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Primary and secondary seismic waves generated by localized seismic sources

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Abstract

The faulting seismic source model is re-analyzed and its representation by means of the seismic moment tensor is re-formulated for spatially localized sources (point sources). A point volume seismic source is also introduced, related to the pressure exerted in a small spherical cavity in an elastic body where the source is localized. It is shown that such a volume source with a δ-pulse time dependence is equivalent with an isotropic faulting source characterized by a scalar seismic moment. A structure factor of the focal region is discussed in this context, which can account for time and spatial extension of the seismic sources (or for multiple sources). The elastic waves produced by point sources with a δ-pulse time dependence are considered in an isotropic elastic half-space bounded by a plane surface, with sources placed beneath the surface. Such a half-space may be viewed as an acceptable model for the Earth. In these circumstances the solution of the elastic waves equation is determined by using the decomposition in Helmholtz potentials. Particular attention is given to the problem of "boundary conditions" (usually for a free surface), and the transient regime is identified for the seismic waves, as a precursor of the stationary regime of vibrations (oscillations). The spherical waves produced by point seismic sources with a δ-pulse time dependence are calculated. The interaction of these primary waves with the surface leads to additional wave sources, lying on the surface, which generate secondary waves (according to Huygens principle). The secondary waves, which may be viewed as waves scattered from the surface, are estimated. It is shown that the secondary waves contribute to the long tail recorded in seismograms.

Good for "boundary conditions";

Introduction. It may be widely accepted that the main problem in Seismology is the generation and propagation of the seismic waves. It gives information about processes occurring in the earthquake focal region, the nature and structure of the Earth's interior, and the effect of the seismic waves on the Earth's surface. The problem originates with the classical works of Rayleigh, Lamb and Love.[1]-[3] In a simplified model, the Earth may be viewed as an isotropic elastic half-space bounded by a plane surface, the seismic sources being localized beneath the surface. For sufficiently long distances the spatial localization of the seismic sources may be represented by δ-functions, or their derivatives (point sources). The double-couple representation of point seismic sources by means of the seismic moment tensor emerged gradually in the first half of the 20th century.[4]-[17]

The standard way of treating the seismic waves is to employ the (formal) Green function for the elastic waves equation and the Green theorem (the so-called Betti's representation) for a general, anisotropic, elastic body.[18]-[21] In this treatment the seismic sources are located on internal surfaces, either as faulting sources or volume sources. The faulting seismic sources are related to the discontinuity occurring in the displacement across the faulting surface (fault slip, dislocation model), while the volume sources are related to the dilatational strain.[6, 7] In both cases equivalent forces are derived for the seismic source representation, and the tensor of the seismic moment is introduced. The interpretation of the double-couple representation of the seismic moment as an extended mechanical torque (a couple of forces), while ensuring the vanishing of the net force and angular momentum, leaves open the question of the uniqueness of the double-couple distribution. We reformulate here the faulting source and the introduction of the seismic moment tensor, by resorting to a point mechanical-torque interpretation, and introduce a volume source model related to the pressure exerted in a small spherical cavity where the source is localized.

The elastic waves equation with localized (point) sources is discussed here for an isotropic half space bounded by a plane surface; a special attention is given to the problem of "boundary conditions" (usually, for a free surface). The problem of the seismic waves with boundary conditions is a standard problem of vibrations (oscillations) of the Earth, viewed as an elastic sphere. The seismic waves produced by a source suffer multiple reflections on the Earth's surface (or on the interfaces of the internal Earth's layers) and the stationary regime of oscillations sets in a finite time interval. The relevant magnitude of this amount of time is of the order R/c, where R is the radius of the Earth and c is the wave velocity. For $R = 6370km$ and a mean velocity $c = 5km/s$ of the elastic waves we get $R/c \simeq 1274s$; this time interval is much longer than the time taken by the seismic waves to propagate from the source to the Earth's surface. We can see that the effects of the seismic waves on the Earth's surface are produced in a much shorter time than the time needed for attaining the stationary regime of vibrations. It follows that we are interested primarily in the transient regime of the seismic waves, where the boundary conditions are practically radiation conditions, and the quasi-spherical Earth may be approximated locally by an elastic half-space. In these circumstances, in a first approximation, the solution consists of P- and S- (double-shock) spherical waves generated by temporal and spatial δ-pulses from the faulting source; a volume source produces only P-wave, as expected. For sources with a finite temporal or spatial extension (or for multiple sources) we discuss the necessity of introducing a structure factor of the focal region, which may be viewed as an inprint of the focal region in recorded seismograms. The local seismic waves are usually treated by a variety of methods, like Fourier or Laplace (or Hilbert) transforms, or the well-known Cagniard-de Hoop method.[22, 23] These methods make use of the reflection (and refraction) coefficients of the approximate plane waves at the Earth's surface (or at interfaces of Earth's internal layers). While, in principle, they may offer an exact solution, in practice the results are often approximate, especially due to the approximate character of the plane waves in comparison with spherical waves generated by localized sources; many such approximate results are known as surface waves, head (or lateral) waves, cylindrical or conical waves, leaking waves, inhomogeneous, damped waves, etc.[24]-[34] We present here an alternate method of treating the local waves by means of the secondary waves generated by the scattering of the primary spherical P- and S-waves off the surface (or layers interfaces). It is shown that these secondary waves contribute to the well-known long tail recorded in seismograms. The secondary waves may be viewed as the next step in the approximate process during which the transient wave regime becomes gradually a vibrations stationary regime.

Seismic sources. Usually, the seismic sources are concentrated in a small volume, which is the earthquake's focus. The linear dimensions of these regions are much smaller than the seismic wavelengths and distances of interest, so we may view them as point sources, in the first approximation. In a simplified representation of a faulting source the slip and the associated force occur, during the earthquake, along one direction, lying on the fault surface, of arbitrary orientation. Let

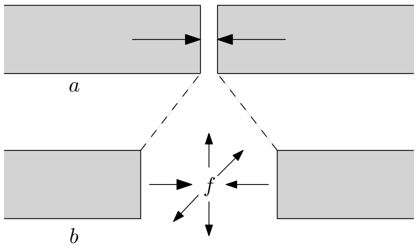

Figure 1: The load accumulation in two elements of tectonic plates in (quasi-) equilibrium (a) may lead to a resistance loss and a localized active focal region f (b).

$\mathbf{n}$ be the unit vector along this direction. The seismic load in the focus consists of two opposite forces, practically at equilbrium, so that the total force and angular momentum are vanishing. The equilibrium can be reached by successive small, (quasi-) static deformations of the crust and tectonic plates. During the earthquake, the resistance of the rocks in the focus yields, such that we have a localized, active distribution of opposite forces which is proportional to $\partial\delta(\mathbf{R}-\mathbf{R}_0)/\partial n$, where $\mathbf{R}_0$ is the position of the focus and n is the (dimensionless) coordinate along the unit vector $\mathbf{n}$. Since the direction of the vector $\mathbf{n}$ is arbitrary, the force distribution can also be written as $fl_i\partial_i\delta(\mathbf{R}-\mathbf{R}_0) = f_il\partial_i\delta(\mathbf{R}-\mathbf{R}_0) = fln_i\partial_i\delta(\mathbf{R}-\mathbf{R}_0)$, where f_i are the components of the force with magnitude f acting along the direction $\mathbf{n}$ and l_i are the components of the spatial extension of the focus (summation over repeated labels is assumed); the function δ in these expressions should be understood as a function localized over the distance of the order l along the direction $\mathbf{n}$, and, similarly, along the other two transverse directions. The quantity fl is proportional to the seismic moment M (we may also take $f = \lambda A$, where λ is the elastic modulus and A is the area of the fault); we prefer to use the seismic moment divided by density ρ, $m = M/\rho$; then, the force distribution per unit mass reads

$$\mathbf{F}(\mathbf{R}, t) = m(t)n_i\partial_i\delta(\mathbf{R}-\mathbf{R}_0)\mathbf{n} \ , \ F_i(\mathbf{R}, t) = m(t)n_in_j\partial_j\delta(\mathbf{R}-\mathbf{R}_0) \ , \tag{1}$$

where $m(t)$ has a certain time dependence during the earthquake. Usually, this function is localized over a finite duration T, such that we may use the δ-pulse time dependence $m(t) = mT\delta(t)$.

