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Deformation of an isotropic elastic half-space under the action of point forces. General solution

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Abstract

The general solution of static equilibrium of an isotropic elastic half-space is provided for point forces of arbitrary orientation and structure, localized either at an inner point (generalized Mindlin problem) or on the surface (generalized Boussinesq-Cerruti problem). The method makes use of generalized Poisson equations and in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). The results are applied to a point tensorial force which may arise from seismic sources governed by the seismic moment tensor. For inner forces explicit results are given for the surface displacement.

Introduction. The static deformation produced in an isotropic elastic half-space by point forces, localized either at an inner point or on the surface, is a classical problem in Elasticity and Geotechnics. The deformation of an infinite, isotropic elastic body under the action of a point force has been calculated as early as 1848 by Kelvin.[1, 2] In the second half of the 19th century the effect of forces localized on the surface of an isotropic elastic half-space have been studied. In the Boussinesq problem[3]-[6] the point force acts perpendicular to the surface, in the Cerruti problem[7] the point force is tangential to the surface, while in the Flamant problem[8] the force perpendicular to the surface is localized along a straight line. The deformation of an isotropic elastic half-space caused by a point force localized at an inner point has been calculated by Mindlin between 1936 and 1953,[9]-[13] while the two dimensional version of the Mindlin problem, known as the Melan problem,[14] was solved in 1932. In all these problems the deformation is calculated by solving the Navier-Cauchy equation of elastic equilibrium with suitable boundary conditions. The particular approaches vary from a direct application of the Green theorem to using Kelvin approach to Grodskii-Neuber-Papkovitch,[15]-[17] or Helmholtz, potentials. Various accounts of these problems, at various levels of complexity, can be found in the classical treatises given in Refs. [18]-[25]. A very interesting, original, heuristic method of solving these problems is described in Ref. [26], where the method consists in guessing at solution by using the underlying symmetries.

We give here the general solution to the problem of static equilibrium of an isotropic elastic half-space with point forces of arbitrary orientation and structure, localized either at an inner point or on the surface. The method makes use of generalized Poisson equations and in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). Beside generalized Mindlin and Boussinesq-Cerruti problems, the method is applied here to point tensorial forces governed by the seismic moment tensor, which may arise from seismic sources.

Internal point force. We consider an isotropic elastic half-space occupying the region $z < 0$, bounded by a plane, free surface $z = 0$ and a point force $\mathbf{f}$ localized at the inner point $\mathbf{R}_0$

with coordinates $(0,0,z_0)$, $z_0<0$. The equation of elastic equilibrium with the force density $\mathbf{F}=\mathbf{f}\delta(\mathbf{R}-\mathbf{R}_0)$ is[27]

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}grad\cdot div\mathbf{u}=-\frac{2(1+\sigma)}{E}\mathbf{F}\ , \tag{1}$$

where $\mathbf{u}$ is the displacement vector, E is the Young modulus and σ is the Poisson ratio. Beside the notation x,y,z for the coordinates of the point with the position vector $\mathbf{R}$ we use also the notation x_i, $i=1,2,3$. In order to simplify the calculations (and notations) it is convenient to absorb the factor $-2(1+\sigma)/E$ in the force density $\mathbf{F}$ and re-write (1) as

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}grad\cdot div\mathbf{u}=\mathbf{F}\ . \tag{2}$$

We introduce the notation

$$D=div\mathbf{u}\ , \tag{3}$$

and write equation (2) as

$$\Delta\mathbf{u}=\mathbf{F}-\frac{1}{1-2\sigma}gradD\ . \tag{4}$$

In order to prepare ourselves for tackling the boundary conditions, it is convenient to extend (4) to its generalized form,[28] by introducing the $\overline{\mathbf{u}}=\mathbf{u}\theta(-z)$, where $\theta(z)=1$ for $z>0$ and $\theta(z)=0$ for $z<0$ is the step function. The vector $\overline{\mathbf{u}}$, which is the restriction of $\mathbf{u}$ to the domain $z<0$, is the solution $\mathbf{u}$ of the original Poisson equation. It is easy to see, by direct calculations, that (4) becomes

$$\Delta\mathbf{u}=\mathbf{F}-\frac{1}{1-2\sigma}gradD-\mathbf{u}^1\delta(z)-\mathbf{u}^0\delta^{'}(z) \tag{5}$$

where $\mathbf{u}^0=\mathbf{u}\mid_{z=0}$, $\mathbf{u}^1=\frac{\partial\mathbf{u}}{\partial z}\mid_{z=0}$; the superscripts 0 and 1 will be used throughout this paper for the values of the functions and, respectively, their derivative with respect to z at $z=0$. The prime over the δ-function in (5) means the derivative with respect to z. We can see that the Green theorem is recovered from (5) for the restriction of the function $\mathbf{u}$ to the domain $z<0$.

