Journal of Theoretical Physics

Founded and Edited by M. Apostol **243** (2016)

ISSN 1453-4428

Vibration eigenfrequencies of a homogeneous elastic sphere with large radius

B. F. Apostol
Department of Engineering Seismology, Institute for Earth's Physics
Magurele-Bucharest MG-6, POBox MG-35, Romania
email: afelix@theory.nipne.ro

Abstract

An estimation is given for the vibration eigenfrequencies of a homogeneous solid sphere with a large radius, with application to the Earth's vibrations. The eigenfrequencies of vibrations of a fluid sphere is derived as a particular case. Various corrections arising from static and dynamic gravitation, rotation and inhomegeneities are estimated, and a tentative notion of an earthquake temperature is introduced.

Key words: elastic vibrations; solid sphere; Earth's vibrations

PACS: 46.40.-f; 91.10.Kg; 91.30.Cd; 91.30.Fn; 46.15.-x

Introduction. The vibrations of a homogeneous solid sphere have been calculated by Lamb as early as 1882,[1, 2] by introducing spheroidal and torsional functions and by numerical calculation; the results were applied to the Earth's vibrations (oscillations) caused by earthquakes. Subsequent approaches made extensive use of numerical calculation.[3] In this Note we give an approximate analytical formula for the eigenfrequencies of a homogeneous solid sphere, which, beside providing a more transparent physical picture, can be extended to other, more complex, related problems.

Solid sphere. We consider the equation of elastic motion[4]

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)grad\cdot div\mathbf{u} + \mathbf{F}\ \ , \tag{1}$$

or

$$\rho\ddot{\mathbf{u}} = -\mu curl\cdot curl\mathbf{u} + (\lambda+2\mu)grad\cdot div\mathbf{u} + \mathbf{F} \tag{2}$$

(by $curl\cdot curl = -\Delta + grad\cdot div$), where ρ is the density, $\mathbf{u}$ is the displacement field, λ and μ are the Lame elastic moduli and $\mathbf{F}$ denotes an external force. The vibrations of the solid sphere are described in terms of the functions[1]

$$\begin{array}{c}\mathbf{R}_{lm} = Y_{lm}\mathbf{e}_r\ , \\ \\ \mathbf{S}_{lm} = \frac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\varphi\ , \\ \\ \mathbf{T}_{lm} = \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\theta - \frac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\varphi\ ,\end{array} \tag{3}$$

[1]H.Lamb, "On the vibrations of an elastic sphere", Proc. London Math. Soc. **XIII** 189-212 (1882) (see also, H. Lamb, "On the oscillations of a viscous spheroid", Proc. London Math. Soc. **XIII** 51-66 (1881)).

$l\neq 0$, where Y_{lm} are the spherical harmonics and $\mathbf{e}_{r,\theta,\varphi}$ are the spherical unit vectors. For obvious reasons, the functions $\mathbf{R}_{lm}$ and $\mathbf{S}_{lm}$ are called spheroidal functions, while $\mathbf{T}_{lm}$ are called toroidal functions. It is easy to see that these function are orthogonal on the sphere,

$$\begin{gathered}\int do\mathbf{R}_{lm}\mathbf{R}_{l'm'}=\delta_{ll'}\delta_{mm'}\ ,\ \int do\mathbf{S}_{lm}\mathbf{S}_{l'm'}=l(l+1)\delta_{ll'}\delta_{mm'}\ ,\\ \int do\mathbf{T}_{lm}\mathbf{T}_{l'm'}=l(l+1)\delta_{ll'}\delta_{mm'}\end{gathered}\tag{4}$$

and

$$\int do\mathbf{R}_{lm}\mathbf{S}_{l'm'}=\int do\mathbf{R}_{lm}\mathbf{T}_{l'm'}=\int do\mathbf{S}_{lm}\mathbf{T}_{l'm'}=0\ .,\tag{5}$$

where do is the element of the solid angle. We note here a few useful relations:

