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General solution to the elastic deformation of the half-space with point forces

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Abstract

The general solution for the elastic displacement in an isotropic half-space is provided for point forces of arbitrary orientation and structure, localized either at an inner point (generalized Mindlin problem) or on the surface (generalized Boussinesq-Cerruti problem). The starting point is the decomposition of the displacement by means of the Helmholtz potentials and use of a simplified Grodskii-Neuber-Papkovitch procedure, followed by generalized Poisson equations and in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). For inner forces explicit results are given for the surface displacement. The method is applied to point tensorial forces which may appear from seismic sources governed by the seismic moment tensor. The application of the method to other similar problems and an alternate starting point to the general solution are discussed.

Introduction. The static deformation produced in an istropic elastic half-space by point forces, localized either at an inner point or on the surface, is a well-known classical problem in static Elasticity and Geotechnics. The deformation of an infinite, isotropic elastic body under the action of a point force has been calculated as early as 1848 by Kelvin [1, 2]. In the second half of the 19th century forces localized on the surface of an isotropic elastic half-space have been studied. In the Boussinesq problem [3]-[6] the point force acts perpendicular to the surface, in the Cerruti problem [7] the point force is tangential to the surface, while in the Flamant problem [8] the force perpendicular to the surface is localized along a straight line. The deformation of an isotropic elastic half-space caused by a point force localized at an inner point has been calculated by Mindlin between 1936 and 1953 [9]-[13], while the two dimensional version of the Mindlin problem, known as the Melan problem [14], was solved in 1932. In all these problems the deformation is calculated by solving the Navier-Cauchy equation of elastic equilibrium with suitable boundary conditions. The particular approaches vary from a direct application of the Green theorem to using Kelvin approach to Grodskii-Neuber-Papkovitch [15]-[17], or Helmholtz, potentials. Various accounts of these problems, at various levels of complexity, can be found in the classical treatises given in Refs. [18]-[25]. A very interesting, original, heuristic method of solving these problems is described in Ref. [26], where the method consists in guessing at solution by using the underlying symmetries. Mindlin and Boussinesq problems have been recently revisited [27], by using generalized Poisson equations and in-plane Fourier transforms, which are convenient tools for treating boundary conditions.

We extend here this method to the whole class of point forces of arbitrary orientation and structure, acting either at an inner point or on the surface of an isotropic elastic half-space. The starting point of the method is the decomposition of the displacement by means of the Helmholtz potentials, followed by a simplified version of the Grodskii-Neuber-Papkovitch procedure. The main features of the method are the use of the generalized Poisson equations and the in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). Beside generalized Mindlin and Boussinesq-Cerruti problems, the method is applied here to point tensorial forces governed by the seismic moment tensor, which may arise from seismic sources.

Helmholtz potentials. First part of solution. The equation of elastic equilibrium with the force density $\mathbf{f}$ is [28]

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}\mathrm{grad}\,\mathrm{div}\,\mathbf{u}=-\frac{2(1+\sigma)}{E}\mathbf{f}\ , \tag{1}$$

where $\mathbf{u}$ is the displacement vector (with components u_i, $i=1,2,3$), E is the Young modulus and σ is the Poisson ratio. The derivatives are taken with respect to the coordinates x_1, x_2, x_3 of a point with position vector $\mathbf{r}$. Beside the notation x_1, x_2, x_3 for coordinates we use also the notation $x=x_1$, $y=x_2$, $z=x_3$. In order to simplify the calculations (and notations) it is convenient to absorb the factor $-2(1+\sigma)/E$ in the force density $\mathbf{f}$ and re-write (1) as

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}\mathrm{grad}\,\mathrm{div}\,\mathbf{u}=\mathbf{f}\ . \tag{2}$$

We represent the solution as a sum of Helmholtz potentials Φ and $\mathbf{a}$,

$$\mathbf{u}=\mathrm{grad}\,\Phi+\mathrm{curl}\,\mathbf{a}\ ,\ \ \mathrm{div}\,\mathbf{a}=0\ , \tag{3}$$

and introduce the vector

$$\mathbf{b}=\frac{2(1-\sigma)}{1-2\sigma}\mathrm{grad}\,\Phi+\mathrm{curl}\,\mathbf{a} \tag{4}$$

which satisfies the equation

$$\Delta\mathbf{b}=\mathbf{f}\ , \tag{5}$$

derived from (2). In addition, by taking the div in equation (4), we get

$$\Delta\Phi=\frac{1-2\sigma}{2(1-\sigma)}\mathrm{div}\,\mathbf{b}\ . \tag{6}$$

Eliminating $\mathrm{curl}\,\mathbf{a}$ from (3) and (4), the solution becomes

$$\mathbf{u}=\mathbf{b}-\frac{1}{1-2\sigma}\mathrm{grad}\,\Phi\ . \tag{7}$$

Basically, this is the starting point of the Grodskii-Neuber-Papkovitch procedure [15]-[17], which continues with using the solution

$$\Phi=\frac{1-2\sigma}{4(1-\sigma)}(\mathbf{r}\cdot\mathbf{b}+\varphi) \tag{8}$$

of (6), where the potential φ satisfies the equation

$$\Delta\varphi=-\mathbf{r}\Delta\mathbf{b}=-\,\mathbf{r}\cdot\mathbf{f}\ . \tag{9}$$

Since the formation $\mathbf{r}\cdot\mathbf{f}$ is not convenient for our use of the Fourier transforms, we prefer to preserve the potential Φ given by (6). This is what we call a simplified version of the Grodskii-Neuber-Papkovitch procedure. In the infinite space, for a point force distribution, (5) and (6) lead immediately to the Kelvin solution. Our strategy for the half-space is to solve (5) for the vector potential $\mathbf{b}$, then use $\mathbf{b}$ to solve (6) for the Helmholtz scalar potential Φ and, finally, obtain the solution $\mathbf{u}$ from (7). For the domain of definition of these equations we assume an isotropic elastic half-space occupying the region $z < 0$ and bounded by the plane surface $z = 0$.