It is also convenient to use the projection $\boldsymbol{r}$ of the position vector $\mathbf{R}$ on the plane $z=0$, corresponding to the coordinates (x_1,x_2), and to introduce the in-plane Fourier transforms of the type

$$\mathbf{u}(\boldsymbol{r},z)=\frac{1}{(2\pi)^2}\int d\mathbf{k}\cdot\tilde{\mathbf{u}}(\mathbf{k},z)e^{i\mathbf{kr}}\ ; \tag{6}$$

for the sake of simplicity we omit the tilde over the Fourier transforms and the arguments $(\boldsymbol{r},z)$ or $(\mathbf{k},z)$, as they can be easily read from the context of the equations. Beside Roman labels $i,j,l...=1,2,3$ for coordinates and vector and tensor components, we use also throughout the paper Greek suffixes $\alpha,\beta,\gamma,...=1,2$ for the coordinates and components labels 1 and 2, summation over such repeated labels being implicit. The in-plane Fourier transform of (5) leads to

$$\begin{gathered}\frac{d^2u_\alpha}{dz^2}-k^2u_\alpha=F_\alpha-\frac{ik_\alpha}{1-2\sigma}D-u_\alpha^1\delta(z)-u_\alpha^0\delta^{'}(z)\ ,\\ \frac{d^2u_3}{dz^2}-k^2u_3=F_3-\frac{1}{1-2\sigma}D^{'}-u_3^1\delta(z)-u_3^0\delta^{'}(z)\end{gathered} \tag{7}$$

for the components of the displacement $\mathbf{u}$. It is well known that the Green function of the one-dimensional Helmholtz operator on the left of (7) is $-(1/2k)e^{-k|z|}$. Making use of this Green function, we get the solution

$$u_\alpha = -\frac{1}{2k}\int_{-\infty}^{0} dz^{'} F_\alpha(z^{'})\left(e^{-k\left|z-z^{'}\right|} - e^{-k\left|z+z^{'}\right|}\right) +$$

$$+\frac{ik_\alpha}{2(1-2\sigma)k}\int_{-\infty}^{0} dz^{'} D(z^{'})\left(e^{-k\left|z-z^{'}\right|} - e^{-k\left|z+z^{'}\right|}\right) + u_\alpha^0 e^{-k|z|} \ , \tag{8}$$

$$u_3 = -\frac{1}{2k}\int_{-\infty}^{0} dz^{'} F_3(z^{'})\left(e^{-k\left|z-z^{'}\right|} - e^{-k\left|z+z^{'}\right|}\right) -$$

$$-\frac{1}{2(1-2\sigma)}\int_{-\infty}^{0} dz^{'} D(z^{'})\left[sgn(z-z^{'})e^{-k\left|z-z^{'}\right|} - e^{-k\left|z+z^{'}\right|}\right] + u_3^0 e^{-k|z|} \ ;$$

in addition, we have

$$u_\alpha^1 = \int_{-\infty}^{0} dz^{'} F_\alpha(z^{'})e^{-k\left|z^{'}\right|} - \frac{ik_\alpha}{1-2\sigma}\int_{-\infty}^{0} dz^{'} D(z^{'})e^{-k\left|z^{'}\right|} + ku_\alpha^0 \tag{9}$$

$$u_3^1 = \int_{-\infty}^{0} dz^{'} F_3(z^{'})e^{-k\left|z^{'}\right|} - \frac{D^0}{1-2\sigma} + \frac{k}{1-2\sigma}\int_{-\infty}^{0} dz^{'} D(z^{'})e^{-k\left|z^{'}\right|} + ku_3^0 \ .$$

We can see the occurrence of the image Green function $-(1/2k)e^{-k\left|z+z^{'}\right|}$, beside the direct Green function $-(1/2k)e^{-k\left|z-z^{'}\right|}$, on the right of (8).

We turn now to the boundary conditions. For a free surface the force (per unit area) with the components $P_i = -n_j\sigma_{ij}$ on the surface $z=0$, where $\mathbf{n}$ is the unit vector normal to the surface $z=0$ (with components $(0,0,1)$) and σ_{ij} is the stress tensor, is vanishing: $\sigma_{i3}=0$ for $z=0$. As it is well-known,[27] the stress tensor is $\sigma_{ij} = \frac{E}{1+\sigma}[u_{ij} + \frac{\sigma}{1-2\sigma}u_{kk}\delta_{ij}]$, where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the strain tensor; the boundary conditions read

$$u_{\alpha 3} = 0 \ , \ (1-\sigma)u_{33} + \sigma u_{\alpha\alpha} = 0 \ , \ z=0 \ . \tag{10}$$