$$\begin{gathered}grad[f(r)Y_{lm}]=f'\mathbf{R}_{lm}+\tfrac{f}{r}\mathbf{S}_{lm}\ ,\\ div[f(r)\mathbf{R}_{lm}]=\tfrac{1}{r^2}\tfrac{d}{dr}(r^2f)Y_{lm}\ ,\ div[f(r)\mathbf{S}_{lm}]=-\tfrac{f}{r}l(l+1)Y_{lm}\quad ,\\ div[f(r)\mathbf{T}_{lm}]=0\end{gathered}\tag{6}$$

and

$$\begin{gathered}curl[f(r)\mathbf{R}_{lm}]=\tfrac{f}{r}\mathbf{T}_{lm}\ ,\ curl[f(r)\mathbf{S}_{lm}]=-\tfrac{1}{r}\tfrac{d}{dr}(rf)\mathbf{T}_{lm}\ ,\\ curl[f(r)\mathbf{T}_{lm}]=\tfrac{1}{r}\tfrac{d}{dr}(rf)\mathbf{S}_{lm}+\tfrac{f}{r}l(l+1)\mathbf{R}_{lm}\end{gathered}\tag{7}$$

for any function $f(r)$; equations (6) and (7) are derived by using the expressions of the differential operators in spherical coordinates.

We prefer to use the form (2) of the equation of the elastic motion; making use of time Fourier transforms this equation reads

$$-\rho\omega^2\mathbf{u}+\mu curl\cdot curl\mathbf{u}-(\lambda+2\mu)grad\cdot div\mathbf{u}=\mathbf{F}\ .\tag{8}$$

We decompose the force as

$$\begin{gathered}\mathbf{F}=\sum_{lm}(F^r_{lm}\mathbf{R}_{lm}+F^s_{lm}\mathbf{S}_{lm}+F^t_{lm}\mathbf{T}_{lm})\ ,\\ F^r_{lm}=\int do\mathbf{F}\mathbf{R}_{lm}\ ,\ l(l+1)F^s_{lm}=\int do\mathbf{F}\mathbf{S}_{lm}\\ l(l+1)F^t_{lm}=\int do\mathbf{F}\mathbf{T}_{lm}\end{gathered}\tag{9}$$

and seek the solution as

$$\mathbf{u}=\sum_{lm}(f_{lm}\mathbf{R}_{lm}+g_{lm}\mathbf{S}_{lm}+h_{lm}\mathbf{T}_{lm})\ ,\tag{10}$$

where f_{lm}, g_{lm} and h_{lm} are functions only of the radius r. Making use of equations (6) and (7), equation (8) leads to

$$\begin{gathered}f''+\tfrac{2}{r}f'+\tfrac{\rho\omega^2}{\lambda+2\mu}f-\left[2+\tfrac{\mu l(l+1)}{\lambda+2\mu}\right]\tfrac{1}{r^2}f+\\ +\tfrac{(\lambda+3\mu)l(l+1)}{(\lambda+2\mu)r^2}g-\tfrac{(\lambda+\mu)l(l+1)}{(\lambda+2\mu)r}g'=-\tfrac{F^r}{\lambda+2\mu}\ ,\\ g''+\tfrac{2}{r}g'+\tfrac{\rho\omega^2}{\mu}g-\tfrac{(\lambda+2\mu)l(l+1)}{\mu r^2}g+\\ +\tfrac{2(\lambda+2\mu)}{\mu r^2}f+\tfrac{\lambda+\mu}{\mu r}f'=-\tfrac{F^s}{\mu}\ ,\\ h''+\tfrac{2}{r}h'+\tfrac{\rho\omega^2}{\mu}h-\tfrac{l(l+1)}{r^2}h=-\tfrac{F^t}{\mu}\quad ,\end{gathered}\tag{11}$$

where we dropped the suffixes lm.