In order to prepare ourselves for tackling the boundary conditions, it is convenient to extend (5) to its generalized form [29], by introducing the vector function $\overline{\mathbf{b}} = \mathbf{b}\theta(-z)$, where $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z < 0$ is the step function. The vector $\overline{\mathbf{b}}$, which is the restriction of $\mathbf{b}$ to the domain $z < 0$, is the solution $\mathbf{b}$ of the original Poisson equation. It is easy to see, by direct calculations, that (5) becomes

$$\Delta \mathbf{b} = \mathbf{f} - \mathbf{b}^{(1)}\delta(z) - \mathbf{b}^{(0)}\delta'(z) \tag{10}$$

where $\mathbf{b}^{(0)} = \mathbf{b} \mid_{z=0}$, $\mathbf{b}^{(1)} = \frac{\partial \mathbf{b}}{\partial z} \mid_{z=0}$ (the prime on the δ-function denotes the derivative with respect to z); the superscripts (0) and (1) will be used throughout this paper for the values of the functions and, respectively, their derivative with respect to z at $z = 0$. We can see that the Green theorem is recovered from (10) for the restriction of the function $\mathbf{b}$ to the domain $z < 0$.

It is also convenient to use the projection $\boldsymbol{\rho}$ of the position vector $\mathbf{r}$ on the plane $z = 0$, corresponding to the coordinates x_1, x_2, and to introduce the in-plane Fourier transforms of the type

$$\mathbf{b}(\boldsymbol{\rho}, z) = \frac{1}{(2\pi)^2} \int dk_1 dk_2 \cdot \widetilde{\mathbf{b}}(\mathbf{k}, z) e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \ , \tag{11}$$

where the integration is extended to the whole plane of $\mathbf{k}$-vectors; k_1, k_2 are the components of the vector $\mathbf{k}$. This is a decomposition in plane waves, where $\mathbf{k}$ plays the role of a wavevector; the wavevector $\mathbf{k}$ is the argument of the Fourier transform $\widetilde{\mathbf{b}}(\mathbf{k}, z)$, and k denotes the magnitude of the vector $\mathbf{k}$. These partial (or mixed) Fourier transformations are performed only with respect to the in-plane coordinates x_1, x_2 (associated with the vector $\boldsymbol{\rho}$), while the perpendicular-to-surface coordinate $x_3 = z$ is not affected. As it is well known, the inverse Fourier transform is

$$\widetilde{\mathbf{b}}(\mathbf{k}, z) = \int dx_1 dx_2 \cdot \mathbf{b}(\boldsymbol{\rho}, z) e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} \ , \tag{12}$$

where the integration extends to the whole (x_1, x_2)-plane (the coordinates of the position vector $\boldsymbol{\rho}$). Such type of Fourier transforms are used throughout this paper for various other functions; symbols endowed with a tilde are Fourier transforms of the type given by (11) and (12).

The in-plane Fourier transform of (10) leads to

$$\frac{d^2\widetilde{\mathbf{b}}}{dz^2} - k^2\widetilde{\mathbf{b}} = \widetilde{\mathbf{f}} - \widetilde{\mathbf{b}}^{(1)}\delta(z) - \widetilde{\mathbf{b}}^{(0)}\delta'(z) \ , \tag{13}$$

where $\widetilde{\mathbf{b}}^{(0)} = \widetilde{\mathbf{b}} \mid_{z=0}$, $\widetilde{\mathbf{b}}^{(1)} = \frac{\partial \widetilde{\mathbf{b}}}{\partial z} \mid_{z=0}$; for the sake of simplicity we may omit the arguments $(\boldsymbol{\rho}, z)$ or $(\mathbf{k}, z)$, as they can be easily read from the context of the equations. Beside Roman labels $i, j, l... = 1, 2, 3$ for coordinates and vector and tensor components, we use also throughout the paper Greek suffixes $\alpha, \beta, \gamma, ... = 1, 2$ for the coordinates and components labels 1 and 2, summation over such repeated labels being implicit. With regard to the Fourier transformations given above, the derivatives ∂_α (with respect to the coordinates x_α, $\alpha = 1, 2$) applied to the δ-function

$$\delta(\boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \int dk_x dk_y \cdot e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \tag{14}$$

(or, in general, to Fourier transforms) yield factors ik_α, while the Laplacian $\partial_\alpha^2 = \partial_\alpha \partial_\alpha$ generates a factor $-k^2$ in the Fourier transform; for example, $\partial_\alpha^2 = \partial_\alpha \partial_\alpha$ applied to $\mathbf{b}$ given by (11) generates $-k^2\widetilde{\mathbf{b}}$.