Making use of the displacement $\mathbf{u}$ given by equations (8) the boundary conditions read

$$ku_\alpha^0 + ik_\alpha u_3^0 = -G_\alpha + \frac{ik_\alpha d}{1-2\sigma} \ , \tag{11}$$

$$i\sigma k_\alpha u_\alpha^0 + (1-\sigma)ku_3^0 = -(1-\sigma)G_3 + \frac{1-\sigma}{1-2\sigma}D^0 - \frac{(1-\sigma)kd}{1-2\sigma} \ ,$$

where we have introduced the notations

$$\mathbf{G} = \int_{-\infty}^{0} dz^{'}\mathbf{F}(z^{'})e^{-k\left|z^{'}\right|} \ , \ d = \int_{-\infty}^{0} dz^{'} D(z^{'})e^{-k\left|z^{'}\right|} \ . \tag{12}$$

The solutions of the system of equations (11) are

$$u_\alpha^0 = -\frac{G_\alpha}{k} + \sigma\frac{k_\alpha k_\beta G_\beta}{k^3} + (1-\sigma)\frac{ik_\alpha G_3}{k^2} - \frac{1-\sigma}{1-2\sigma}\frac{ik_\alpha D^0}{k^2} + \frac{2(1-\sigma)}{1-2\sigma}\frac{ik_\alpha d}{k} \ , \tag{13}$$

$$u_3^0 = \sigma\frac{ik_\alpha G_\alpha}{k^2} - (1-\sigma)\frac{G_3}{k} + \frac{1-\sigma}{1-2\sigma}\frac{D^0}{k} - d \ .$$

We can see that the solution $\mathbf{u}$ given by equations (8) and (13) includes the unknowns D^0 and d; these can be obtained from equation (3), which becomes

$$D = ik_\alpha u_\alpha + u_3^{'} \ . \tag{14}$$

Taking $z=0$ in this equation and using equations (9) and (13) we get

$$ik_\alpha u_\alpha^0 + ku_3^0 = -G_3 + \frac{2(1-\sigma)}{1-2\sigma}D^0 - \frac{kd}{1-2\sigma} \ . \quad (15)$$

Now we multiply equation (14) by $e^{-k|z|}$ and use the displacement **u** given by equations (8) and the integrals

$$\begin{gathered}\int_{-\infty}^0 dz\cdot e^{-k|z|}e^{-k\left|z-z^{'}\right|} = \left(\tfrac{1}{2k}-z^{'}\right)e^{-k\left|z^{'}\right|} \ , \\ \int_{-\infty}^0 dz\cdot e^{-k|z|}e^{-k\left|z+z^{'}\right|} = \tfrac{1}{2k}e^{-k\left|z^{'}\right|} \ , \\ \int_{-\infty}^0 dz\cdot sgn(z-z^{'})e^{-k|z|}e^{-k\left|z-z^{'}\right|} = -\left(\tfrac{1}{2k}+z^{'}\right)e^{-k\left|z^{'}\right|} \ ;\end{gathered} \quad (16)$$

we get

$$ik_\alpha u_\alpha^0 + ku_3^0 = \tfrac{3-4\sigma}{1-2\sigma}kd - ik_\alpha A_\alpha + kA_3 \ \ , \quad (17)$$

where

$$\mathbf{A} = \int_{-\infty}^0 dz^{'}z^{'}\mathbf{F}(z^{'})e^{-k\left|z^{'}\right|} \ . \quad (18)$$

Using $\mathbf{u}^0$ given by equations (13) we get

$$\begin{gathered}\tfrac{2(1-\sigma)}{1-2\sigma}D^0 = -2(1-\sigma)\tfrac{ik_\alpha G_\alpha}{k} - (1-2\sigma)G_3 - ik_\alpha A_\alpha + kA_3 \ , \\ 2kd = -(1-2\sigma)\left(\tfrac{ik_\alpha G_\alpha}{k} + G_3\right)\end{gathered} \quad (19)$$

from equations (15) and (17). With D^0 and d given above and $\mathbf{u}^0$ given by equations (13) the displacement **u** given by equations (8) is completely determined for any force density **F**. It remains to perform the inverse Fourier transforms.

For the generalized Mindlin problem we take $\mathbf{F} = \mathbf{f}\delta(\mathbf{R}-\mathbf{R}_0)$, $\mathbf{R}_0 = (0,0,z_0)$, for which we have

$$\mathbf{G} = \mathbf{f}e^{-k|z_0|} \ , \ \mathbf{A} = z_0\mathbf{f}e^{-k|z_0|} \ . \quad (20)$$

We limit ourselevs here to give only the surface displacement $\mathbf{u}^0$. From equations (13) we get

$$\begin{gathered}u_\alpha^0 = -\left[\tfrac{f_\alpha}{k} - \tfrac{\sigma k_\alpha k_\beta f_\beta}{k^3} - \tfrac{i(1-2\sigma)k_\alpha f_3}{2k^2} + \tfrac{z_0 k_\alpha k_\beta f_\beta}{2k^2} + \tfrac{iz_0 k_\alpha f_3}{2k}\right]e^{-k|z_0|} \ , \\ u_3^0 = -\left[\tfrac{(1-\sigma)f_3}{k} + \tfrac{i(1-2\sigma)k_\alpha f_\alpha}{2k^2} + \tfrac{iz_0 k_\alpha f_\alpha}{2k} - \tfrac{1}{2}z_0 f_3\right]e^{-k|z_0|} \ \ ,\end{gathered} \quad (21)$$