We turn now to the boundary conditions. The force $\mathbf{P}$ acting outward on the surface $r = R$ of the sphere, where R is the radius of the sphere, with the spherical components P_α $(\alpha = r, \theta, \varphi)$ is $P_\alpha = -n_\beta \sigma_{\alpha\beta} = -\sigma_{\alpha r}$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma}\delta_{\alpha\beta}$; we get

$$
\begin{gathered}
2\mu u_{\theta r} = -P_\theta \ , \ 2\mu u_{\varphi r} = -P_\varphi \ , \\
2\mu u_{rr} + \lambda div\mathbf{u} = -P_r \ ,
\end{gathered}
\tag{12}
$$

where $div\mathbf{u}$ is written in spherical coordinates,

$$
div\mathbf{u} = \sum_{lm} \left[\frac{1}{r^2}\frac{d}{dr}(r^2 f_{lm}) - \frac{g_{lm}}{r} l(l+1) \right] Y_{lm}
\tag{13}
$$

(by using equations (6)). We compute the strain tensor $u_{\alpha\beta}$ in spherical coordinates[4]

$$
\begin{gathered}
u_{rr} = \frac{\partial u_r}{\partial r} \ , \ u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \ , \ u_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} \ , \\
2u_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta} - u_\varphi \cot\theta \right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} \ , \ 2u_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \ , \\
2u_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}
\end{gathered}
\tag{14}
$$

by using the spherical components

$$
\begin{gathered}
u_r = \sum_{lm} f_{lm} Y_{lm} \ , \\
u_\theta = \sum_{lm} g_{lm}\frac{\partial Y_{lm}}{\partial \theta} + \sum_{lm} \frac{h_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} \ , \\
u_\varphi = \sum_{lm} \frac{g_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} - \sum_{lm} h_{lm}\frac{\partial Y_{lm}}{\partial \theta}
\end{gathered}
\tag{15}
$$

of the displacement vector given by equation (10) and the definition of the spheroidal and toroidal functions. Similarly, we decompose the force $\mathbf{P}$ in spheroidal and toroidal functions and identify its spherical components; the boundary conditions given by equations (12) lead to

$$
\begin{gathered}
2\mu f' + \lambda \left[\frac{2}{r} f + f' - \frac{g}{r} l(l+1) \right] |_{r=R} = -P^r \ , \\
\mu \left(\frac{g}{r} - g' - \frac{f}{r} \right) |_{r=R} = P^s \ , \\
\mu \left(\frac{h}{r} - h' \right) |_{r=R} = P^t \ ,
\end{gathered}
\tag{16}
$$

where we dropped the subscripts lm.

Let us first discuss the toroidal component which implies the function h in equations (11) and (16). If the volume force F^t and the surface force P^t are both zero, we have free (toroidal) vibrations of the sphere; the third equation (11) is the Bessel equation for spherical Bessel functions $j_l(kr)$, $k = \sqrt{\rho\omega^2/\mu} = \omega/c_2$, where $c_2 = \sqrt{\mu/\rho}$ is the velocity of the "t-waves". The third equation in the boundary conditions (16) gives

$$
j_l(kR) = kR j_l'(kR) \ ;
\tag{17}
$$

this equation has an infinity of solutions β_{ln}, labelled by integer n, such that we get the eigenfrequencies

$$
\omega_{ln} = \frac{c_2}{R}\beta_{ln} \ .
\tag{18}
$$

We can get a representation of the numbers β_{ln} by using the asymptotic expression of the spherical Bessel functions[5]

$$j_l(kr) \simeq \frac{1}{kr}\cos\left[kr-(l+1)\frac{\pi}{2}\right] \ , \ kr \gg 1 \ ; \tag{19}$$

for $kR \gg 1$ equation (17) becomes

$$\tan\left[kR-(l+1)\frac{\pi}{2}\right] = -\frac{2}{kR} \ , \tag{20}$$

which have the approximate zeroes

$$\beta_{ln} \simeq n\pi + (l+1)\frac{\pi}{2} \ , \tag{21}$$

where n is any (large) integer. We can see that the frequencies are dense for large R ($\Delta\omega_{ln} = \pi c_2/R$); the wavelengths are given by $\lambda_{ln} = 2\pi R/\beta_{ln}$. The toroidal vibrations imply only the shear modulus μ; their frequencies do not depend on the azimuthal number m. It is worth noting that the solution h is a superposition of $j_l(k_{ln}r)$, where $k_{ln} = \omega_{ln}/c_2$, with undetermined coefficients.