It is well known that the Green function of the one-dimensional Helmholtz operator on the left of (13) is $-(1/2k)e^{-k|z|}$. Making use of this Green function, we get the solution

$$\widetilde{\mathbf{b}} = -\tfrac{1}{2k}\widetilde{\mathbf{c}} + \widetilde{\mathbf{b}}^{(0)}e^{-k|z|} \ , \tag{15}$$

where

$$\widetilde{\mathbf{c}} = \int_{-\infty}^{0} dz'\widetilde{\mathbf{f}}(z')\left[e^{-k\left|z-z'\right|} - e^{-k\left|z+z'\right|}\right] \tag{16}$$

and

$$\widetilde{\mathbf{b}}^{(1)} = k\mathbf{b}^{(0)} + \widetilde{\mathbf{g}} \ , \ \widetilde{\mathbf{g}} = \int_{-\infty}^{0} dz'\widetilde{\mathbf{f}}(z')e^{-k\left|z'\right|} \ ; \tag{17}$$

in addition we have $\widetilde{\mathbf{b}}^{(2)} == \frac{\partial^2\widetilde{\mathbf{b}}}{\partial z^2}\mid_{z=0} = k^2\widetilde{\mathbf{b}}^{(0)} + \widetilde{\mathbf{f}}^{(0)}$ directly from (13), where $\widetilde{\mathbf{b}}^{(2)}$ is the second-order derivative of $\widetilde{\mathbf{b}}$ with respect to z for $z = 0$. We can see the occurrence of the image Green function $-(1/2k)e^{-k\left|z+z'\right|}$, beside the direct Green function $-(1/2k)e^{-k\left|z-z'\right|}$, on the right of (16). We note also the useful relations $\widetilde{\mathbf{c}}^{(0)} = 0$, $\widetilde{\mathbf{c}}^{(1)} = -2k\mathbf{g}$ and $\widetilde{\mathbf{c}}^{(2)} = -2k\widetilde{\mathbf{f}}^{(0)}$, which can be obtained directly from (16).

We turn now to solving (6). It is not necessary to extend this equation to its generalized form, by introducing the boundary values $\Phi^{(0)}$and $\Phi^{(1)}$, since $\mathbf{b}^{(0)}$ with its three components (which play the role of "constants of integration") suffices to satisfy the boundary conditions; indeed, the boundary conditions consist of three equations, as many as the number of the functions $b_i^{(0)}$, $i = 1, 2, 3$; the system of equations generated by the boundary conditions is determined for the unknowns $b_i^{(0)}$; introducing $\Phi^{(0)}$and $\Phi^{(1)}$ would be superfluos. By Fourier transforming (6), with a technique similar with that used above for (5) and making use of $\widetilde{\mathbf{b}}$ given by (15), we get

$$\begin{gathered}\widetilde{\Phi} = \tfrac{(1-2\sigma)ik_\alpha}{8(1-\sigma)k^2}\int_{-\infty}^{0} dz'\widetilde{c}_\alpha(z')e^{-k\left|z-z'\right|} - \\ -\tfrac{1-2\sigma}{8(1-\sigma)k}\int_{-\infty}^{0} dz' sgn(z-z')\widetilde{c}_3(z')e^{-k\left|z-z'\right|} - \\ -\tfrac{1-2\sigma}{8(1-\sigma)k^2}\left(ik_\alpha\widetilde{b}_\alpha^{(0)} + \widetilde{b}_3^{(0)}\right)(1-2kz)\,e^{-k|z|}\end{gathered} \tag{18}$$

(by Fourier transforming, div $\mathbf{b}$ yields $ik_\alpha\widetilde{b}_\alpha + \widetilde{b}_3'$, where prime means the derivative with respect to z). In deriving (18) we use some particular forms of the integrals

$$\begin{gathered}J_+ = \int_{-\infty}^{0} dz'\,e^{-k\left|z-z'\right|}e^{-k\left|z'+z_0\right|} = \left(\tfrac{1}{2k} - z\right)e^{-k|z+z_0|} \ , \\ J_- = \int_{-\infty}^{0} dz'\,e^{-k\left|z-z'\right|}e^{-k\left|z'-z_0\right|} = \tfrac{1}{k}e^{-k|z-z_0|} + \\ + |z-z_0|\,\tfrac{1}{k}e^{-k|z-z_0|} - \tfrac{1}{2k}e^{-k|z+z_0|}\end{gathered} \tag{19}$$

and

$$\begin{gathered}J_-^s = \int_{-\infty}^{0} dz' sgn(z'-z_0)e^{-k\left|z-z'\right|}e^{-k\left|z'-z_0\right|} = \\ = (z-z_0)\,e^{-k|z-z_0|} - \tfrac{1}{2k}e^{-k|z+z_0|} \ , \\ J_0^s = \int_{-\infty}^{0} dz' sgn(z-z')e^{-k\left|z-z'\right|}e^{-k\left|z'\right|} = \left(\tfrac{1}{2k} + z\right)e^{-k|z|} \ ,\end{gathered} \tag{20}$$

valid for $z \leq 0$ and the parameter $z_0 \leq 0$.

Second part of solution. The boundary conditions. We consider a free surface $z = 0$. Consequently, the force (per unit area) with the components $p_i = -n_j\sigma_{ij}$ on the surface $z = 0$, where $\mathbf{n}$ is the unit vector normal to the surface $z = 0$ (with components $0, 0, 1$) and σ_{ij} is the stress tensor, is vanishing: $\sigma_{i3} = 0$ for $z = 0$. As it is well-known [28], the stress tensor is $\sigma_{ij} = \frac{E}{1+\sigma}[u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}]$, where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain tensor; the boundary conditions read

$$u_{\alpha 3} = 0 \ , \ (1-\sigma)u_{33} + \sigma u_{\alpha\alpha} = 0 \ , \ z = 0 \ . \tag{21}$$