or, with the reverse Fourier transforms,

$$\begin{gathered}2\pi\cdot u_\alpha^0 = -f_\alpha I^{(1)} - \tfrac{1}{2}f_\beta\partial_\beta\left[2\sigma I_\alpha^{(3)} - z_0 I_\alpha^{(2)}\right] + \\ +\tfrac{1}{2}f_3\left[(1-2\sigma)I_\alpha^{(2)} - z_0 I_\alpha^{(1)}\right] \ , \\ 4\pi\cdot u_3^0 = -f_3\left[2(1-\sigma)I^{(1)} - z_0 I^{(0)}\right] - \\ -f_\alpha\left[(1-2\sigma)I_\alpha^{(2)} + z_0 I_\alpha^{(1)}\right] \ \ ,\end{gathered} \quad (22)$$

where

$$I_\alpha^{(n)}=\partial_\alpha I^{(n)}\ ,\ I^{(n)}=\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{k}\mathbf{r}}}{k^n}e^{-k|z_0|}\ ,\ n=0,1,2,3\ . \tag{23}$$

The integral $I^{(1)}=1/R_1$ is the Sommerfeld integral,[29] where $R_1=(r^2+z_0^2)^{1/2}$. By differentiating $I^{(1)}$ with respect to z_0 we get $I^{(0)}=-z_0/R_1$. The integral $I^{(2)}$ is singular; in Ref. [13] the function $-\ln(R_1+|z_0|)$ is used for it. However, we need $I_\alpha^{(2)}$, which is finite and can be computed by means of the Bessel function $J_0(kr)$; we get

$$I_\alpha^{(2)}=-\frac{x_\alpha}{R_1\left(R_1+|z_0|\right)}\ . \tag{24}$$

Similarly, the integral $I^{(3)}$ is singular, but $I_\alpha^{(3)}$ is finite and can be calculated by means of the Bessel function $J_1(kr)$; indeed, from (23) we have

$$\begin{gathered}I_\alpha^{(3)}=\frac{1}{2\pi}\partial_\alpha\int d\mathbf{k}\frac{e^{i\mathbf{k}\mathbf{r}}}{k^3}e^{-k|z_0|}=\partial_\alpha\int_0^\infty dk\frac{J_0(kr)}{k^2}e^{-k|z_0|}=\\ =-\frac{x_\alpha}{r}\int_0^\infty dk\frac{J_1(kr)}{k}e^{-k|z_0|}=-\frac{x_\alpha}{R_1+|z_0|}\ ,\end{gathered} \tag{25}$$

the last integral being given in Ref. [30]. This completes the solution of the generalized Mindlin problem given by (22) for the surface displacement. It is easy to recover from (22) the two particular cases usually presented in literature.[13, 26] Indeed, for $f_3^0\neq 0$, $f_\alpha^0=0$ we get from (22)

$$u_r^0=-\frac{f_3}{4\pi}\left[\frac{|z_0|}{R_1^2}+\frac{1-2\sigma}{R_1+|z_0|}\right]\frac{r}{R_1}\ ,\ u_3^0=-\frac{f_3}{4\pi}\left[2(1-\sigma)+\frac{z_0^2}{R_1^2}\right]\frac{1}{R_1}\ , \tag{26}$$

where u_r^0 is the radial component of the displacement; similarly, for $f_1\neq 0$, $f_2=f_3=0$ we get from (22)

$$u_3^0=-\frac{f_1}{4\pi}\left[\frac{|z_0|}{R_1^2}-\frac{1-2\sigma}{R_1+|z_0|}\right]\frac{x}{R_1} \tag{27}$$

(it is usual to give only the u_3^0 component, because it has a simple expression). The results (26) and (27) coincide with those given in Refs. [13, 26]. We can see in (22) the separate contributions of the in-plane components f_α and the perpendicular-to-surface component f_3. For a full comparison we note that the force $\mathbf{f}$ in the equations given above includes the factor $-2(1+\sigma)/E$.

Tensorial force. In Seismology we need concentrated load distributions which have a total vanishing force and angular momentum. In a simplified model, the Earth may be viewed as an isotropic elastic half-space bounded by a plane surface, the seismic sources being localized beneath the surface. For sufficiently long distances the spatially localized seismic sources may be represented as point sources. The "double-couple" representation of point seismic sources by means of the seismic moment tensor emerged gradually in the first half of the 20th century [31]-[43]. Let $\mathbf{F}(\mathbf{R})=\mathbf{f}g(\mathbf{R})$ be a force density, where $\mathbf{f}$ is the force and $g(\mathbf{R})$ is a distribution function; a point couple along the i-th direction ($i=1,2,3$) can be represented as