If the boundary force P^t is different from zero (but the body force F^t is still zero), the solution is $C_l j_l(kr)$, where the constant C_l is determined from the boundary condition; the vibrations are driven by the boundary force. If the body force F^t is different from zero and the surface force P^t is zero, the third equation (11) is a second-order ordinary differential equation whose solution $h(r)$ depends on two integration constants; the boundary condition gives a relationship between these constants; an additional boundary condition, like, for instance, $h(0) = 0$, determines completely the solution; the vibrations are driven by the body force. The same is true for both forces (F^t and P^s) diferent from zero; it is worth noting that the full solution of the differential equation implies also the solution of the homogeneous equation, beside a particular solution of the inhomogeneous equation.

We turn now to the spheroidal component which involves the functions f and g in equations (11) and (16). We note that the two coupled equations (11) for the functions f and g include Bessel operators for spherical Bessel functions. We can get a simplified picture of these equations for large values of r. First, let us consider the Bessel equation

$$j_l^{''} + \frac{2}{r}j_l^{'} + k^2 j_l - \frac{l(l+1)}{r^2}j_l = 0 \tag{22}$$

and assume the form

$$j_l = \frac{1}{kr}J_l \tag{23}$$

for its solution; we get

$$\frac{J_l^{''}}{r} + k^2\frac{J_l}{r} - \frac{l(l+1)}{r^3}J_l = 0 \ , \tag{24}$$

which shows that $J_l = \cos(kr + \varphi_l)$ for large values of r, where φ_l is an undetermined phase; indeed, this is the asymptotic expression of the spherical Bessel functions. We define the velocities $c_1 = \sqrt{(\lambda+2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$ and $k_{1,2} = \omega/c_{1,2}$ and assume the asymptotic expressions

$$f = \frac{1}{k_1 r}F \ , \ g = \frac{1}{k_2 r}G \ ; \tag{25}$$

the two equations (11) for f and g read

$$F'' + k_1^2 F = -\tfrac{k_1 r F^r}{\lambda+2\mu} \ , \ G'' + k_2^2 G = -\tfrac{k_2 r F^s}{\mu} \ , \tag{26}$$

up to corrections of the order $1/r$; similarly, the corresponding boundary conditions (16) become

$$\rho c_1^2 \frac{F'}{k_1 R} \mid_{r=R} = -P^r \ , \ \rho c_2^2 \frac{G'}{k_2 R} \mid_{r=R} = -P^s \tag{27}$$

up to corrections of the order $1/R^2$. Now we can see that for free oscillations the asymptotic functions F and G are $F = \cos(k_1 r + \varphi_1)$ and $G = \cos(k_2 r + \varphi_2)$ (and the solutions are approximately spherical Bessel functions), where the phases $\varphi_{1,2}$ remain undetermined; the frequencies are given approximately by

$$\omega_{nl}^{(1)} \simeq \frac{c_1}{R}(n\pi - \varphi_{1l}) \ , \ \omega_{nl}^{(2)} \simeq \frac{c_2}{R}(n\pi - \varphi_{2l}) \ , \tag{28}$$

where we have restored the suffix l (the frequencies do not depend on the suffix m). We can see that there are two branches of spheroidal eigenfrequencies (corresponding to the velocities $c_{1,2}$), which are dense (continuous) for large R, very similar with the infinite space (as expected for large R); the $\omega^{(2)}$-branch, although close to the toroidal branch, is distinct (there is a total of three branches of eigenfrequencies, corresponding to the three degrees of freedom; in the limit of the rotations of the sphere as a whole their frequencies go to zero (acoustic modes)). For non-vanishing forces we have spheroidal vibrations driven by these forces, as discussed for the previous cases. Equations (26) and (27) can be used to compute approximately such vibrations for large values of the radius R of the sphere. The set of all eigenfrequencies is called the (seismic) spectrum. Earth's eigenmodes with eigenfrequencies of the order $10^{-3} - 10^{-4}s^{-1}$, excited by earthquakes, have been discussed in Refs. [6]-[8].