We calculate the strain tensor from the Fourier transform of (7), by making use of $\widetilde{\mathbf{b}}$ given by (15) and $\widetilde{\Phi}$ given by (18). We give here the boundary values of $\widetilde{\Phi}$ and its derivatives on the surface (which enter the expressions of the strain tensor):

$$\begin{gathered}
\widetilde{\Phi}^{(0)} = \frac{(1-2\sigma)ik_\alpha \widetilde{d}_\alpha}{8(1-\sigma)k^2} - \frac{(1-2\sigma)\widetilde{d}_3}{8(1-\sigma)k} - \frac{1-2\sigma}{8(1-\sigma)k^2}\left(ik_\alpha \widetilde{b}_\alpha^{(0)} + k\widetilde{b}_3^{(0)}\right) \ , \\
\widetilde{\Phi}^{(1)} = -\frac{(1-2\sigma)ik_\alpha \widetilde{d}_\alpha}{8(1-\sigma)k} + \frac{(1-2\sigma)\widetilde{d}_3}{8(1-\sigma)} + \frac{1-2\sigma}{8(1-\sigma)k}\left(ik_\alpha \widetilde{b}_\alpha^{(0)} + k\widetilde{b}_3^{(0)}\right) \ , \\
\widetilde{\Phi}^{(2)} = \frac{(1-2\sigma)ik_\alpha \widetilde{d}_\alpha}{8(1-\sigma)} - \frac{(1-2\sigma)k\widetilde{d}_3}{8(1-\sigma)} + \frac{(1-2\sigma)\widetilde{g}_3}{2(1-\sigma)} + \\
+\frac{3(1-2\sigma)}{8(1-\sigma)}\left(ik_\alpha \widetilde{b}_\alpha^{(0)} + k\widetilde{b}_3^{(0)}\right) \ ,
\end{gathered} \tag{22}$$

where

$$\widetilde{\mathbf{d}} = \int_{-\infty}^{0} dz' \widetilde{\mathbf{c}}(z') e^{-k\left|z'\right|} \ . \tag{23}$$

Making use of these expressions the boundary conditions become

$$\begin{gathered}
k\widetilde{b}_\alpha^{(0)} + \frac{k_\alpha k_\beta \widetilde{b}_\beta^{(0)}}{4(1-\sigma)k} + \frac{i(3-4\sigma)k_\alpha \widetilde{b}_3^{(0)}}{4(1-\sigma)} = -\widetilde{g}_\alpha + \frac{k_\alpha k_\beta \widetilde{d}_\beta}{4(1-\sigma)k} + \frac{ik_\alpha \widetilde{d}_3}{4(1-\sigma)} \ , \\
i(3-4\sigma)k_\alpha \widetilde{b}_\alpha^{(0)} - (5-4\sigma)k\widetilde{b}_3^{(0)} = 4(1-\sigma)\widetilde{g}_3 - ik_\alpha \widetilde{d}_\alpha + k\widetilde{d}_3 \ .
\end{gathered} \tag{24}$$

The solutions of this algebraic system of equations are given by

$$\begin{gathered}
ik_\alpha \widetilde{b}_\alpha^{(0)} = -\frac{i(5-4\sigma)k_\alpha \widetilde{g}_\alpha}{4k} - \frac{(3-4\sigma)\widetilde{g}_3}{4} + \frac{i}{2}k_\alpha \widetilde{d}_\alpha - \frac{1}{2}k\widetilde{d}_3 \ , \\
k\widetilde{b}_3^{(0)} = -\frac{i(3-4\sigma)k_\alpha \widetilde{g}_\alpha}{4k} - \frac{(5-4\sigma)\widetilde{g}_3}{4} + \frac{i}{2}k_\alpha \widetilde{d}_\alpha - \frac{1}{2}k\widetilde{d}_3 \ , \\
\widetilde{b}_\alpha^{(0)} = -\frac{(1-4\sigma)k_\alpha k_\beta \widetilde{g}_\beta}{4k^3} + \frac{i(3-4\sigma)k_\alpha \widetilde{g}_3}{4k^2} - \frac{\widetilde{g}_\alpha}{k} + \frac{k_\alpha k_\beta \widetilde{d}_\beta}{2k^2} + \frac{ik_\alpha \widetilde{d}_3}{2k} \ .
\end{gathered} \tag{25}$$

With the functions $\widetilde{b}_i^{(0)}$ given above the solution of the problem, given by (7), (15) and (18), is completely determined; it remains to perform the reverse Fourier transforms.

Surface displacement. Mindlin general solution. We limit ourselves to give here the surface displacement

$$\widetilde{u}_\alpha^{(0)} = \widetilde{b}_\alpha^{(0)} - \frac{ik_\alpha \widetilde{\Phi}^{(0)}}{1-2\sigma} \ , \ \widetilde{u}_3^{(0)} = \widetilde{b}_3^{(0)} - \frac{\widetilde{\Phi}^{(1)}}{1-2\sigma} \ , \tag{26}$$

derived from (7). Making use of $\widetilde{b}_i^{(0)}$ from (25) and $\widetilde{\Phi}^{(0)}$and $\widetilde{\Phi}^{(1)}$ from (22), we get

$$\begin{gathered}\widetilde{u}_\alpha^{(0)}=-\frac{\widetilde{g}_\alpha}{k}+\frac{\sigma k_\alpha k_\beta \widetilde{g}_\beta}{k^3}+\frac{i(1-2\sigma)k_\alpha\widetilde{g}_3}{2k^2}+\frac{k_\alpha k_\beta\widetilde{d}_\beta}{2k^2}+\frac{ik_\alpha\widetilde{d}_3}{2k}\ ,\\ \widetilde{u}_3^{(0)}=-\frac{(1-\sigma)\widetilde{g}_3}{k}-\frac{i(1-2\sigma)k_\alpha\widetilde{g}_\alpha}{2k^2}+\frac{ik_\alpha\widetilde{d}_\alpha}{2k}-\frac{1}{2}\widetilde{d}_3\ .\end{gathered}\tag{27}$$