$$f_ig(x_1+h_1,x_2+h_2,x_3+h_3)-f_ig(x_1,x_2,x_3)\simeq f_ih_j\partial_jg(x_1,x_2,x_3)\ , \tag{28}$$

where h_j, $j=1,2,3$, are the components of an infinitesimal displacement $\mathbf{h}$. The moment f_ih_j is generalized to a symmetric tensor M_{ij}, which in Seismology is called the seismic moment;[42] in addition, the distribution $g(\mathbf{R})$ is replaced by $\delta(\mathbf{R}-\mathbf{R}_0)$, where δ denotes the Dirac function localized at the point with the position vector $\mathbf{R}_0$. Thus, we get a tensorial force density with the components

$$F_i=M_{ij}\partial_j\delta(\mathbf{R}-\mathbf{R}_0)\ ; \tag{29}$$

we can check immediately that the total force is vanishing and so is the total angular momentum (due to the symmetry of the tensor M_{ij}).

We apply the results obtained above to compute the surface displacement of the isotropic elastic half-space under the action of this tensorial force. We note that the solution can be obtained by the indirect method of taking the moment of forces as given by (28) in the solution of the Mindin problem of point forces. Beside the particular character of this method, which requires calculations for each component of the force moment, it may be unpracticable, due to the calculation complexity. This is why we prefer to use here the direct method described above, which leads to an elegant and compact form of solution.

The tensorial force given by (29), placed at the point with the position vector $\mathbf{R}_0$ of coordinates $0, 0, z_0$, $z_0 < 0$, has the components

$$F_j = M_{jl}\partial_l\delta(\mathbf{R}-\mathbf{R}_0) = M_{j\alpha}\partial_\alpha\delta(\mathbf{r})\delta(z-z_0) + M_{j3}\delta(\mathbf{r})\delta^{'}(z-z_0) \; ; \tag{30}$$

their in-plane Fourier transform are

$$F_j = iM_{j\alpha}k_\alpha\delta(z-z_0) + M_{j3}\delta^{'}(z-z_0) \; . \tag{31}$$

From (12) and (18) we get

$$\begin{gathered} G_j = (iM_{j\alpha}k_\alpha - M_{j3}k)\,e^{-k|z_0|} \; , \\ A_j = [iz_0M_{j\alpha}k_\alpha - M_{j3}(1+kz_0)]\,e^{-k|z_0|} \; . \end{gathered} \tag{32}$$

Inserting these quantities in (13) and (19) we get

$$\begin{gathered} u_\alpha^0 = [M_{\alpha 3} - \tfrac{iM_{\alpha\beta}k_\beta}{k} + \tfrac{iM_{\beta\gamma}k_\alpha k_\beta k_\gamma}{2k^3}(2\sigma - kz_0) + \\ + \tfrac{z_0M_{3\beta}k_\alpha k_\beta}{k} + \tfrac{iM_{33}k_\alpha}{2k}(2\sigma + kz_0)]e^{-k|z_0|} \; , \\ u_3^0 = [\tfrac{M_{\alpha\beta}k_\alpha k_\beta}{2k^2}(1-2\sigma+kz_0) + iz_0M_{3\alpha}k_\alpha + \\ + \tfrac{1}{2}M_{33}(1-2\sigma-kz_0)]e^{-k|z_0|} \end{gathered} \tag{33}$$

and, with the reverse Fourier transforms,

$$\begin{gathered} 2\pi\cdot u_\alpha^0 = -M_{\alpha\beta}I_\beta^{(1)} + M_{\alpha 3}I^{(0)} - \tfrac{1}{2}M_{\beta\gamma}\partial_\beta\partial_\gamma[2\sigma I_\alpha^{(3)} - z_0I_\alpha^{(2)}] - \\ -z_0M_{3\beta}\partial_\beta I_\alpha^{(1)} + \tfrac{1}{2}M_{33}[2\sigma I_\alpha^{(1)} + z_0I_\alpha^{(0)}] \; , \\ 2\pi\cdot u_3^0 = -\tfrac{1}{2}M_{\alpha\beta}\partial_\beta[(1-2\sigma)I_\alpha^{(2)} + z_0I_\alpha^{(1)}] + \\ +z_0M_{3\alpha}I_\alpha^{(0)} + \tfrac{1}{2}M_{33}[(1-2\sigma)I^{(0)} - z_0\tfrac{\partial}{\partial z_0}I^{(0)}] \; . \end{gathered} \tag{34}$$

Making use of the integrals $I_\alpha^{(n)}$ and $I^{(0)}$ given above, we get from (34) the components u_α^0 and u_3^0 of the surface displacement caused by a point tensorial force localized beneath the surface of an isotropic elastic half-space. We can see from (34) that u_α^0 is vanishing for $r \to 0$ and goes like $1/r^2$ for $r \to \infty$; it attains a maximum value for distances r of the order of $|z_0|$. The component u_3^0 goes like $1/z_0^2$ for $r \to 0$ and like $1/r^2$ for $r \to \infty$. A simplified version of (34) is obtained for $M_{ij} = M\delta_{ij}$; noting that $\partial_\beta^2 I_\alpha^{(n)} = -I_\alpha^{(n-2)}$, we get

$$u_\alpha^0 = -\frac{1-2\sigma}{2\pi}MI_\alpha^{(1)} = \frac{1-2\sigma}{2\pi}M\frac{x_\alpha}{R_1^3} \; , \; u_3^0 = \frac{1-2\sigma}{2\pi}MI^{(0)} = \frac{1-2\sigma}{2\pi}M\frac{|z_0|}{R_1^3} \tag{35}$$