The force distribution given by equation (1) represents a point linear dipole; since a strain occurring along a direction $\mathbf{n}$ generates forces directed both along $\mathbf{n}$ and along the two directions perpendicular to $\mathbf{n}$, the force distribution given by equation (1) should be generalized by replacing $m(t)n_in_j$ by the symmetric tensor $m_{ij}(t)$ of the seismic moment:

$$F_i(\mathbf{R}, t) = m_{ij}(t)\partial_j\delta(\mathbf{R}-\mathbf{R}_0) \tag{2}$$

(the transverse components of the seismic moment involve the shear elastic modulus μ); we can see that the force generated by a faulting source is a tensorial force. It is easy to see that the total force and angular momentum associated with the force distribution given by equation (2) are zero (the latter by the symmetry of the tensor m_{ij}). According to our definition, the moment tensor is positive definite for an "implosion", and negative definite for an "explosion" (in general, it is an indefinite tensor). A schematic representation of a faulting-source force distribution is shown in Fig.1.

A more direct derivation of the seismic tensorial forces is obtained by estimated the couple produced by a force density $\mathbf{F}(\mathbf{R}, t) = \mathbf{f}(t)g(\mathbf{R})$, where $\mathbf{f}$ is the force and $g(\mathbf{R})$ is a distribution function; a point couple along the i-th direction can be represented as

$$f_ig(x_1+h_1, x_2+h_2, x_3+h_3) - f_ig(x_1, x_2, x_3) \simeq f_ih_j\partial_jg(x_1, x_2, x_3) \ , \tag{3}$$

where h_j are the components of an infinitesimal displacement $\mathbf{h}$. The moment $f_i h_j$ are generalized to a symmetric tensor M_{ij}, which is the seismic moment; in addition, the distribution $g(\mathbf{R})$ is replaced by $\delta(\mathbf{R}-\mathbf{R}_0)$, where δ denotes the Dirac function localized at the point with the position vector $\mathbf{R}_0$. Thus, we get the tensorial force density which is given in equation (2).[35]

Similarly, the force distribution localized in a volume source with a small radius a can be written as

$$\mathbf{F}(\mathbf{R},t) = p(t)\frac{\mathbf{R}-\mathbf{R}_0}{|\mathbf{R}-\mathbf{R}_0|}\theta\left(a-|\mathbf{R}-\mathbf{R}_0|\right) \ , \tag{4}$$

where $p(t) = f(t)/a^3$ is force divided by density per unit volume (force per unit mass) and $\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$ is the step function. We show in this paper that the waves produced by this volume source in the limit $a \to 0$ (for a δ-pulse time dependence) can be obtained from the faulting source given by equation (2) by replacing formally the tensor m_{ij} by $-m\delta_{ij}$, where the scalar seismic moment is of the order $m \simeq fa$.

It is worth giving a numerical estimation of the localization length l of the focal region. We note that the seismic moment M has the dimension of a mechanical work (energy); it is reasonable to admit that this energy is spent to destroy the elastic consistency of the material which is ruptured in the focal volume V during the earthquake; this energy density is of the order of the elastic energy density of the material ρc^2, where ρ is the material density and c is a mean value of the velocity of the elastic waves. Therefore, the equality $M/V \simeq \rho c^2$ may hold. For $M = 10^{26} dyn \cdot cm$ (corresponding to an earthquake magnitude $M_w = 7$, from the Gutenberg-Richter definition[36]-[40] $\lg M = 1.5M_w + 16.1$), $\rho = 5g/cm^3$ for the average Earth's density and $c = 5km/s$ for a mean value of the velocity of the elastic waves we get a volume $V = 8 \times 10^{13} cm^3$ of the focal region and a localization length $l = V^{1/3} \simeq 1km$. This spatial uncertainty leads to a time uncertainty in the spherical waves of the order $T = l/c = 0.2s$ (for a mean velocity $c = 5km/s$).

Primary P- and S-waves. The equation of the elastic waves reads

$$\ddot{\mathbf{u}} - c_t^2\Delta\mathbf{u} - (c_l^2 - c_t^2) grad \cdot div\mathbf{u} = \mathbf{F} \ , \tag{5}$$

where $\mathbf{u}$ is the displacement, $c_{l,t}$ are the wave velocities and $\mathbf{F}$ is the force (per unit mass).[41] We consider this equation in an isotropic elastic half-space extending in the region $z < 0$ and bounded by the flat surface $z = 0$. The faulting source, which generates the force $\mathbf{F}$, is placed at $\mathbf{R}_0 = (0,0,z_0)$, $z_0 < 0$; the force is given by equation (2) above with $m_{ij}(t) = m_{ij}T\delta(t)$, where m_{ij} is the seismic moment tensor (divided by density) and T is the duration of the timeδ-impulse. The coordinates of the position vector $\mathbf{R}$ are denoted by (x_1,x_2,x_3); the notation $x = x_1$, $y = x_2$, $z = x_3$ is also used. We use the Helmholtz decomposition $\mathbf{F} = grad\phi + curl\mathbf{H}$ ($div\mathbf{H} = 0$), whence

$$\Delta\phi = div\mathbf{F} \ , \ \Delta\mathbf{H} = -curl\mathbf{F} \ ; \tag{6}$$

similarly, the displacement is decomposed as $\mathbf{u} = grad\Phi + curl\mathbf{A}$, with $\mathbf{u}^l = grad\Phi$ and $\mathbf{u}^t = curl\mathbf{A}$, by using the Helmholtz potentials Φ and $\mathbf{A}$ ($div\mathbf{A} = 0$); equation (5) is transformed into two standard wave equations

$$\ddot{\Phi} - c_l^2\Delta\Phi = \phi \ , \ \ddot{\mathbf{A}} - c_t^2\Delta\mathbf{A} = \mathbf{H} \ ; \tag{7}$$

we can see that the l,t-waves are separated.

From equations (6), and making use of the force distribution given by equation (2), we get immediately

$$\begin{gathered}\phi = -\tfrac{1}{4\pi}m_{ij}T\delta(t)\int d\mathbf{R}'\tfrac{1}{|\mathbf{R}-\mathbf{R}'|}\partial_i'\partial_j'\delta(\mathbf{R}'-\mathbf{R}_0) = \\ = -\tfrac{1}{4\pi}m_{ij}T\delta(t)\partial_i\partial_j\tfrac{1}{|\mathbf{R}-\mathbf{R}_0|}\end{gathered} \tag{8}$$

and

$$H_i = \frac{1}{4\pi}\varepsilon_{ijk}m_{kl}T\delta(t)\int d\mathbf{R}'\frac{1}{\left|\mathbf{R}-\mathbf{R}'\right|}\partial'_j\partial'_l\delta(\mathbf{R}'-\mathbf{R}_0) = = \frac{1}{4\pi}\varepsilon_{ijk}m_{kl}T\delta(t)\partial_j\partial_l\frac{1}{\left|\mathbf{R}-\mathbf{R}_0\right|} \ , \tag{9}$$

where ε_{ijk} is the totally antisymmetric tensor of rank three. Making use of these sources in equations (7), we get the potentials

$$\Phi = -\frac{T}{(4\pi c_l)^2}m_{ij}\int d\mathbf{R}'\frac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_l)}{\left|\mathbf{R}-\mathbf{R}'\right|}\partial'_i\partial'_j\frac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = = -\frac{T}{(4\pi c_l)^2}m_{ij}\partial_i\partial_j\int d\mathbf{R}'\frac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_l)}{\left|\mathbf{R}-\mathbf{R}'\right|}\frac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} \tag{10}$$

and

$$A_i = \frac{T}{(4\pi c_t)^2}\varepsilon_{ijk}m_{kl}\int d\mathbf{R}'\frac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_t)}{\left|\mathbf{R}-\mathbf{R}'\right|}\partial'_j\partial'_l\frac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = = \frac{T}{(4\pi c_t)^2}\varepsilon_{ijk}m_{kl}\partial_j\partial_l\int d\mathbf{R}'\frac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_t)}{\left|\mathbf{R}-\mathbf{R}'\right|}\frac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} \ . \tag{11}$$

We extend the integral

$$I = \int d\mathbf{R}'\frac{\delta(t-\left|\mathbf{R}-\mathbf{R}'\right|/c)}{\left|\mathbf{R}-\mathbf{R}'\right|}\frac{1}{\left|\mathbf{R}'-\mathbf{R}_0\right|} = = \int d\mathbf{r}\frac{\delta(t-r/c)}{r}\frac{1}{\left|\mathbf{R}-\mathbf{R}_0-\mathbf{r}\right|} \tag{12}$$