The numerical solution of equations (11) indicates that the lowest mode (the fundamental mode) is $\mathbf{S}_{lm}$ with $l = 2$ and $n = 0$ (therefore, we may denote it as $S_{l=2,m}^{(n=0)}$);[3] it is denoted by ${}_0S_2$, and its eigenfrequency is denoted ω_{20}; the corresponding period is approximately 1 hour. Much later, the Earth's crust was modelled as a series of superposed layers, with welded interfaces; the vibrations of such a stack of layers can be computed and long periods of the fundamental modes have been obtained; the dispersion relation of these modes (*i.e.*, the dependence of the frequency on their label n) can give information about the inner crustal structure.[9, 10] The first observation of "free oscillations of the Earth as a whole" was made for the Kamchatka earthquake of November 4, 1952;[11] they were followed by many observations of the Earth's vibrations caused by the great Chile earthquake of May 22, 1960[12]-[14] (with magnitude greater than 8, which saturated the scales[15]). Today, eigenoscillations of the Earth can be recorded even for small earthquakes.[16]

From studies of propagation of the seismic waves it was inferred the Earth's solid inner core[17, 18] of radius $\simeq 1000km$ and the outer liquid core of radius $\simeq 2000km$. The inner-outer core discontinuity is called the Bullen, or Lehmann, discontinuity. The temperature of the inner core is $\simeq 6000K$ (iron and nickel) and the pressure is $\simeq 10^{12}dyn/cm^2$. The buoyancy at this boundary could be the source of convection currents which generate the Earth's magnetic field (geodynamo effect). The next layers are a viscous mantle of thickness $\simeq 3000km$ and the solid crust of thickness $\simeq 70km$. The boundary between mantle and crust is known as the Mohorovic discontinuity.

Fluid Sphere. For a fluid sphere the shear modulus μ is zero ($\mu = 0$); equations (11) become

$$\begin{array}{c} f'' + \frac{2}{r}f' + k^2 f - \frac{2}{r^2}f - \frac{d}{dr}\left[\frac{l(l+1)g}{r}\right] = -\frac{F^r}{\lambda} \\ \frac{1}{r}f' + \frac{2}{r^2}f - \frac{l(l+1)}{r^2}g + k^2 g = -\frac{F^s}{\lambda} \ , \end{array} \tag{29}$$

where $k^2 = \rho\omega^2/\lambda = \omega^2/c^2$; the boundary condition reads

$$\left[\frac{2}{r}f + f' - \frac{g}{r}l(l+1)\right] \mid_{r=R} = -\frac{P^r}{\lambda} \ . \tag{30}$$

Let us introduce $div\mathbf{u}$, given by equation (13), which, analyzed with respect to the subscripts lm, amounts to

$$d = f' + \frac{2}{r}f - \frac{g}{r}l(l+1) \ . \tag{31}$$

Then the boundary condition becomes

$$d\mid_{r=R} = -\frac{P^r}{\lambda} \ , \tag{32}$$

the second equation (29) reads

$$\frac{d}{r} + k^2 g = -\frac{F^s}{\lambda} \tag{33}$$

and the first equation (29) is

$$d' + k^2 f = -\frac{F^r}{\lambda} \ . \tag{34}$$

Hence, we have

$$g = -\frac{d}{k^2 r} - \frac{F^s}{\lambda k^2} \ , \ f = -\frac{d'}{k^2} - \frac{F^r}{\lambda k^2} \ . \tag{35}$$

Now we introduce these functions in equation (31) and get

$$d'' + \frac{2d'}{r} + k^2 d - \frac{l(l+1)}{r^2}d = -\frac{(F^r)'}{\lambda} - \frac{2F^r}{\lambda r} + \frac{F^s}{\lambda r}l(l+1) \ . \tag{36}$$

For free vibrations this is the Bessel equation for spherical Bessel functions $d = j_l(kr)$; the boundary condition (32) leads to the eigenfrequencies $\omega_{ln} = (c/R)\beta_{ln}$, $j_l(\beta_{ln}) = 0$. In a fluid we have only pressure p, and the stress tensor is $\sigma_{ij} = -p\delta_{ij}$ ($\sigma_{ij} = 2\mu u_{ij} + \lambda u_{kk}\delta_{ij}$ with $\mu = 0$); therefore, for a fluid $p = -\lambda u_{ii} = -\lambda div\mathbf{u}$; equations written above for d are in fact equations for the pressure p. It is convenient to introduce the decomposition in Helmholtz potentials $\mathbf{u} = grad\Phi + curl\mathbf{A}$, $div\mathbf{A} = 0$ and $\mathbf{F} = grad\varphi + curl\mathbf{H}$, $div\mathbf{H} = 0$; then, $p = -\lambda\Delta\Phi$ and the equation of motion $\rho\ddot{\mathbf{u}} = \lambda grad \cdot div\mathbf{u} + \mathbf{F} = -gradp + \mathbf{F}$ becomes $\rho\ddot{\Phi} = \lambda\Delta\Phi + \varphi$.