(in deriving (35) $\partial_\alpha^2 I^{(1)}$ occurs, which may be written as $-I^{(-1)} = -\partial I^{(0)}/\partial z_0$ by extending the definition of the integrals $I^{(n)}$ to $n=-1$; the same integral appears also in (34)).

Force on the surface. If the force density $\mathbf{P}$ is applied on the surface $z=0$, (volume) force terms do not appear anymore in the generalized Poisson equation, but the external loads appear in the boundary conditions, which read

$$u_{\alpha 3} = -\frac{1+\sigma}{E}P_\alpha \ , \ (1-\sigma)u_{33} + \sigma u_{\alpha\alpha} = -\frac{(1+\sigma)(1-2\sigma)}{E}P_3 \ , \ z=0 \ ; \tag{36}$$

it is convenient to absorb the factor $(1+\sigma)/E$ in the force density $\mathbf{P}$.

Making use of equations (8) (without volume force terms), equation (14) becomes

$$\frac{2(1-\sigma)}{1-2\sigma}D = \frac{k}{1-2\sigma}\int_{-\infty}^{0} dz' D(z')e^{-k\left|z+z'\right|} + (ik_\alpha u_\alpha^0 + ku_3^0)e^{-k|z|} \ ; \tag{37}$$

the solution of this integral equation is

$$D = \frac{2(1-2\sigma)}{3-4\sigma}(ik_\alpha u_\alpha^0 + ku_3^0)e^{-k|z|} \ ; \tag{38}$$

we can see that D^0 derived from this equation coincides with D^0 given by equations (15) and (17) in the absence of volume forces, as it should. Now we can compute $\mathbf{u}$ given by equations (8). The boundary conditions given above become

$$\begin{gathered} ku_\alpha^0 + \tfrac{k_\alpha k_\beta u_\beta^0}{(3-4\sigma)k} + i\tfrac{2(1-2\sigma)}{3-4\sigma}k_\alpha u_3^0 = -P_\alpha \ , \\ i(1-2\sigma)k_\alpha u_\alpha^0 - 2(1-\sigma)ku_3^0 = (3-4\sigma)P_3 \ ; \end{gathered} \tag{39}$$

the solutions of this system of equations are

$$\begin{gathered} u_\alpha^0 = -\tfrac{P_\alpha}{k} + \sigma\tfrac{k_\alpha k_\beta P_\beta}{k^3} + (1-2\sigma)\tfrac{ik_\alpha P_3}{k^2} \ , \\ u_3^0 = -(1-2\sigma)\tfrac{ik_\alpha P_\alpha}{2k^2} - 2(1-\sigma)\tfrac{P_3}{k} \ , \end{gathered} \tag{40}$$

which, introduced in equations (8) and making use of the integrals given by equations (16), give

$$\begin{gathered} u_\alpha = \left[-\tfrac{P_\alpha}{k} + \tfrac{1}{2}(2\sigma - kz)\tfrac{k_\alpha k_\beta P_\beta}{k^3} + (1-2\sigma+kz)\tfrac{ik_\alpha P_3}{k^2}\right]e^{-k|z|} \ , \\ u_3 = -\left\{\tfrac{1}{2}(1-2\sigma-kz)\tfrac{ik_\alpha P_\alpha}{k^2} + \left[2(1-\sigma) - kz\right]\tfrac{P_3}{k}\right\}e^{-k|z|} \ . \end{gathered} \tag{41}$$

For the generalized Boussinesq-Cerruti problem the surface force density is $\mathbf{P} = \mathbf{p}\delta(\mathbf{r})$; by reverse Fourier transforming we get from equations (41) the displacement

$$\begin{gathered} 2\pi \cdot u_\alpha = -p_\alpha \overline{I}^{(1)} - \tfrac{1}{2}p_\beta\partial_\beta\left[2\sigma\overline{I}_\alpha^{(3)} - z\overline{I}_\alpha^{(2)}\right] + \\ +p_3\left[(1-2\sigma)\overline{I}_\alpha^{(2)} + z\overline{I}_\alpha^{(1)}\right] \ , \\ 2\pi \cdot u_3 = -\tfrac{1}{2}p_\alpha\left[(1-2\sigma)\overline{I}_\alpha^{(2)} - z\overline{I}_\alpha^{(1)}\right] - \\ -p_3\left[2(1-\sigma)\overline{I}^{(1)} - z\overline{I}^{(0)}\right] \ , \end{gathered} \tag{42}$$