(where c stands for $c_{l,t}$) occurring in the above equations to the whole space, where it can be effected straightforwardly by using spherical coordinates; we get

$$I = 4\pi c\left[\theta\left(ct-\left|\mathbf{R}-\mathbf{R}_0\right|\right)+\frac{ct}{\left|\mathbf{R}-\mathbf{R}_0\right|}\theta\left(\left|\mathbf{R}-\mathbf{R}_0\right|-ct\right)\right] \ ; \tag{13}$$

inserting this result in equations (10) and (11) we get the Helmholtz potentials

$$\Phi = -\frac{T}{4\pi c_l}m_{ij}\partial_i\partial_j\left[\theta\left(c_lt-\left|\mathbf{R}-\mathbf{R}_0\right|\right)+\frac{c_lt}{\left|\mathbf{R}-\mathbf{R}_0\right|}\theta\left(\left|\mathbf{R}-\mathbf{R}_0\right|-c_lt\right)\right] \ , \quad A_i = \frac{T}{4\pi c_t}\varepsilon_{ijk}m_{kl}\partial_j\partial_l\left[\theta\left(c_tt-\left|\mathbf{R}-\mathbf{R}_0\right|\right)+\frac{c_tt}{\left|\mathbf{R}-\mathbf{R}_0\right|}\theta\left(\left|\mathbf{R}-\mathbf{R}_0\right|-c_tt\right)\right] \ . \tag{14}$$

Making use of the notation

$$F_{l,t} = \frac{T}{4\pi c_{l,t}}\left[\theta\left(c_{l,t}t-\left|\mathbf{R}-\mathbf{R}_0\right|\right)+\frac{c_{l,t}t}{\left|\mathbf{R}-\mathbf{R}_0\right|}\theta\left(\left|\mathbf{R}-\mathbf{R}_0\right|-c_{l,t}t\right)\right] \ , \tag{15}$$

the potentials can be written as

$$\Phi = -m_{ij}\partial_i\partial_jF_l \ , \ A_i = \varepsilon_{ijk}m_{kl}\partial_j\partial_lF_t \ ; \tag{16}$$

it follows the displacement

$$u_i^l = \partial_i\Phi = -m_{jk}\partial_i\partial_j\partial_kF_l \ , \quad u_i^t = \varepsilon_{ijk}\partial_jA_k = m_{jk}\partial_i\partial_j\partial_kF_t - m_{ij}\partial_j\Delta F_t \ . \tag{17}$$

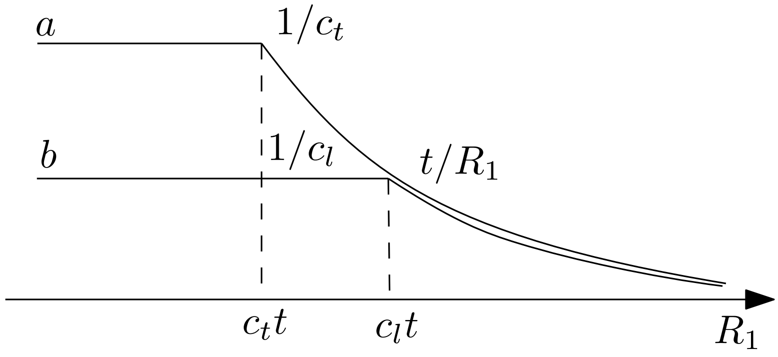

Figure 2: The functions F_t (a) and F_l (b) *vs* R_1.

We can see that these solutions consist of two parts: spherical waves propagating with velocities $c_{l,t}$, given by δ-functions and derivatives of δ-functions (arising from the derivatives of the θ-functions in equation (15)), and a quasi-static displacement which includes the functions $\theta(|\mathbf{R}-\mathbf{R}_0|-c_{l,t}t)$ and extends over the distance $\Delta R_1=(c_l-c_t)t$ ($\mathbf{R}_1=\mathbf{R}-\mathbf{R}_0$). The quasi-static contributions, being proportional to third-order derivatives of t/R_1, are solutions of homogeneous wave equations. In the transient regime, the quasi-static contributions are omitted, and we limit ourselves to the δ-functions and derivatives of δ-functions arising from the derivatives of the θ-functions in equation (15). Outside the support of the δ-functions and their derivatives (*i.e.*, for $R_1=|\mathbf{R}-\mathbf{R}_0|\neq c_{l,t}t$) the displacement is zero. This point can be made more technical by the following considerations.

For $R_1\neq c_t t$ the function F_t in equation (15) is either $T/4\pi c_t$ or $Tt/4\pi R_1$; in both cases the term with the laplacian in the second equation (17) cancels out, and $\mathbf{u}^t$ acquires the same expression as $-u_i^l$ with c_l replaced by c_t. Therefore, in what follows we may restrict ourselves to only one kind of solution, for instance $\mathbf{u}^l$; for simplicity, we give up for the moment the label l in $\mathbf{u}^l$, F_l and c_l, and write simply $\mathbf{u}$, F and c. The functions $F_{l,t}(R_1)$ are shown in Fig.2.

The strain tensor is

$$u_{ij}=\partial_j u_i=-m_{kn}\partial_i\partial_j\partial_k\partial_n F \tag{18}$$

(the moment tensor m_{ij} is symmetric) and the trace of the strain tensor reads

$$u_{ii}=-\partial_k\partial_n m_{kn}\Delta F\ ; \tag{19}$$

for $R_1\neq ct$, the laplacian in equation (19) is zero; it follows that the trace of the strain tensor is vanishing ($u_{ii}=0$). The boundary conditions for a free surface $z=0$ are $u_{i3}=0$ for $i=1,2$ and $(1-2\sigma)u_{33}+\sigma u_{ii}=0$ for $z=0$ (where σ is Poisson's ratio), *i.e.* $u_{i3}=0$ for $i=1,2,3$ and $z=0$ (since $u_{ii}=0$); we get

$$u_{i3}|_{z=0}=-\ m_{kn}\partial_i\partial_3\partial_k\partial_n F|_{z=0}=0\ . \tag{20}$$

It is worth noting that for $R_1<ct$ (behind the wavefront), $F=T/4\pi c$ and equation (20) is satisfied (actually, it is satisfied outside the region $\Delta R_1=(c_l-c_t)t$, where the displacement is in fact vanishing). For $R_1>ct$ (beyond the wavefront; actually inside the region $\Delta R_1=(c_l-c_t)t$) it is convenient to introduce Greek labels α,β,γ for the coordinates $i=1,2$; we have $m_{kn}\partial_k\partial_n=m_{\beta\gamma}\partial_\beta\partial_\gamma+2m_{\beta 3}\partial_\beta\partial_3+m_{33}\partial_{33}^2$ and

$$\begin{aligned}u_{\alpha 3}|_{z=0}&=-\ (m_{\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\partial_3+2m_{\beta 3}\partial_\alpha\partial_\beta\partial_3^2+m_{33}\partial_\alpha\partial_{33}^3)\,F|_{z=0}=0\ ,\\ u_{33}|_{z=0}&=-\ (m_{\beta\gamma}\partial_\beta\partial_\gamma\partial_3^2+2m_{\beta 3}\partial_\beta\partial_3^3+m_{33}\partial_{33}^4)\,F|_{z=0}=0\ .\end{aligned} \tag{21}$$

In order to satisfy such conditions it is usual to seek a reflected wave[42] given by $\Phi_r=-n_{ij}\partial_i\partial_j\widetilde{F}$, where $\widetilde{F}$ is given by equations (15) with $\mathbf{R}_0$ replaced by $\widetilde{\mathbf{R}}_0=(\mathbf{r},-z_0)$ and n_{ij} is a moment tensor to be determined. We note that Φ_r is a solution of the homogeneous wave equation $\ddot{\widetilde{\Phi}} - c^2\Delta\widetilde{\Phi} = 0$ (for $z < 0$) and preserves the vanishing trace of the strain tensor, since $\widetilde{F}$ is either $T/4\pi c$ or $(Tt/4\pi)\frac{1}{R_2}$, where $R_2 = \left|\mathbf{R} - \widetilde{\mathbf{R}}_0\right|$. It is easy to see that there is no such tensor n_{ij} to satisfy the conditions given by equations (21). For instance, we may choose $n_{\beta\gamma} = m_{\beta\gamma}$, $n_{33} = m_{33}$ and $n_{\beta 3} = -m_{\beta 3}$, and the first equation (21) is satisfied, but the second equation (21) is not satisfied (except for $m_{\beta\gamma} = n_{\beta\gamma} = 0$, $m_{33} = n_{33} = 0$); similarly, we can satisfy this latter equation (21) by $n_{\beta\gamma} = -m_{\beta\gamma}$, $n_{33} = -m_{33}$ and $n_{\beta 3} = m_{\beta 3}$, but not the former (except for $m_{\beta 3} = n_{\beta 3} = 0$).