For the Mindlin problem we assume a point force density of the form

$$\mathbf{f}=\mathbf{f}^{(0)}\delta(\mathbf{r}-\mathbf{r}_0)=\mathbf{f}^{(0)}\delta(\boldsymbol{\rho})\delta(z-z_0)\ \ ,\tag{28}$$

where $\mathbf{f}_0^{(0)}$ is the force localized at the point with the position vector $\mathbf{r}_0$ of coordinates $0,0,z_0$, $z_0<0$. The in-plane Fourier transform of this force is

$$\widetilde{\mathbf{f}}=\mathbf{f}^{(0)}\delta(z-z_0)\ .\tag{29}$$

From (16), (17) and (23) we get

$$\begin{gathered}\widetilde{\mathbf{g}}=\mathbf{f}^{(0)}e^{-k|z_0|}\ ,\ \widetilde{\mathbf{c}}=\mathbf{f}^{(0)}\left(e^{-k|z-z_0|}-e^{-k|z+z_0|}\right)\ ,\\ \widetilde{\mathbf{d}}=-\mathbf{f}^{(0)}z_0e^{-k|z_0|}\ ;\end{gathered}\tag{30}$$

the surface displacement given by (27) becomes

$$\begin{gathered}\widetilde{u}_\alpha^{(0)}=-\left[\frac{f_\alpha^{(0)}}{k}-\frac{\sigma k_\alpha k_\beta f_\beta^{(0)}}{k^3}-\frac{i(1-2\sigma)k_\alpha f_3^{(0)}}{2k^2}+\frac{z_0k_\alpha k_\beta f_\beta^{(0)}}{2k^2}+\frac{iz_0k_\alpha f_3^{(0)}}{2k}\right]e^{-k|z_0|}\ ,\\ \widetilde{u}_3^{(0)}=-\left[\frac{(1-\sigma)f_3^{(0)}}{k}+\frac{i(1-2\sigma)k_\alpha f_\alpha^{(0)}}{2k^2}+\frac{iz_0k_\alpha f_\alpha^{(0)}}{2k}-\frac{1}{2}z_0f_3^{(0)}\right]e^{-k|z_0|}\ \ ,\end{gathered}\tag{31}$$

where $f_\alpha^{(0)}$, $\alpha=1,2$, $f_3^{(0)}$ are the components of the force $\mathbf{f}^{(0)}$. Since the multiplication by k_α of the Fourier transform is associated with the operator ∂_α, it is easy to see that the reverse Fourier transforms of (31) are given by

$$\begin{gathered}2\pi\cdot u_\alpha^{(0)}=-f_\alpha^{(0)}I^{(1)}-\frac{1}{2}f_\beta^{(0)}\partial_\beta\left[2\sigma I_\alpha^{(3)}-z_0I_\alpha^{(2)}\right]+\\ +\frac{1}{2}f_3^{(0)}\left[(1-2\sigma)I_\alpha^{(2)}-z_0I_\alpha^{(1)}\right]\ ,\\ 4\pi\cdot u_3^{(0)}=-f_3^{(0)}\left[2(1-\sigma)I^{(1)}-z_0I^{(0)}\right]-\\ -f_\alpha^{(0)}\left[(1-2\sigma)I_\alpha^{(2)}+z_0I_\alpha^{(1)}\right]\ \ ,\end{gathered}\tag{32}$$

where

$$I_\alpha^{(n)}=\partial_\alpha I^{(n)}\ ,\ I^{(n)}=\frac{1}{2\pi}\int dk_1dk_2\frac{e^{i\mathbf{k}\cdot\boldsymbol{\rho}}}{k^n}e^{-k|z_0|}\ ,\ n=0,1,2,3\ .\tag{33}$$

The integral $I^{(1)}=1/r_1$ is the Sommerfeld integral [30], where $r_1=(\rho^2+z_0^2)^{1/2}$ (ρ being the magnitude of the vector $\boldsymbol{\rho}$). By differentiating $I^{(1)}$ with respect to z_0 we get $I^{(0)}=-z_0/r_1^3$. The integral $I^{(2)}$ is singular; in Ref. [13] the function $-\ln(r_1+|z_0|)$ is used for it. However, we need $I_\alpha^{(2)}$, which is finite and can be computed by means of the Bessel function $J_0(k\rho)$; we get

$$I_\alpha^{(2)}=-\frac{x_\alpha}{r_1\left(r_1+|z_0|\right)}\ .\tag{34}$$

Similarly, the integral $I^{(3)}$ is singular, but $I_\alpha^{(3)}$ is finite and can be calculated by means of the Bessel function $J_1(k\rho)$; indeed, from (33) we have

$$I_\alpha^{(3)} = \frac{1}{2\pi}\partial_\alpha \int dk_1 dk_2 \frac{e^{i\mathbf{k}\cdot\boldsymbol{\rho}}}{k^3} e^{-k|z_0|} = \partial_\alpha \int_0^\infty dk \frac{J_0(k\rho)}{k^2} e^{-k|z_0|} =$$
$$= -\frac{x_\alpha}{\rho}\int_0^\infty dk \frac{J_1(k\rho)}{k} e^{-k|z_0|} = -\frac{x_\alpha}{r_1+|z_0|} \ , \tag{35}$$

the last integral being given in Ref. [31]. This completes the solution of the generalized Mindlin problem given by (32) for the surface displacement. It is easy to recover from (32) the two particular cases usually presented in literature [13, 27]. Indeed, for $f_3^{(0)} \neq 0$, $f_\alpha^{(0)} = 0$ we get from (32)

$$u_\rho^{(0)} = -\frac{f_3^{(0)}}{4\pi}\left[\frac{|z_0|}{r_1^2} + \frac{1-2\sigma}{r_1+|z_0|}\right]\frac{\rho}{r_1} \ , \ u_3^{(0)} = -\frac{f_3^{(0)}}{4\pi}\left[2(1-\sigma) + \frac{z_0^2}{r_1^2}\right]\frac{1}{r_1} \ , \tag{36}$$

where $u_\rho^{(0)}$ is the radial component of the displacement (along the vector $\boldsymbol{\rho}$); similarly, for $f_1^{(0)} \neq 0$, $f_2^{(0)} = f_3^{(0)} = 0$ we get from (32)

$$u_3^{(0)} = -\frac{f_1^{(0)}}{4\pi}\left[\frac{|z_0|}{r_1^2} - \frac{1-2\sigma}{r_1+|z_0|}\right]\frac{x}{r_1} \tag{37}$$