Static self-gravitation. A gravitational force

$$FdV = G\rho dV\frac{m}{r^2} = \frac{4\pi}{3}G\rho^2 r dV \tag{37}$$

acts upon a volume element dV placed at distance r from the centre of a sphere, where $G = 6.67 \times 10^{-8} cm^3/g \cdot s^2$ is the universal constant of gravitation, ρ is the density of the sphere (assumed incompressible) and $m = (4\pi/3)\rho r^3$ is the mass of the sphere with radius r. If the sphere is compressible, the gravitational potential φ is given by the Poisson equation $\Delta\varphi = 4\pi G\rho$ and the gravitational force per unit mas is $\mathbf{F} = -grad\varphi$; the condition of (hydrostatic) equilibrium (for a non-rotationg sphere) reads $gradp = \rho\mathbf{F} = -\rho grad\varphi$, such that $div\left[(gradp)/\rho\right] = -4\pi G\rho$; the dependence of the pressure on the density is given by the equation of state; for a constant density the pressure for a self-gravitating sphere of radius R at rest with free surface is $p = (2\pi/3)G\rho^2(R^2 - r^2)$ (it seems that the pressure in the inner Earth's (solid) core is $\simeq 300GPa = 3 \times 10^{12} dyn/cm^2$). Making use of equation (37), the equation of the elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda + \mu)grad \cdot div\mathbf{u} = \mathbf{F} = -\gamma\mathbf{r} \ , \tag{38}$$

where $\gamma = (4\pi/3)G\rho^2$. Since $Y_{00} = 1/\sqrt{4\pi}$, we may write

$$\mathbf{F} = -\gamma\mathbf{r} = -\sqrt{4\pi}\gamma r Y_{00}\mathbf{e}_r \ , \tag{39}$$

whence we can see that $\mathbf{F}$ has an expansion is series of spheroidal and toroidal functions with all the coefficients zero, except the coefficient $F^r_{00} = -\sqrt{4\pi}\gamma r$ of the function $\mathbf{R}_{00}$; it follows that the motion may include all the eigenmodes $\mathbf{S}_{lm}$ and $\mathbf{T}_{lm}$, as well as all the eigenmodes $\mathbf{R}_{lm}$, the latter with $l \neq 0$; for $l = 0$, $m = 0$ the motion, described by $f = f_{00}$, is driven by the gravitational force. We note also that the force in equation (38) is static, which means that its Fourier transform is proportional to $\delta(\omega)$. For $l = 0$ the first equation (11) includes only the function f, *i.e.* $f\delta(\omega)$; this equation reads

$$f'' + \tfrac{2}{r}f' - \tfrac{2}{r^2}f = \tfrac{\sqrt{4\pi}\gamma}{\lambda+2\mu}r \ . \tag{40}$$

It is easy to see that a particular solution of this equation is $[\sqrt{4\pi}\gamma/10(2\mu+\lambda)]r^3$, while the homogeneous part of this equation has the solution $C_1 r + C_2/r^2$, where $C_{1,2}$ are constants of integration; we must take $C_2 = 0$, because the solution is finite at the origin. We are left with the solution

$$u_r = Ar^3 + C_1 r \ , \ A = \frac{\gamma}{10(2\mu+\lambda)} \ . \tag{41}$$

This solution must satisfy the boundary conditions at the surface of the sphere; making use of equations (15), we have the strain tensor $u_{rr} = u_r^{'}$ and $u_{\theta\theta} = u_{\varphi\varphi} = u_r/r$; the force on the surface is $-\sigma_{\alpha r} \mid_R$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma\delta_{\alpha\beta}}$; for a free surface we get the boundary condition