In fact, we note that $\overline{F} = -T/4\pi c$ for $R_1 < ct$ and $\overline{F} = -Tt/4\pi R_1$ for $R_1 > ct$ generate a potential $\overline{\Phi} = -m_{ij}\partial_i\partial_j\overline{F}$ which is a solution of the homogeneous wave equation $\ddot{\overline{\Phi}} - c^2\Delta\overline{\Phi} = 0$ (in particular, it is worth noting the linear time dependence in $\overline{F} = -Tt/4\pi R_1$). Consequently, we may add $\overline{F}$ to F, resulting in $\overline{F} + F = 0$ (for $R_1 \neq ct$) and a displacement $\overline{u}_{i3} + u_{i3}$ which satisfies the boundary conditions given by equation (21) (obviously, outside the support $R_1 = ct$ of the δ-functions and their derivatives). A similar argument applies to the displacement $\mathbf{u}^t$. Moreover, it is worth noting that under these conditions the Helmholtz potentials are identically vanishing everywhere for $R_1 \neq ct$, *i.e.* we have the trivial solution $\mathbf{u} = 0$ in these regions (and the problem of boundary conditions becomes in fact meaningless, as expected). We have nontrivial solutions only on the support of the functions $\delta(c_{l,t} - R_1)$ and their derivatives, as expected.[43]

The solution is given by the potentials in equation (16), provided we leave aside the quasi-static displacement; we get

$$\begin{gathered}
u_i^l = -\frac{T}{4\pi c_l}[(m_{jj}x_i + 2m_{ij}x_j)(1 - 2c_l t/R)\frac{1}{R^3} - \\
-m_{jk}x_i x_j x_k(3 - 8c_l t/R)\frac{1}{R^5}]\delta(R - c_l t) + \\
+\frac{T}{4\pi c_l}[(m_{jj}x_i + 2m_{ij}x_j)(1 - c_l t/R)\frac{1}{R^2} - \\
-m_{jk}x_i x_j x_k(3 - 5ct/R)\frac{1}{R^4}]\delta'(R - c_l t) + \\
+\frac{T}{4\pi c_l}m_{jk}x_i x_j x_k(1 - c_l t/R)\delta''(R - c_l t) \ ,
\end{gathered} \tag{22}$$

where $\mathbf{R}_1 = \mathbf{R} - \mathbf{R}_0$ is denoted here by $\mathbf{R} = (x_1, x_2, x_3)$; we may put $R = c_l t$ in this equation, and get

$$\begin{gathered}
u_i^l = \frac{T}{4\pi c_l}[(m_{jj}x_i + 2m_{ij}x_j)\frac{1}{R^3} - 5m_{jk}x_i x_j x_k\frac{1}{R^5}]\delta(R - c_l t) + \\
+\frac{T}{2\pi c_l}m_{jk}x_i x_j x_k\frac{1}{R^4}\delta'(R - c_l t) \ ;
\end{gathered} \tag{23}$$

similarly, from equations (17) we get

$$\begin{gathered}
u_i^t = -\frac{T}{4\pi c_t}[(m_{jj}x_i + 2m_{ij}x_j)\frac{1}{R^3} - 5m_{jk}x_i x_j x_k\frac{1}{R^5}]\delta(R - c_t t) - \\
-\frac{T}{2\pi c_t}\left(m_{jk}x_i x_j x_k\frac{1}{R^4} - m_{ij}x_j\frac{1}{R^2}\right)\delta'(R - c_t t) \ .
\end{gathered} \tag{24}$$

We can see that in the far-field region the faulting source generates two (double-shock) spherical waves (derivatives of the δ-function), propagating with velocities $c_{l,t}$, given by

$$u_i^f \simeq \frac{Tm_{ij}x_j}{2\pi c_t R^2}\delta'(R - c_t t) + \frac{Tm_{jk}x_i x_j x_k}{2\pi R^4}\left[\frac{1}{c_l}\delta'(R - c_l t) - \frac{1}{c_t}\delta'(R - c_t t)\right] \ ; \tag{25}$$

these are the leading contributions to the solution in the far-field region.

The waves propagating with velocity c_l are the primary P waves (compressional waves), while the waves propagating with velocity c_t are the primary S-waves (they include the shear contribution). The second term on the right in equation (25) is longitudinal ($\sim \mathbf{R}$), while the polarization of the first term depends on the moment tensor. The magnitude of these waves is of the order $u^f \simeq mT/cRl^2$, where m is the seismic moment (divided by density), c is a mean wave velocity and $l = cT$ is the linear dimension of the localization of the δ-function (linear dimension of the earthquake's focus). Making use of a seismic moment $M = 10^{26} dyn \cdot cm$ (earthquake's magnitude 7), density $\rho = 5g/cm^3$ ($m = M/\rho$), a mean velocity $c = 5km/s$, $l = 1km$, for an earthquake's duration $T = 0.2s$, we get at distance $R = 100km$ a far-field wave u^f of the order $1m$.

Structure factor. It is worth noting that the spherical-wave character of the displacement (involving δ- and δ'-functions) is closely connected to the localization of the source, *i.e.* to the functions $\delta(t)$ and $\delta(\mathbf{R} - \mathbf{R}_0)$ occurring in the mathematical expression of the source (equation (6)). For instance, let us assume that we have a succession of shocks in the source, labelled by i, occurring at times t_i, with duration T_i; then, the displacement given by equations (23) and (24) includes summations of the type

$$\sum_i T_i \delta \left(R - c(t - t_i)\right) \ , \ \sum_i T_i \delta' \left(R - c(t - t_i)\right) \ , \tag{26}$$

where c is a generic notation for the velocities $c_{l,t}$; for a sufficiently dense distribution of such shocks, we may replace the summations over i by integrals:

$$\begin{gathered} \sum_i T_i \delta \left(R - c(t - t_i)\right) = \frac{1}{\Delta T} \int dt' T(t') \delta(R - ct + ct') = \frac{1}{c\Delta T} T(t - R/c) \ , \\ \sum_i T_i \delta' \left(R - c(t - t_i)\right) = \frac{1}{\Delta T} \int dt' T(t') \delta'(R - ct + ct') = -\frac{1}{c^2 \Delta T} T'(t - R/c) \ , \end{gathered} \tag{27}$$

where ΔT is the mean separation between the pulses. We can see that the displacement has not a spherical-wave character anymore, but instead it is given now by the functions T and its derivative T' (at the retarded time), which play the role of time signatures of the source. A similar analysis can be done for shocks distributed spatially; we have, for instance

$$\begin{gathered} \sum_{ij} T_i \delta' \left(|\mathbf{R} - \mathbf{R}_j| - c(t - t_i)\right) f \left(\mathbf{R} - \mathbf{R}_j\right) = \\ = -\frac{1}{c^2 \Delta T \Delta v} \int d\mathbf{R}' T'(t - \left|\mathbf{R} - \mathbf{R}'\right| /c) f \left(\mathbf{R} - \mathbf{R}'\right) \ , \end{gathered} \tag{28}$$

where $f(\mathbf{R})$ represents the spatial dependence in equation (25) (except the δ'-functions) and Δv is the mean volume associated with individual shocks. The integral in equation (28) reflects the time-space structure of the earthquake's focal region. The factor $1/\Delta v$ can be replaced by a spatial distribution weight $w_s(\mathbf{R}')$, a procedure which is also valid for the factor $1/\Delta T$, which may be replaced by a weight function $w_t(t')$; a more general situation would imply a weight function $w(t', \mathbf{R}')$ instead $T_i/\Delta T \Delta v$, which plays the role of a structure factor for the focal region; then, the displacement can be represented as

$$\begin{gathered} \int d\mathbf{R}' dt' w(t', \mathbf{R}') \delta' \left(\left|\mathbf{R} - \mathbf{R}'\right| - c(t - t')\right) f \left(\mathbf{R} - \mathbf{R}'\right) = \\ = -\frac{1}{c^2} \int d\mathbf{R}' w'(t - \left|\mathbf{R} - \mathbf{R}'\right| /c, \mathbf{R}') f \left(\mathbf{R} - \mathbf{R}'\right) \ , \end{gathered} \tag{29}$$

where the weight function w is localized over the focal region and over the time duration of the earthquake; such weight functions can be derived, in principle, from recorded seismograms, as an inprint of the structure of the focal region, by de-convoluting equations of the type given by equation (29). The occurence of shocks in succession is reflected in the irregular oscillations exhibited usually by the weight function (and by the displacement, velocity and acceleration recorded in seismograms). The succession (i, j) of shocks arising at t_i and R_j, of the form $\delta(t - t_i)\delta(\mathbf{R} - \mathbf{R}_j)$, may be viewed as a series of elementary (primitive) earthquakes.[3, 44]