where the integrals $\overline{I}^{(n)}$, $\overline{I}_\alpha^{(n)}$ are the corresponding integrals $I^{(n)}$, $I_\alpha^{(n)}$ given by (23) with z_0 replaced by z. This is the general solution of the Boussinesq-Cerruti problem for a point force of arbitrary orientation acting on the surface of the half-space. For the particular case $p_\alpha = 0$ (force perpendicular to the surface, Boussinesq problem) we obtain form (42)

$$\begin{gathered} u_\alpha = -\frac{p_3}{2\pi}\left[(1-2\sigma)\frac{1}{R+|z|}+\frac{z}{R^2}\right]\frac{x_\alpha}{R} \ , \\ u_3 = -\frac{p_3}{2\pi}\left[2(1-\sigma)+\frac{z^2}{R^2}\right]\frac{1}{R} \ , \end{gathered} \tag{43}$$

which coincide with the well-known results given in Refs. [19] and [44]. The solution for the Cerruti problem is obtained from (42) for $p_3 = p_2 = 0$ and $p_1 \neq 0$.

Final remarks. The deformation of an isotropic elastic half-space is calculated in this paper for a point force of arbitrary orientation and structure concentrated either at an inner point or on the surface. This is a generalization of the well-known Mindlin and Boussinesq-Cerruti problems. The results are used to get the deformation produced in the half-space by a tensorial force localized beneath the surface, which may arise from seismic sources. For Mindlin problem and the tensorial force explicit results are given for surface deformation. The problem is solved by a special method which implies the generalized Poisson equation and the use of in-plane Fourier transforms (*i.e.*, Fourier transforms with respect to the coordinates parallel with the surface). The method used here is particularly convenient for accounting for the boundary conditions. For inhomogeneous boundary conditions, the (volume) force is absent in the generalized Poisson equation, which includes only the values of the functions and their normal derivatives at the surface as "force" terms, but the external loads appear in the boundary conditions. In Fourier transforms, the boundary conditions generate a system of algebraic equations which can be solved for the values of the functions at the surface, thus providing a completely determined solution. The method can be extended to other geometries, or other similar problems (like a thick plate, for instance). The solution to the problems regarding forces concentrated along infinite lines (Melan and Flamant problems) can be obtained immediately by a direct integration of the solutions of the corresponding point forces.

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References

[1] Sir W. Thomson (Lord Kelvin), On the equations of equilibrium of an elastic solid, Cambridge Dublin Mathematical Journal **3** 87-89 (1848).

[2] Sir W. Thomson (Lord Kelvin), *Mathematical and Physical Papers*, vol.1, p. 97, Cambridge Univ. Press, London (1982).

[3] J. Boussinesq, Equilibre d'elasticite d'un sol isotrope sans pesanteur, supportant differents poids, CR Acad. Sci. **86** 1260-1263 (1878).

[4] J. Boussinesq, *Applications des potentiels a l'etude de l'equilibre et du mouvement des solides elastiques*, Gauthier-Villars, Paris (1885).

[5] J. Boussinesq, Equilibre d'elasticite d'un solide sans pesanteur, homogene et isotrope, dont les parties profondes sont maintenues fixes, pendant que sa surface eprouve des pressions ou de deplacements connus, s'annullant hors d'une region restreinte ou ils sont arbitrares, CR Acad. Sci. **106** 1043-1048, 1119-1123 (1888).

[6] J. Boussinesq, Des perturbations locales qui produit au-dessous d'elle une forte charge, repartie uniformement le long d'une droite normale aux deux bords, a la surface superieure d'une poutre rectangulaire et de longuer indefinie posee de champ soit sur un sol horizontal, soit sur deux appuis transversaux equidistants de la charge, CR Acad. Sci. **114** 1510-1516 (1892).

[7] V. Cerruti, Ricerche intorno all'equilibrio de' corpi elastici isotropi, Rend. Accad. Lincei **3** 81-122 (1882).

[8] A. Flamant, Sur la repartition des pressions dans un solide rectangulaire charge tranversalement, CR Acad. Sci. **114** 1465-1468 (1892).

[9] R. D. Mindlin, Force at a point in the interior of a semi-infinite solid, Physics **7** 195-202 (1936).

[10] R. D. Mindlin, Note on the Galerkin and Papkovitch stress functions, Bull. Am. Math. Soc. **42** 373-376 (1936).

[11] R. D. Mindlin, Nuclei of strain in the semi-infinite solid, J. Appl. Phys. **21** 926-930 (1950).

[12] R. D. Mindlin, Force at a point in the interior of a semi-infinite solid, in Proceedings of 1st Midwest Conference on Solid Mechanics, pp. 56-69 (1953).

[13] R. D. Mindlin, Force at a point in the interior of a semi-infinite solid, Tech. Note #8, Columbia Univ. (1953).