$$(2\mu+\lambda)u_r^{'} + 2\lambda\frac{u_r}{r} \mid_{r=R} = 0 \tag{42}$$

($\sigma_{\alpha r} \mid_R = 0$), whence we determine the constant $C_1 = -[(6\mu+5\lambda)/(2\mu+3\lambda)]AR^2$ and, finally, the radial displacement

$$u_r = Ar\left(r^2 - \frac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right) = \frac{\gamma}{10(2\mu+\lambda)}r\left(r^2 - \frac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right) \ ; \tag{43}$$

we note that the radial displacement u_r is negative, as expected. It is worth estimating the radial displacement at the surface due to gravitation

$$u_r \mid_{r=R} = -\frac{\gamma}{5(2\mu+3\lambda)}R^3 \ ; \tag{44}$$

making use of $\rho = 5g/cm^3$, λ, $\mu \simeq 10^{11}dyn/cm^2$ (parameters for Earth), we get $\gamma \simeq 10^{-6}g/cm^3s^2$and $u_r \mid_R \simeq 10^{-18}R^3 cm \simeq 10^8 cm = 10^3 km$, for the Earth's radius $R \simeq 6\times10^8 cm$; this is a distance of the order of the Earth radius. Moreover, the strain is of the order 1/6, which may cast doubts on the validity of the linear elasticity used in this estimation. In addition, we note that the density suffers an important change due to the static gravitational field. Indeed, the change in density is $\delta\rho = -div(\rho\mathbf{u}) = -\rho_0 div\mathbf{u}$, where ρ_0 is the uniform initial density; with $\mathbf{u}$ given by equation (43) we get

$$\frac{\delta\rho}{\rho_0} = A(3\alpha R^2 - 5r^2) \ , \ \alpha = \frac{6\mu+5\lambda}{2\mu+3\lambda} \ , \tag{45}$$

which is of the order unity. The proper estimation of the static effect of the self-gravitational field on the elastic sphere is to solve simultaneously the equation of elastic equilibrium (38) with $\mathbf{F} = -\rho grad\varphi$ and the Poisson equation for the gravitational field φ, $\Delta\varphi = 4\pi G\rho$. With spherical symmetry we have

$$\mathbf{F} = -\frac{4\pi}{3r^2}G\rho\int_{0<r'<r} d\mathbf{r}'\rho\frac{\mathbf{r}}{r} \ ; \tag{46}$$

the Poisson equation for the gravitational potential may be written as $\Delta(\mathbf{F}/\rho) = -4\pi G grad\rho$, such that the problem involves two equations and unknowns, $\mathbf{u}$ and ρ. Since this is a more difficult problem it is preferable to consider the density ρ as an empirically known function of r

(a parametrization in powers of r can be used for ρ and a variational approach can be applied to the problem). Even so, the equations governing the influence of the gravitational field upon the elasticity of a self-gravitating sphere are difficult.

Dynamic self-gravitation. Let us assume a spheric, non-rotating, homogeneous, elastic Earth at equilibrium under the action of its own gravitational field; we consider small elastic deformations of this equilibrium state; in first approximation, we have a small change denoted by K in the gravitational potential as a consequence of the small changes in density $-div(\rho\mathbf{u})$, *i.e.,* we have

$$\Delta K = -4\pi G div(\rho\mathbf{u}) \ , \tag{47}$$

where ρ is a known function of r. The equation of elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu) grad \cdot div\mathbf{u} = -\rho grad K \ . \tag{48}$$

These two coupled (vectorial) equations are difficult to be treated by an analytical method, due to the non-uniformity of the density. For a uniform density, taking the *div* in equation (48) and using equation (47) we get for $D = div\mathbf{u}$