Energy balance. Multiplying the wave equation (5) by $\dot{\mathbf{u}}$ we get the energy conservation law

$$\frac{\partial \mathcal{E}}{\partial t} = -div\mathbf{S} + w \ , \tag{30}$$

where

$$\mathcal{E} = \frac{1}{2}\dot{u}_i^2 + \frac{1}{2}c_t^2(\partial_j u_i)^2 + \frac{1}{2}(c_l^2 - c_t^2)(\partial_i u_i)^2 \tag{31}$$

is the energy density (per unit mass),

$$S_i = -c_t^2(\dot{u}_j \partial_i u_j) - (c_l^2 - c_t^2)(\dot{u}_i \partial_j u_j) \tag{32}$$

are the components of the energy flux density (per unit mass) and $w = \dot{u}_i F_i$ is the density of the mechanical work done by the external force $\mathbf{F}$ per unit time (and unit mass). It is easy to see that for a force localized in the focal point the mechanical work w is nonvanishing only at this point and for a short time T, while for a spherical wave the continuity equation $\partial \mathcal{E} + div\mathbf{S} = 0$ is satisfied identically at any point outside the focus, the energy density $\mathcal{E}$ and the energy flux density $\mathbf{S}$ being zero outside the support of the wave. The mechanical work done by the external force for a short period of time in the focus is transferred to the wave energy, which is carried through the space by the propagating wave without loss.

In order to have a numerical estimate we use $F \simeq M/\rho l^4$ for the force given by equation (2) and $u \simeq M/\rho c^2 Rl$ for a spherical wave of the form $u = (MT/\rho cR)\delta'(R - ct)$, with $l = cT$. The density of the mechanical work per unit time is $w \simeq M^2/\rho^2 cl^7$ and the total mechanical work is $W \simeq M^2/\rho c^2 l^3$. The energy density is $\mathcal{E} \simeq M^2/\rho^2 c^2 R^2 l^4$ and the total energy is $E_0 = M^2/\rho c^2 l^3 = W$ (similarly, the energy flux density is $S \simeq M^2/\rho^2 cR^2 l^4$, $div\mathbf{S} \simeq M^2/\rho^2 cR^2 l^5$, and we can check the continuity equation $\partial \mathcal{E}/\partial t + div\mathbf{S} = 0$). It is worth noting that the energy $E_0 = W$ transferred to the waves is smaller than the energy M released in the focal region by the factor $W/M = M/\rho c^2 l^3 = u_0/l$, where $u_0 = M/\rho c^2 l^2$ is the displacement in the focal region (at distance $R = l$). Making use of $M = 10^{26} dyn \cdot cm$, $\rho = 5g/cm^3$, $c = 5km/s$ and $l = 1km$, we get a focal displacement of the order $u_0 \simeq 80m$.

Secondary waves. The primary P- and S-waves given above are only an approximation to our problem. Without resorting to the formal theory of the generalized functions (distributions),[43] we may admit that the support of the δ'-functions which define the primary P- and S- (double-shock) waves is small, but finite. For instance, for a duration T we may view the time δ-pulse as extending over this duration T; in this case the spatial extension of the primary waves is $\Delta R_{l,t} = c_{l,t}T$; for $T = 0.2s$ we may have spatial extensions as large as $0.6km - 1.2km$ (for $c_l = 3km/s$ and $c_t = 6km/s$).

The wavefront of the spherical waves given by equation (25) intersects the surface $x_3 = z = 0$ along a circular line defined by $\overline{\mathbf{R}} = (x_1, x_2, -z_0)$, $\overline{R} = (r^2 + z_0^2)^{1/2}$, where $r = (x_1^2 + x_2^2)^{1/2}$ is the distance from the origin (placed on the surface) to the intersection points (we recall that $\mathbf{R}$ and $\overline{\mathbf{R}}$ are in fact $\mathbf{R} - \mathbf{R}_0$ and $\overline{\mathbf{R}} - \mathbf{R}_0$). The radius $\overline{R}$ moves with velocity c, $\overline{R} = ct$, $t > |z_0|/c$, and the in-plane radius r moves according to the law $r = \sqrt{\overline{R}^2 - z_0^2} = \sqrt{c^2t^2 - z_0^2}$, where c stands for the velocities $c_{l,t}$; its velocity $v = dr/dt = c^2 t/r$ is infinite for $r = 0$ ($\overline{R} = ct = |z_0|$) and tends to c for large distances.

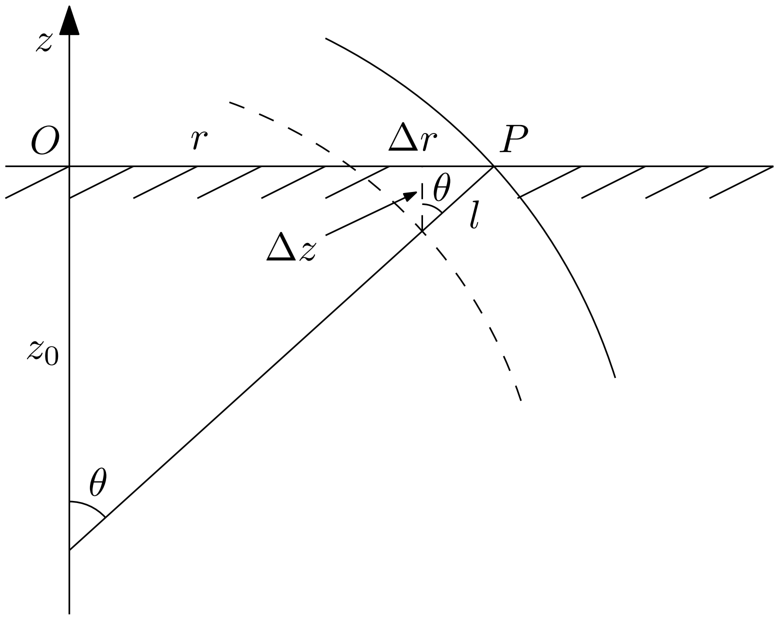

Figure 3: Spherical wave intersecting the surface $z=0$ at P. The notations are given in text.

The finite duration T of the source makes the δ'-functions in equation (25) to be viewed as functions with a finite spread $l=\Delta R=cT\ll R$; consequently, the intersection line of the waves with the surface has a finite spread Δr, which can be calculated from

$$\overline{R}^2=r^2+z_0^2\ ,\ (\overline{R}+l)^2=(r+\Delta r)^2+z_0^2\ ; \tag{33}$$

hence,

$$\Delta r\simeq\frac{2\overline{R}l}{r+\sqrt{r^2+2\overline{R}l}}\ . \tag{34}$$

we can see that for $r\rightarrow 0$ the width $\Delta r\simeq\sqrt{2\left|z_0\right|l}$ of the seismic spot on the surface is much larger than the width of the spot for large distances $\Delta r\simeq l$ $(2\left|z_0\right|\gg l)$. For values of r not too close to the origin (epicentre) we may use the approximation $\Delta r\simeq\overline{R}l/r$.

As long as the spherical wave is fully included in the half-space its total energy E_0 is given by the energy density $\mathcal{E}$ integrated over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy E is given by the energy density integrated over the spherical sector which subtends the solid angle $2\pi(1+\cos\theta)$, where $\cos\theta=\left|z_0\right|/\overline{R}$. It follows that $E=\frac{1}{2}E_0\left(1+\left|z_0\right|/ct\right)$ for $ct>\left|z_0\right|$. We can see that the energy of the wave decreases by the amount $E_s=\frac{1}{2}E_0\left(1-\left|z_0\right|/ct\right)$, $ct>\left|z_0\right|$. This amount of energy is transferred to the surface, which generates secondary waves (according to Huygens principle). A spherical wave intersecting the surface $z=0$ is shown in Fig.3.