(it is usual to give only the $u_3^{(0)}$ component, because it has a simple expression). The results (36) and (37) coincide with those given in Refs. [13, 27]. We can see in (32) the separate contributions of the in-plane components $f_\alpha^{(0)}$ and the perpendicular-to-surface component $f_3^{(0)}$. For a full comparison we note that the force $\mathbf{f}^{(0)}$ in the equations given above includes the factor $-2(1+\sigma)/E$.

Tensorial force. In Seismology we need concentrated load distributions which have a total vanishing force and angular momentum. In a simplified model, the Earth may be viewed as an isotropic elastic half-space bounded by a plane surface, the seismic sources being localized beneath the surface. For sufficiently long distances the spatially localized seismic sources may be represented as point sources. The "double-couple" representation of point seismic sources by means of the seismic moment tensor emerged gradually in the first half of the 20th century [32]-[44]. Let $\mathbf{f}(\mathbf{r}) = \mathbf{f}^0 w(\mathbf{r})$ be a force density, where $\mathbf{f}^0$ is the force, $w(\mathbf{r})$ is a distribution function and $\mathbf{r}$ is the position vector of a point with coordinates x_1, x_2, x_3; a point couple along the i-th direction ($i = 1, 2, 3$) can be represented as

$$f_i^0 w(x_1+h_1, x_2+h_2, x_3+h_3) - f_i^0 w(x_1, x_2, x_3) \simeq f_i^0 h_j \partial_j w(x_1, x_2, x_3) \ , \tag{38}$$

where f_i^0, $i = 1, 2, 3$, are the components of the force, h_j, $j = 1, 2, 3$, are the components of an infinitesimal displacement $\mathbf{h}$, ∂_j denotes the derivative with respect to the coordinate x_j and summation over repeated labels is assumed. The moment $f_i^0 h_j$ is generalized to a symmetric tensor M_{ij}, which in Seismology is called the seismic moment [43]; in addition, the distribution $w(\mathbf{r})$ is replaced by $\delta(\mathbf{r}-\mathbf{r}_0)$, where δ denotes the Dirac function localized at the point with the position vector $\mathbf{r}_0$. Thus, we get a tensorial force density with the components

$$f_i = M_{ij}\partial_j \delta(\mathbf{r}-\mathbf{r}_0) \ ; \tag{39}$$

we can check immediately that the total force is vanishing and so is the total angular momentum (due to the symmetry of the tensor M_{ij}).

We apply the results obtained above to compute the surface displacement of the isotropic elastic half-space under the action of this tensorial force. We note that the solution can be obtained by the indirect method of taking the moment of forces as given by (38) in the solution of the Mindin problem of point forces. Beside the particular character of this method, which requires calculations for each component of the force moment, it may be unpracticable, due to the calculation complexity. This is why we prefer to use here the direct method described above, which leads to an elegant and compact form of solution.

The tensorial force given by (39), placed at the point with the position vector $\mathbf{r}_0$ of coordinates $0, 0, z_0$, $z_0 < 0$, has the components

$$f_j = M_{jl}\partial_l\delta(\mathbf{r}-\mathbf{r}_0) = M_{j\alpha}\partial_\alpha\delta(\boldsymbol{\rho})\delta(z-z_0) + M_{j3}\delta(\boldsymbol{\rho})\delta^{'}(z-z_0) \; ; \tag{40}$$

their in-plane Fourier transform are

$$\widetilde{f}_j = iM_{j\alpha}k_\alpha\delta(z-z_0) + M_{j3}\delta^{'}(z-z_0) \; . \tag{41}$$

From (16), (17) and (23) we get

$$\begin{gathered} \widetilde{g}_j = (iM_{j\alpha}k_\alpha - M_{j3}k)\, e^{-k|z_0|} \; , \\ \widetilde{c}_j = iM_{j\alpha}k_\alpha \left(e^{-k|z-z_0|} - e^{-k|z+z_0|}\right) - \\ -M_{j3}k\left[sgn(z-z_0)e^{-k|z-z_0|} - e^{-k|z+z_0|}\right] \; , \\ \widetilde{d}_j = \left[-iz_0M_{j\alpha}k_\alpha + M_{j3}(1+kz_0)\right] e^{-k|z_0|} \; . \end{gathered} \tag{42}$$

Inserting these quantities in (31) we get

$$\begin{gathered} \widetilde{u}_\alpha^{(0)} = [M_{\alpha 3} - \tfrac{iM_{\alpha\beta}k_\beta}{k} + \tfrac{iM_{\beta\gamma}k_\alpha k_\beta k_\gamma}{2k^3}(2\sigma - kz_0) + \\ + \tfrac{z_0M_{3\beta}k_\alpha k_\beta}{k} + \tfrac{iM_{33}k_\alpha}{2k}(2\sigma + kz_0)]e^{-k|z_0|} \; , \\ \widetilde{u}_3^{(0)} = [\tfrac{M_{\alpha\beta}k_\alpha k_\beta}{2k^2}(1 - 2\sigma + kz_0) + iz_0M_{3\alpha}k_\alpha + \\ + \tfrac{1}{2}M_{33}(1 - 2\sigma - kz_0)]e^{-k|z_0|} \end{gathered} \tag{43}$$