[14] E. Melan, Der Spannungzustand der durch eine Einzelkraft im Innern beanspruchten Halbscheibe, Z. Angew. Math. Mech. **12** 343-346 (1932) (correction Z. Angew. Math. Mech. **20** 368 (1940)).

[15] G. D. Grodskii, Integration of general equations of equilibrium of an isotropic elastic body by means of Newtonian potentials and harmonic functions, Izv. Akad. Nauk SSSR, Mat. Estestv. Nauk, **4** 587-614 (1935) (in Russian).

[16] H. Neuber, Ein neuer Ansatz zur Losung raumlicher Probleme der Elastizitatstheorie. Der Hohlkegel unter Einzellast als Beispiel, Z. Angew. Math. Mech. **14** 203-212 (1934).

[17] P. F. Papkovitch, A review of some general solutions of basic differential equations of rest for an isotropic elastic body, PPM (Appl. Math. Mech) **1** 117-132 (1937) (in Russian).

[18] A. E. H. Love, *A treatise on the Mathematical Theory of Elasticity*, 4th ed. Dover, NY (1944).

[19] A. I. Lurie, *Spatial Problems in the Theorie of Elasticity*, Gostehizdat, Moscow (1955) (in Russian).

[20] I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd ed. McGraw-Hill, NY (1956).

[21] L. E. Malvern, *Introduction to the Mechanics of a Continuous Medium*, Prentice-Hall, Englewood Cliffs, NJ (1969).

[22] G. M. L. Gladwell, *Contact Problems in the Classical Theory of Elasticity*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands (1980).

[23] R. O. Davis and A. P. S. Selvadurai, *Elasticity and Geomechanics*, Cambridge University Press, Cambridge (1996).

[24] J. Barber, *Elasticity*, Kluwer, Dordrecht (2002).

[25] M. H. Sadd, *Elasticity: Theory, Applications and Numerics*, Elsevier Butterworth Heinemann, Amsterdam (2009).

[26] P. Podio-Guidugli and A. Favata, *Elasticity for Geotechnicians*, Springer, NY (2014).

[27] L. Landau and E. Lifshitz, *Theory of Elasticity*, *Course of Theoretical Physics*, vol. 7. Elsevier, Oxford (1986).

[28] V. S. Vladimirov, Equations of Mathematical Physics, ed. A. Jeffrey, Marcel Dekker, NY (1971).

[29] A. Sommerfeld, *Partielle Differentialgleichungen der Physik*, *Vorlesungen uber Theoretische Physik*, Bd. VI. Akad. Verlag, Leipzig (1966).

[30] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products, eds. A. Jeffrey and D. Zwillinger, 6th edition, Acdemic, NY (2000), p.694, 6.623(3).

[31] H. Nakano, Notes on the nature of the forces which give rise to the earthquake motions, Seism. Bull. Central Metrological Observatory of Japan **1** 92-120 (1923).

[32] H. Honda, Earthquake mechanism and seismic waves, J. Phys. Earth **10** 1-98 (1962).

[33] R. Burridge and L. Knopoff, Body force equivalents for seismic dislocations, Bull. Seism. Soc. Am. **54** 1875-1888 (1964).

[34] J. D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems, Proc. Roy. Soc. London **A241** 376-396 (1957).

[35] L. R. Sykes, Mechanism of earthquakes and nature of faulting on the mid-oceanic ridges, J. Geophys. Res. **72** 2131-2153 (1967).

[36] T. Maruyama, On force equivalents of dynamic elastic dislocations with reference to the earthquake mechanism, Bull. Earthq. Res. Inst., Tokyo Univ., **41** 467-486 (1963).

[37] A. Ben Menahem, Radiation of seismic surface waves from finite moving sources, Bull. Seism. Soc. Am. **51** 401-435 (1961).

[38] A. Ben Menahem, Radiation of seismic body waves from finite moving sources, J. Geophys. Res. **67** 345-350 (1962).

[39] G. Backus and M. Mulcahy, Moment tensors and other phenomenological descriptions of seismic sources. I. Continuous displacements, Geophys. J. Roy. Astron. Soc. **46** 341-361 (1976).

[40] G. Backus and M. Mulcahy, Moment tensors and other phenomenological descriptions of seismic sources. II. Discontinuous displacements, Geophys. J. Roy. Astron. Soc. **47** 301-329 (1976).

[41] B. V. Kostrov and S. Das, *Principles of Earthquake Source Mechanics*, Cambridge University Press, NY (1988).

[42] K. Aki and P. G. Richards, Q*uantitative Seismology*, University Science Books, Sausalito, CA (2009).

[43] R. Madariaga, Seismic Source Theory, in *Treatise of Geophysics*, ed. H. Kanamori, vol. 4, *Earthquake Seismology*, Elsevier (2015).

[44] A. Sommerfeld, *Mechanik der Deformierbaren Medien*, *Vorlesungen uber Theoretische Physik*, vol. II, Akademie Verlag, Leipzig (1970).