In the seismic spot of width Δr generated on the surface by the far-field primary P- and S-waves given by equation (25) we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves scattered from the surface. Strictly speaking, if the reaction force is limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer of thickness Δz $(\Delta z\ll\left|z_0\right|)$, where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave can be estimated from $\Delta z = l\cos\theta = l\left|z_0\right|/\overline{R}$, where $\overline{R} = ct > |z_0|$, or $\Delta z = l\left|z_0\right|/\left(r^2+z_0^2\right)^{1/2}$. We can see that Δz depends on $ct = \overline{R}$ (and on r). For a limited range of variation of r about $|z_0|$, and in virtue of its small values, we may assume the parameter Δz as being a constant of the order l.

The volume force per unit mass is given by $\partial_j\sigma_{ij}/\rho$, where $\sigma_{ij} = \rho\left[2c_t^2u_{ij} + (c_l^2-2c_t^2)u_{kk}\delta_{ij}\right]$ is the stress tensor, u_{ij} is the strain tensor and ρ is the density of the body. At the surface we may take approximately the reaction force which compensates the elastic force as

$$f_i = -\partial_j\sigma_{ij}/\rho = -\partial_j\left[2c_t^2u_{ij} + (c_l^2-2c_t^2)u_{kk}\delta_{ij}\right] \ . \tag{35}$$

The strain tensor calculated from the displacement given by equation (25) (in the far-field region) is

$$\begin{gathered}u_{ij} \simeq \tfrac{T}{4\pi c_t}\tfrac{m_{ik}x_jx_k+m_{jk}x_ix_k}{R^3}\delta''(R-c_tt)+\\ +\tfrac{T}{2\pi}\tfrac{m_{kn}x_ix_jx_kx_n}{R^5}\left[\tfrac{1}{c_l}\delta''(R-c_lt)-\tfrac{1}{c_t}\delta''(R-c_tt)\right] \ ,\end{gathered} \tag{36}$$

and its trace is

$$u_{ii} = \frac{T}{2\pi c_l}\frac{m_{ij}x_ix_j}{R^3}\delta''(R-c_lt) \ . \tag{37}$$

In order to compute the secondary waves we use the decomposition in Helmholtz potentials. We denote by $\mathbf{u}'$ the displacement in the secondary waves, and introduce the Helmholtz potentials Φ and $\mathbf{A}$ ($div\mathbf{A} = 0$) by $\mathbf{u}' = grad\Phi + curl\mathbf{A}$; then, we decompose the force $\mathbf{f}$ according to $\mathbf{f} = grad\varphi + curl\mathbf{H}$ ($div\mathbf{H} = 0$), where $\Delta\varphi = div\mathbf{f}$ and $\Delta\mathbf{H} = -curl\mathbf{f}$; by the equation of the elastic waves, the Helmholtz potentials satisfy the wave equations $\ddot{\Phi} - c_l^2\Delta\Phi = \varphi$, $\ddot{\mathbf{A}} - c_t^2\Delta\mathbf{A} = \mathbf{H}$. From the force given by equation (35) we get by straightforward calculations

$$\varphi = -c_l^2u_{ii} \ , \ \mathbf{H} = c_t^2curl\mathbf{u} \ . \tag{38}$$

Secondary l- and t-waves. Let us focus first on the potential φ, given by equations (37) and (38),

$$\varphi = -\frac{cT}{2\pi}\frac{m_{ij}x_ix_j}{R^3}\delta''(R-ct) \ , \tag{39}$$

where, for simplicity, we put c instead of c_l. The potential φ is limited to a surface layer of thickness Δz at the surface $z=0$, such that we introduce

$$\overline{\varphi} = -\Delta z\frac{cT}{2\pi}\left.\frac{m_{ij}x_ix_j}{R^3}\right|_{z=0}\delta''(\overline{R}-ct)\delta(z) \ , \tag{40}$$

which can also be written as

$$\overline{\varphi} = -\Delta z\frac{cT}{2\pi}\frac{m_{\alpha\beta}x_\alpha x_\beta - 2m_{\alpha3}x_\alpha z_0 + m_{33}z_0^2}{\overline{R}^3}\delta''(\overline{R}-ct)\delta(z) \ , \tag{41}$$

where $\alpha,\beta = 1,2$. From the wave equation $\ddot{\Phi} - c_l^2\Delta\Phi = \varphi$ we have the Kirchhoff solution

$$\begin{gathered}\Phi = -\Delta z\tfrac{T}{8\pi^2c}\int d\mathbf{R}'\tfrac{m_{\alpha\beta}x'_\alpha x'_\beta-2m_{\alpha3}x'_\alpha z_0+m_{33}z_0^2}{\overline{R}'^3\left|\mathbf{R}-\mathbf{R}'\right|}\delta''(\overline{R}'-ct')\delta(z') =\\ = -\Delta z\tfrac{T}{8\pi^2c}\int d\mathbf{r}'\tfrac{m_{\alpha\beta}x'_\alpha x'_\beta-2m_{\alpha3}x'_\alpha z_0+m_{33}z_0^2}{\overline{R}'^3\left|\mathbf{R}-\overline{\mathbf{R}}'\right|}\delta''(\overline{R}'-ct') \ ,\end{gathered} \tag{42}$$

where $t^{'} = t - \left|\mathbf{R} - \overline{\mathbf{R}}^{'}\right| / c$.[45]

The point of localization of the function $\delta^{''}(\overline{R}^{'} - ct^{'})$ in equation (42) gives the propagation equation

$$ct^{'} = ct - \left|\mathbf{R} - \overline{\mathbf{R}}^{'}\right| = \overline{R}^{'} \ . \tag{43}$$

The distances R, $\left|\mathbf{R} - \overline{\mathbf{R}}^{'}\right|$ and $\overline{R}^{'}$ are the sides of a triangle; we can see that the secondary waves have to go along the distance $\left|\mathbf{R} - \overline{\mathbf{R}}^{'}\right| + \overline{R}^{'}$ up to the observation point $\mathbf{R}$, which is longer than the distance R covered by the primary waves (except for $\overline{\mathbf{R}}^{'} = \mathbf{R}$). Let us take an observation point at $\mathbf{R} = (\mathbf{r}, -z_0)$ on the surface $z = 0$; for $|z_0| < ct < R$ the primary wave intersects the plane $z = 0$ along a circle with radius $r_P < r$ ($r_P = \sqrt{c^2t^2 - z_0^2}$). The primary wave has passed through all the points inside this circle, so all these points may act as sources of secondary waves. However, it is easy to see that the propagation condition (43) is not fulfilled for these points ($\overline{R}^{'} + \left|\mathbf{R} - \overline{\mathbf{R}}^{'}\right| > R$, triangle inequality). On the contrary, for $ct > R$ there exist points on the surface which satisfy the propagation condition. It follows that for $t > t_a = R/c$ secondary waves may begin to arrive at the observation point; we note that the moment of arrival of the secondary waves at the observation point is delayed in comparison with the moment the primary wave passes over the observation point, since the velocity v of the primary waves on the surface is higher than velocity c. Since the wavefront of the secondary waves arriving at the observation point is produced by sources in the proximity of the observation point, we may expect a large contribution at the arrival moment. This is the main shock, well documented in all seismic records.[3, 46] The secondary waves continue to arrive at the observation point at all the later times. Such a long tail of the seismic waves is also well documented (see, for instance, Refs. [18, 44, 47]).

Using cartesian coordinates $\mathbf{R} = (x, 0, -z_0)$, $\overline{\mathbf{R}}^{'} = (x^{'}, y^{'}, -z_0)$, the propagation condition (43) leads to the equation of a displaced ellipse

$$\frac{(x^{'} - x_0)^2}{a^2} + \frac{y^{'2}}{b^2} = 1 \ , \tag{44}$$

where

$$\begin{gathered} x_0 = \frac{A^2 x}{c^2t^2 - x^2} \ , \ a^2 = \frac{c^2t^2(c^2t^2 - R^2)^2}{4(c^2t^2 - x^2)^2} \ , \\ b^2 = \frac{(c^2t^2 - R^2)^2}{4(c^2t^2 - x^2)} \ , \ 2A^2 = c^2t^2 - x^2 + z_0^2 \ ; \end{gathered} \tag{45}$$

the eccentricity parameter of this ellipse is

$$e = \sqrt{1 - b^2/a^2} = |x|/ct \ . \tag{46}$$

The points of this ellipse are sources of the secondary waves. The ellipse of the secondary waves sources is shown in Fig. 4

For long times, $ct \gg R$, the ellipse given by equation (44) becomes a circle, with $a = b \simeq ct/2$, and the distance $\overline{R}^{'}$ tends to large values R_0, $\overline{R}^{'} \to R_0 = ct/2$; in this case, the propagation condition does not depend anymore on the angle of integration in equation (42). Since the largest part of the integration domain corresponds to $\overline{R}^{'} \simeq R_0$, we use this approximation to estimate the integral in equation (42).