and, with the reverse Fourier transforms,

$$\begin{gathered} 2\pi \cdot u_\alpha^{(0)} = -M_{\alpha\beta}I_\beta^{(1)} + M_{\alpha 3}I^{(0)} - \tfrac{1}{2}M_{\beta\gamma}\partial_\beta\partial_\gamma[2\sigma I_\alpha^{(3)} - z_0I_\alpha^{(2)}] - \\ -z_0M_{3\beta}\partial_\beta I_\alpha^{(1)} + \tfrac{1}{2}M_{33}[2\sigma I_\alpha^{(1)} + z_0I_\alpha^{(0)}] \; , \\ 2\pi \cdot u_3^{(0)} = -\tfrac{1}{2}M_{\alpha\beta}\partial_\beta[(1-2\sigma)I_\alpha^{(2)} + z_0I_\alpha^{(1)}] + \\ +z_0M_{3\alpha}I_\alpha^{(0)} + \tfrac{1}{2}M_{33}[(1-2\sigma)I^{(0)} - z_0\tfrac{\partial}{\partial z_0}I^{(0)}] \; . \end{gathered} \tag{44}$$

Making use of the integrals $I_\alpha^{(n)}$ and $I^{(0)}$ given above, we get from (44) the components $u_\alpha^{(0)}$ and $u_3^{(0)}$ of the surface displacement caused by a point tensorial force localized beneath the surface of an isotropic elastic half-space. We can see from (44) that $u_\alpha^{(0)}$ is vanishing for $\rho \to 0$ and goes like $1/\rho^2$ for $\rho \to \infty$; it attains a maximum value for distances ρ of the order of $|z_0|$. The component

$u_3^{(0)}$ goes like $1/z_0^2$ for $\rho \to 0$ and like $1/\rho^2$ for $\rho \to \infty$. A simplified version of (44) is obtained for $M_{ij} = M\delta_{ij}$; noting that $\partial_\beta^2 I_\alpha^{(n)} = -I_\alpha^{(n-2)}$, we get

$$u_\alpha^{(0)} = -\frac{1-2\sigma}{2\pi} M I_\alpha^{(1)} = \frac{1-2\sigma}{2\pi} M \frac{x_\alpha}{r_1^3} \ , \ u_3^{(0)} = \frac{1-2\sigma}{2\pi} M I^{(0)} = \frac{1-2\sigma}{2\pi} M \frac{|z_0|}{r_1^3} \tag{45}$$

(in deriving (45) $\partial_\alpha^2 I^{(1)}$ occurs, which may be written as $-I^{(-1)} = -\partial I^{(0)}/\partial z_0$ by extending the definition of the integrals $I^{(n)}$ to $n = -1$; the same integral appears also in (44)).

Force acting on the surface. If the force density $\mathbf{p}$ is applied on the surface $z = 0$, (volume) force terms do not appear anymore in the generalized Poisson equation, but the external loads appear in the boundary conditions, which read

$$u_{\alpha 3} = -\frac{1+\sigma}{E} p_\alpha \ , \ (1-\sigma)u_{33} + \sigma u_{\alpha\alpha} = -\frac{(1+\sigma)(1-2\sigma)}{E} p_3 \ , \ z = 0 \ ; \tag{46}$$

it is convenient to absorb the factor $(1+\sigma)/E$ in the force density $\mathbf{p}$. The quantities $\widetilde{\mathbf{g}}$, $\widetilde{\mathbf{c}}$ and $\widetilde{\mathbf{d}}$ introduced above (in (16), (17) and (23)) are vanishing, and the expressions for $\widetilde{\mathbf{b}}$ and $\widetilde{\Phi}$ (given by (15) and, respectively, (18)) are simplified. For a general point force density on the surface given by $\mathbf{p} = \mathbf{p}^{(0)}\delta(\boldsymbol{\rho})$, where $\mathbf{p}^{(0)}$ is the force, the boundary conditions (24) become

$$\begin{gathered} k\widetilde{b}_\alpha^{(0)} + \frac{k_\alpha k_\beta \widetilde{b}_\beta^{(0)}}{4(1-\sigma)k} + \frac{i(3-4\sigma)k_\alpha \widetilde{b}_3^{(0)}}{4(1-\sigma)} = -p_\alpha^{(0)} \ , \\ i(3-4\sigma)k_\alpha \widetilde{b}_\alpha^{(0)} - (5-4\sigma)k\widetilde{b}_3^{(0)} = 8(1-\sigma)p_3^{(0)} \ , \end{gathered} \tag{47}$$

where p_α, $\alpha = 1,2$, p_3 are the components of the force $\mathbf{p}_0$. The solutions of the system of equations (47) are

$$\begin{gathered} ik_\alpha \widetilde{b}_\alpha^{(0)} = -\frac{i(5-4\sigma)k_\alpha p_\alpha^{(0)}}{4k} - \frac{(3-4\sigma)p_3^{(0)}}{2} \ , \\ k\widetilde{b}_3^{(0)} = -\frac{i(3-4\sigma)k_\alpha p_\alpha^{(0)}}{4k} - \frac{(5-4\sigma)p_3^{(0)}}{2} \ , \\ \widetilde{b}_\alpha^{(0)} = -\frac{p_\alpha^{(0)}}{k} - \frac{(1-4\sigma)k_\alpha k_\beta p_\beta^{(0)}}{4k^3} + \frac{i(3-4\sigma)k_\alpha p_3^{(0)}}{2k^2} \ . \end{gathered} \tag{48}$$

