \tag{37}$$

for $y=0$ and $|x| \gg |z_0|$ $u_x \simeq [(3-4\sigma)(1+\sigma)f/4\pi E(1-\sigma)]/\left|x\right|$. We can see that u_x has a maximum value of the order $\simeq f/E\left|z_0\right|$ for $r\rightarrow 0$, while $u_y \sim xy/\left|z_0\right|^3$, $u_z \sim x/z_0^2$ for $r \rightarrow 0$ and attains a maximum value $\simeq f/E\left|z_0\right|$ for r of the order $|z_0|$. It is worth noting that u_z is vanishing for a distance r of the order of $|z_0|$.

Force acting on the surface. We consider now a force $F=(0,0,f)\delta(\mathbf{r})$ acting on the surface $z=0$ at the origin. Equations (20) and (21) for the Grodskii functions are now free of body force, but the surface force appears in the boundary conditions which read, in Fourier transforms,

$$\begin{gathered} (1-2\sigma)B_z^{(0)}-\beta^{(1)}=0\ , \\ 2(1-\sigma)B_z^{(1)}-k^2\beta^{(0)}=-4(1+\sigma)(1-\sigma)\tfrac{f}{E}\ . \end{gathered} \tag{38}$$

Making use of relations (20) and (21) this system of equations is solved immediately, leading to

$$\begin{gathered} B_z=-\tfrac{2(1-\sigma^2)f}{\pi E}\tfrac{1}{R}\ , \\ \beta=-\tfrac{2(1-\sigma^2)(1-2\sigma)f}{\pi E}G\ , \end{gathered} \tag{39}$$

where $R=\left(r^2+z^2\right)^{1/2}$ and G is the function given by equation (28) for $z_0=0$. From equations (8) we get the well-known displacement for the Boussinesq problem[5, 6]

$$\begin{gathered} u_r=-\tfrac{(1+\sigma)f}{2\pi E}\left(\tfrac{z}{R^2}+\tfrac{1-2\sigma}{R+|z|}\right)\tfrac{r}{R}\ , \\ u_z=-\tfrac{(1+\sigma)f}{2\pi E}\left[\tfrac{z^2}{R^2}+2(1-\sigma)\right]\tfrac{1}{R}\ . \end{gathered} \tag{40}$$

The case of a force parallel to the surface can be treated in the same manner.

Conclusion. In conclusion, we may say that the displacement of an isotropic elastic half-space has been calculated in this paper, as caused by a force localized on or beneath the surface, by a new method. The original solution of these problems, known as Boussinesq and, respectively, Mindlin problems,[4, 5],[12]-[14] include some artificial devices. The solution given here is obtained by using in-plane Fourier transforms and by including the values of the functions and their derivatives on the boundary in a generalized Poisson equation. This method can be extended to other problems boundary-value problems, like a half-space with fixed surface,[18, 19] or elastic (thick) plates, or elastic bodies with cylindrical or spherical geometry, etc.

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