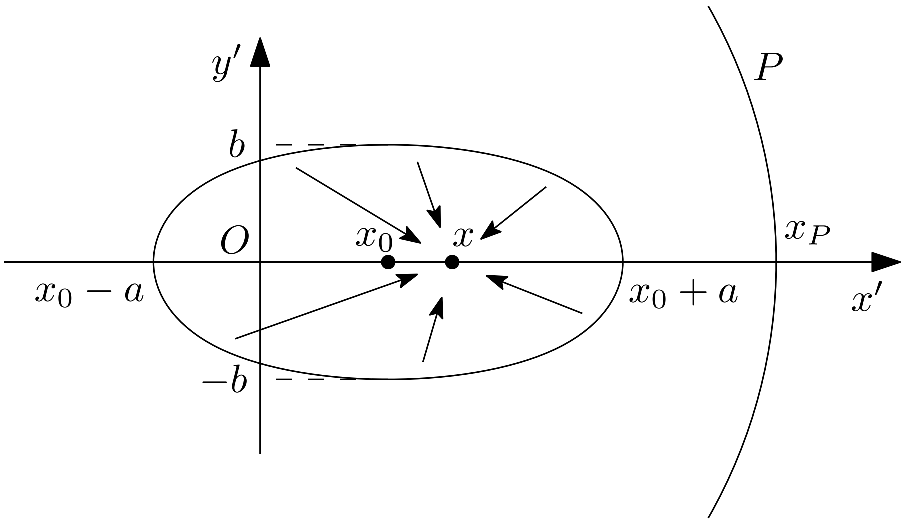

Figure 4: The ellipse of the secondary waves sources. The observation point on the surface is at $(x,0)$ and the primary spherical wave P intersects the surface on a circle of radius $x_P = r_P = (c^2t^2 - z_0^2)^{1/2}$, for large ct. The rest of notations are given in text.

It is convenient to use the displacement $\mathbf{u}' = grad\Phi$ from equation (42); in the far-field region we get

$$\mathbf{u}' \simeq -\Delta z \tfrac{T}{8\pi^2 c}\int d\mathbf{r}' \frac{m_{\alpha\beta}x'_\alpha x'_\beta - 2m_{\alpha 3}x'_\alpha z_0 + m_{33}z_0^2}{\overline{R}'^3\left|\mathbf{R}-\overline{\mathbf{R}}'\right|}\frac{\mathbf{R}-\overline{\mathbf{R}}'}{\left|\mathbf{R}-\overline{\mathbf{R}}'\right|}\delta'''(\overline{R}' - ct') \simeq$$
$$\simeq \Delta z \tfrac{T}{8\pi^2 c}\int d\mathbf{r}' \frac{m_{\alpha\beta}x'_\alpha x'_\beta - 2m_{\alpha 3}x'_\alpha z_0 + m_{33}z_0^2}{\overline{R}'^5}\overline{\mathbf{R}}'\delta'''(\overline{R}' - ct') \ ; \tag{47}$$

the integration over the angle gives

$$u'_\alpha = \Delta z \tfrac{T|z_0|}{4\pi c} m_{\alpha 3}\int dr' \cdot r' \tfrac{r'^2}{\overline{R}'^5}\delta'''(\overline{R}' - ct') \ ,$$
$$u'_3 = \Delta z \tfrac{T|z_0|}{8\pi c}\int dr' \cdot r' \tfrac{r'^2(m_{11}+m_{22})+2z_0^2 m_{33}}{\overline{R}'^5}\delta'''(\overline{R}' - ct') \ . \tag{48}$$

The integration with respect to the variable r' is performed in the vicinity of $r_0 = \sqrt{R_0^2 - z_0^2}$; to this end we insert $r' = r_0 + \rho$ in the argument $\overline{R}' - ct'$ of the function δ''' and expand it in powers of ρ; we get

$$\overline{R}' - ct' = \overline{R}' - ct + \left|\mathbf{R} - \overline{\mathbf{R}}'\right| \simeq$$
$$\simeq 2\sqrt{(r_0+\rho)^2 + z_0^2} - ct \simeq \tfrac{2r_0}{R_0}\rho \ ; \tag{49}$$

the function $\delta'''(\overline{R}' - ct')$ becomes

$$\delta'''(\overline{R}' - ct') \simeq \delta'''(2r_0\rho/R_0) = \frac{R_0^4}{16r_0^4}\frac{d^3}{d\rho^3}\delta(\rho) \ . \tag{50}$$

Similarly, we put $r' = r_0 + \rho$ in the rest of the integrands in equations (47) and expand these functions in powers of ρ; for integration we need to carry out this expansion up to the third order. Making use of

$$\overline{R}' \simeq \left|\mathbf{R} - \overline{\mathbf{R}}'\right| = R_0 + \tfrac{r_0}{R_0}\rho + \tfrac{z_0^2}{2R_0^3}\rho^2 + ... \ , \tag{51}$$

we get finally

$$u'_\alpha = \Delta z \frac{3T|z_0|m_{\alpha 3}}{32\pi c r_0^4 R_0} A \ , \ \ u'_3 = \Delta z \frac{3T|z_0|(m_{11}+m_{22})}{64\pi c r_0^4 R_0} A + \Delta z \frac{15T|z_0|^3 m_{33}}{64\pi c r_0^4 R_0^3} B \ , \tag{52}$$

where

$$A = \frac{105 r_0^6}{2R_0^6} + \frac{35 r_0^4}{R_0^4} + \frac{15 r_0^2}{2R_0^2} - 1 \ ,$$
$$B = \frac{35 r_0^4}{R_0^4} - \frac{14 r_0^2}{R_0^2} + 1 \ . \tag{53}$$

An estimation of the order of magnitude of the displacement $\mathbf{u}'$ can be obtained by noticing that the brackets in equation (53) have a weak variation with $R_0 \simeq r_0 \simeq ct/2 \gg |z_0|$; we get

$$u'_\alpha \simeq \Delta z \frac{282 T m_{\alpha 3}}{\pi c} \frac{|z_0|}{(ct)^5} \ ,$$
$$u'_3 \simeq \Delta z \frac{3T}{\pi c} \frac{|z_0|}{(ct)^5} \left[47(m_{11}+m_{22}) + 220 m_{33} \frac{z_0^2}{c^2 t^2} \right] \ ; \tag{54}$$

we can see that the displacement in the secondary waves has a long tail (t^{-5}-dependence). The greatest value of the displacement may be estimated by extending the above calculations to the time given by $ct_a = R$, when R_0 is of the order R (and $r_0 \simeq r$); for $r \ll |z_0|$ we get

$$u'_\alpha \simeq -\Delta z \frac{3T m_{\alpha 3}}{32\pi c r^4} \ ,$$
$$u'_3 \simeq -\Delta z \frac{3T}{64\pi c r^4} (m_{11} + m_{22} - 5m_{33}) \ . \tag{55}$$

In order to get a numerical estimation for the magnitude of the secondary waves we may take the thickness Δz of the order $l = cT$; then, making use of a seismic moment $M = 10^{26} dyn \cdot cm$ (earthquake's magnitude 7), density $\rho = 5g/cm^3$ ($m = M/\rho$), a mean velocity $c = 5km/s$, $r = 50km$, for an earthquake's duration $T = 0.2s$, we get a displacement of the order $10^{-3}cm$. It is worth noting that the inclusion of the structure factor of the seismic source leads to larger displacements and to irregular oscillations in the amplitude of the secondary waves, corresponding to the succession of shocks in the focal region.