Making use of $\widetilde{\mathbf{b}}$ and $\widetilde{\Phi}$ given by (15) and, respectively, (18), we obtain the Fourier transform of the displacement

$$\begin{gathered} \widetilde{u}_\alpha = \left[-\frac{p_\alpha^{(0)}}{k} + \frac{\sigma k_\alpha k_\beta p_\beta^{(0)}}{k^3} + \frac{i(1-2\sigma)k_\alpha p_3^{(0)}}{k^2} + \frac{izk_\alpha}{2k^2}\left(ik_\beta p_\beta^{(0)} + 2kp_3^{(0)} \right) \right] e^{-k|z|} \ , \\ \widetilde{u}_3 = \left[-\frac{i(1-2\sigma)k_\alpha p_\alpha^{(0)}}{2k^2} - \frac{2(1-\sigma)p_3^{(0)}}{k} + \frac{z}{2k}\left(ik_\alpha p_\alpha^{(0)} + 2kp_3^{(0)} \right) \right] e^{-k|z|} \end{gathered} \tag{49}$$

and, by reverse Fourier transforming,

$$\begin{gathered} 2\pi \cdot u_\alpha = -p_\alpha^{(0)} \overline{I}^{(1)} - \tfrac{1}{2} p_\beta^{(0)} \partial_\beta \left[2\sigma \overline{I}_\alpha^{(3)} - z\overline{I}_\alpha^{(2)} \right] + \\ + p_3^{(0)} \left[(1-2\sigma)\overline{I}_\alpha^{(2)} + z\overline{I}_\alpha^{(1)} \right] \ , \\ 2\pi \cdot u_3 = -\tfrac{1}{2} p_\alpha^{(0)} \left[(1-2\sigma)\overline{I}_\alpha^{(2)} - z\overline{I}_\alpha^{(1)} \right] - \\ - p_3^{(0)} \left[2(1-\sigma)\overline{I}^{(1)} - z\overline{I}^{(0)} \right] \ , \end{gathered} \tag{50}$$

where the integrals $\overline{I}^{(n)}$, $\overline{I}_\alpha^{(n)}$ are the corresponding integrals $I^{(n)}$, $I_\alpha^{(n)}$ given by (33) with z_0 replaced by z. This is the general solution of the Boussinesq-Cerruti problem for a point force of arbitrary orientation acting on the surface of the half-space. For the particular case $p_\alpha^{(0)} = 0$ (force perpendicular to the surface, Boussinesq problem) we obtain from (50)

$$\begin{aligned} u_\alpha &= -\tfrac{p_3^{(0)}}{2\pi}\left[(1-2\sigma)\tfrac{1}{r+|z|} + \tfrac{z}{r^2}\right]\tfrac{x_\alpha}{r} \ , \\ u_3 &= -\tfrac{p_3^{(0)}}{2\pi}\left[2(1-\sigma) + \tfrac{z^2}{r^2}\right]\tfrac{1}{r} \ , \end{aligned} \tag{51}$$

where $r = (\rho^2 + z^2)^{1/2}$, which coincide with the well-known results given in Refs. [19], [27] and [45]. The solution for the Cerruti problem is obtained from (50) for $p_3^{(0)} = p_2^{(0)} = 0$ and $p_1^{(0)} \neq 0$.

Discussion and final remarks. The deformation of an isotropic elastic half-space is calculated in this paper for a point force of arbitrary orientation and structure concentrated either at an inner point or on the surface. This is a generalization of the well-known Mindlin and Boussinesq-Cerruti problems. The results are used to get the deformation produced in the half-space by a tensorial force localized beneath the surface, which may arise from seismic sources. For Mindlin problem and the tensorial force explicit results are given for surface deformation. The problem is solved by a special method which implies the generalized Poisson equation and the use of in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). The starting point is the decomposition of the displacement by means of the Helmholtz potentials and the use of a simplified version of the Grodskii-Neuber-Papkovitch procedure. The method used here is particularly convenient for accounting for the boundary conditions. For inhomogeneous boundary conditions, the (volume) force is absent in the generalized Poisson equation, which includes only the values of the functions and their normal derivatives at the surface as "force" terms, but the external loads appear in the boundary conditions. In Fourier transforms, the boundary conditions generate a system of algebraic equations which can be solved for the values of the functions at the surface, thus providing a completely determined solution. The method can be extended to other geometries, or other similar problems (like a thick plate, for instance). The solution to the problems regarding forces concentrated along infinite lines (Melan and Flamant problems) can be obtained immediately by a direct integration of the solutions of the corresponding point forces.

Finally we comment upon another starting point to the general solution of equilibrium of the isotropic elastic half-space. In (2) we introduce the notation

$$D = div\mathbf{u} \ , \tag{52}$$

such that (2) becomes

$$\Delta\mathbf{u} = \mathbf{f} - \frac{1}{1-2\sigma}\mathrm{grad}\, D \ . \tag{53}$$

These equations may be considered as an alternate starting point with respect to the simplified version of the Grodskii-Neuber-Papkovitch procedure used here. In order to satisfy the boundary conditions it suffices to generalize (53) to

$$\Delta\mathbf{u} = \mathbf{f} - \frac{1}{1-2\sigma}\mathrm{grad}\, D - \mathbf{u}^{(1)}\delta(z) - \mathbf{u}^{(0)}\delta'(z) \ , \tag{54}$$

while maintaining (52); we solve (54) for $\mathbf{u}$ and use this solution to compute D from (52), which provides a self-consistency relation which gives D and, finally, $\mathbf{u}$. The parameters $\mathbf{u}^{(0)}$ ($\mathbf{u}^{(1)}$ are given in terms of $\mathbf{u}^{(0)}$ by the solution of (54)) are sufficient to satisfy the boundary conditions, which become an algebraic system of equations with unknowns $\mathbf{u}^{(0)}$. This D-procedure is entirely equivalent with the $(\mathbf{b}, \Phi)$- procedure used here.

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