The secondary t-waves are derived in the same manner, by using the ptential $\mathbf{H}$ given by equation (38) (and the displacement given by equation (25)). We get

$$H_i = \frac{cT}{2\pi} \frac{\varepsilon_{ijk} m_{kl} x_j x_l}{R^3} \delta''(R - ct) \tag{56}$$

and

$$\overline{H}_i = \Delta z \frac{cT}{2\pi} \frac{C_i}{\overline{R}^3} \delta''(R - ct)\delta(z) \ , \tag{57}$$

where

$$C_i = \varepsilon_{i\alpha k} m_{k\beta} x_\alpha x_\beta - (\varepsilon_{i\alpha k} m_{k3} + \varepsilon_{i3k} m_{k\alpha}) x_\alpha z_0 + \varepsilon_{i3k} m_{k3} z_0^2 \tag{58}$$

and c stands for c_t; the solution of the wave equation$\ddot{\mathbf{A}} - c^2 \Delta \mathbf{A} = \mathbf{H}$ is

$$A_i = \Delta z \frac{T}{8\pi^2 c} \int d\mathbf{r}' \frac{C'_i}{\overline{R}'^3 \left| \mathbf{R} - \overline{\mathbf{R}}' \right|} \delta''(\overline{R}' - ct') \ . \tag{59}$$

In the expression of the displacement $\mathbf{u}' = curl\mathbf{A}$ we perform the integrations over φ' and r' and get

$$u'_i = \Delta z \frac{3T\,|z_0|}{64\pi c r_0^4 R_0} (3m_{i3} - \delta_{i3} m_{jj}) A + \Delta z \frac{15T\,|z_0|^3}{64\pi c r_0^4 R_0^3} (m_{i3} - \delta_{i3} m_{33}) B \ , \tag{60}$$

where A and B are given by equations (53). By comparing equations (60) with equations (52), we can see that the displacement in the secondary l,t-waves is of the same order of magnitude.

Volume source. For a volume source of the form $\mathbf{F}=p(t)(\mathbf{R}/R)\theta\left(a-R\right)$ (equation (4)) we have $curl\mathbf{F}=0$; therefore, $\mathbf{H}=0$ and $\mathbf{u}_t=0$. For such a volume force we have only l-waves (dilatational waves), given by $\ddot{\mathbf{u}}_l-c_l^2\Delta\mathbf{u}_l=grad\phi$, where $\Delta\phi=div\mathbf{F}$; we may take $grad\phi=\mathbf{F}$, such that we have

$$\mathbf{u}_l=\frac{pT}{4\pi c_l^2}\int d\mathbf{R}'\frac{\delta\left(t-\left|\mathbf{R}-\mathbf{R}'\right|/c_l\right)}{\left|\mathbf{R}-\mathbf{R}'\right|}\frac{\mathbf{R}'}{R'}\theta(a-R') \tag{61}$$

for $p(t)=pT\delta(t)$. It is easy to see that $\mathbf{u}_l=u_l\mathbf{R}/R$, *i.e.* a volume source generates only longitudinal waves, as expected. From equation (61) we get

$$u_l=\frac{pT}{2c_l^2}\int_0^a dR'R'^2\int_{-1}^1 du\cdot u\frac{\delta\left(t-\sqrt{R^2+R'^2-2RR'u}/c_l\right)}{\sqrt{R^2+R'^2-2RR'u}}\ . \tag{62}$$

The argument of the δ-function has a zero for

$$-1\leq u_0=\frac{R^2+R'^2-c_l^2t^2}{2RR'}\leq 1\ , \tag{63}$$

which gives

$$u_l=\frac{pT}{4c_lR^2}\int dR'(R^2+R'^2-c_l^2t^2)\ . \tag{64}$$

The function $u_0(R')$ given by equation (63) has a minimum for $R'=\sqrt{R^2-c_l^2t^2}$ for $R>c_lt$ and $u_0(R')=1$ for $R'=R\mp c_lt$; for $R<c_lt$ the function $u_0(R')$ has a zero for $R'=\sqrt{c_l^2t^2-R^2}$ and $u_0(R')=\mp1$ for $R'=c_lt\mp R$; taking into account these conditions, we get

$$\begin{aligned}u_l=&\tfrac{pT}{4c_lR^2}[\tfrac{1}{3}a^3+(R^2-c_l^2t^2)a-(R^2-c_l^2t^2)\left|R-c_lt\right|-\\&-\tfrac{1}{3}\left|R-c_lt\right|^3]\theta\left(a-\left|R-c_lt\right|\right)\ .\end{aligned} \tag{65}$$

This wave extends within the region $-a<R-c_lt<a$ and exhibits a wavefront which moves with velocity c_l ($R=c_lt$). The function in the brackets has two extrema $\mp(a^3/24-2cta^2)$ at $R-c_lt=\mp a/2$. Making use of $p=f/a^3$ (where f is force divided by density), it is easy to see that in the limit $a\rightarrow0$ the displacement u_l given by equation (65) may be approximated by

$$u_l\simeq-\frac{mT}{2\pi c_lR}\delta'(R-c_lt)\ , \tag{66}$$

where we introduced the scalar seismic moment m (of the order $m\simeq fa$). We can see that the displacement caused by such a volume source can be obtained from the far-field displacement caused by a faulting source (equations (25)) by replacing formally in the latter the tensor m_{ij} of the seimic moment by an isotropic (scalar) seismic moment m, $m_{ij}\rightarrow-m\delta_{ij}$. The secondary waves produced by a volume source are obtained from those produced by a faulting source by the same formal procedure.

Secondary waves from plane waves. The calculations described above can be repeated for incident plane waves of the form $\mathbf{v}e^{i(\mathbf{KR}-\omega t)}$; the secondary waves displacement is given by

$$\mathbf{u}'=\frac{i(\mathbf{Kv})\mathbf{K}_l}{2\kappa_l}\Delta ze^{i(\mathbf{K}_l\mathbf{R}-\omega t)}-\frac{i\mathbf{K}_t\times(\mathbf{K}\times\mathbf{v})}{2\kappa_t}\Delta ze^{i(\mathbf{K}_t\mathbf{R}-\omega t)}\ , \tag{67}$$

where $\mathbf{K}_l = (\mathbf{k}, -\kappa_l)$ and $\mathbf{K}_t = (\mathbf{k}, -\kappa_t)$. We can see that for a longitudinal incident (primary) wave ($\mathbf{K}\times\mathbf{v} = 0$) the secondary (scattered, reflected) wave is also longitudinal; and for a transverse incident wave the secondary wave is transverse. This is different from the reflection of the plane waves from a flat surface, where the reflected wave has in general mixed polarizations. At the same time, the reflection coefficients given by equation (67) are different from the plane wave reflection coefficients. However, if the incident beam is sufficiently narrow to be approximated by a pulse, the results given by equation (67) hold. The absence of the mixed polarization is caused by the fact that in a very narrow pulse on a free surface the other polarization component has not enough time to develop. For a fixed surface mixed polarization occurs; the fixed surface is extended, while the free surface takes the localized shape of the pulse.

Discussion and conclusions. The point faulting seismic source (spatially localized source) is re-formulated in this paper, by using its localized nature and the symmetry of the seismic moment tensor. The expression obtained this way for the faulting force, by means of the spatial derivatives of the δ-function, is more convenient for the calculation of the seismic waves. Similarly, the volume seismic source is represented in this paper by means of the pressure developed in a small spherical cavity where the source is localized. It is shown that such a volume source is formally equivalent with faulting source with an isotropic (scalar) seismic moment for a δ-pulse time dependence. For extended sources, both in space and time, a structure factor of the focal region is discussed, which is responsible to a large extent for the detailed, complex structure (fine structure) of the seismograms.

For spatially localized sources with a δ-pulse time dependence the seismic waves are calculated in the transient regime (prior to the setting-up of the stationary vibrations regime), by decomposing both the displacement and the forces in the elastic waves equation by means of the Helmholtz potentials; the calculations are carried out for an isotropic elastic half-space bounded by a plane surface, which may be viewed as an approximate model for the Earth. Particular attention is paid to the problem of boundary conditions. It is shown that, for such a faulting source, the primary waves are double-shock P- and S-spherical waves, propagating with elastic wave velocities c_l and, respectively, c_t (primary l,t-waves); as expected, a volume source produces only longitudinal l-waves. The Dirac δ-function and its derivatives, appearing in the expression of these waves, are only approximate representations of functions with a small, but finite extension in space and time. On the Earth's surface the spherical waves generate a circular seismic spot of finite spread. It is shown here that the interaction of the primary waves with the surface in this region, affected by the primary waves (and propagating on the Earth's surface), the surface reaction produces additional waves sources, localized on a superficial layer, which generate secondary waves. The l,t-secondary waves have been explicitly calculated. They can be viewed as waves scattered from the Earth's surface (or from interfaces of Earth's internal layers). It is shown that the secondary waves contribute a long tail to the seismic movement, which is a well-known feature of the recorded seismograms.

The paper puts forward convenient mathematical expressions for point faulting and volume seismic sources, and the solution of the seismic wave equation in an elastic isotropic half-space in the transient-wave time interval in two steps: primary spherical waves (P- and S-waves) and the secondary waves generated by the primary waves by their interaction with the surface (waves scattered by the surface), for sources with a δ-pulse time dependence.

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