$$\rho\ddot{D} - (\lambda+2\mu)\Delta D = 4\pi G\rho^2 D \ , \tag{49}$$

an equation which indicates that the frequency ω changes by

$$\Delta(\omega^2) = -4\pi G\rho \ ; \tag{50}$$

for frequencies as low as $\omega = 10^{-4}s^{-1}$ the variation given by equation (50) is large. Let us use the Helmholtz decomposition $\mathbf{u} = grad\Phi + curl\mathbf{A}$, $div\mathbf{A} = 0$; then, from equation (47) we have $K = -4\pi G\rho\Phi$ and from equation (48) we get $\Delta\Phi + k_1^2\Phi = 0$, $\Delta\mathbf{A} + k_2^2\mathbf{A} = 0$. These are the same equations as those which hold in the absence of the gravitational field, except that k_1^2 is changed into $k_1^2 \to k_1^2 + 4\pi\rho G/c_1^2$. Moreover, we can see that only the spheroidal modes are affected by gravitation (since $div\mathbf{T}_{lm} = 0$). It follows that the spheroidal frequencies (*i.e.,* the branches $\omega^{(1,2)}$) are given by the same relations of the type $\omega = (c/R)\beta$, where β denote the zeroes the spherical Bessel functions in the limit of large R; for $c = c_1$, this relation reads $\omega^2 + 4\pi\rho G = (c_1^2/R^2)\beta^2$. Hence, we may see that we should have the inequality $(c_1^2/R^2)\beta^2 > 4\pi\rho G$, or $(\lambda+2\mu)\beta^2 > 4\pi\rho^2 GR^2$. The term in the right side of this inequality is, up to an immaterial numerical factor, the pressure due to the gravitation at the origin; it is much larger than the elastic pressure $\lambda + 2\mu$. The inequality is not satisfied for small values of β (as required by experimental observations). It follows that the model of an elastic solid Earth is not valid for the interior of the Earth. In those central regions the elasticity is not able to sustain the gravitational pressure. Likely, an additional pressure exists there, which compensates the gravitational pressure. The large dimensions of the mantle and liquid outer core complicates the matter, and such an Earth's model may exhibit very low frequencies (undertones);[19] If so, we may leave aside the effects of the gravitation in estimating the elastic vibrations of the Earth. In this case, with $c = 5km/s$ we get a period $T \simeq (2.2/\beta)$ hours; the smallest zero of j_2' (corresponding approximately to the mode ${}_0S_2$) is $\beta = 3.6$;[5] we get $T \simeq 37$ minutes (for a velocity $c = 3km/s$ the period is $T = 61$ minutes, which agrees with the experimental observations).

Rotation effect. If a vector $\mathbf{a}$ rotates, its change is $\delta\mathbf{a} + \delta\boldsymbol{\alpha} \times \mathbf{a}$, where $\delta\boldsymbol{\alpha}$ is the infinitesimal rotation angle; therefore, its velocity is $\dot{\mathbf{a}} + \boldsymbol{\Omega} \times \mathbf{a}$, where $\boldsymbol{\Omega}$ is the angular velocity; its acceleration is $\ddot{\mathbf{a}} + \dot{\boldsymbol{\Omega}} \times \mathbf{a} + 2\boldsymbol{\Omega} \times \dot{\mathbf{a}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{a})$. Let us apply this relation to the displaced position $\mathbf{a} = \mathbf{r} + \mathbf{u}$; we get the acceleration $\ddot{\mathbf{u}} + \dot{\boldsymbol{\Omega}} \times (\mathbf{r} + \mathbf{u}) + 2\boldsymbol{\Omega} \times \dot{\mathbf{u}} + \boldsymbol{\Omega} \times [\boldsymbol{\Omega} \times (\boldsymbol{r} + \boldsymbol{u})]$; we can see that additional forces appear in rotation: $-2\boldsymbol{\Omega} \times \dot{\mathbf{u}}$ is the Coriolis acceleration and $-\boldsymbol{\Omega} \times [\boldsymbol{\Omega} \times (\mathbf{r} + \mathbf{u})]$ is the centrifugal acceleration. The Earth rotates with a constant angular velocity $\Omega = 2\pi/T$, $T = 24$ hours, oriented along the z-axis. We write the equation of elastic motion as

$$\rho\ddot{\mathbf{u}} + 2\rho\boldsymbol{\Omega} \times \dot{\mathbf{u}} = \mathbf{F} \ , \tag{51}$$

where $\mathbf{F}$ includes the elastic force ($i.e.$, $F_i = \partial_j\sigma_{ij}$) and other external forces and the centrifugal force is omitted since Ω is much smaller than the eigenfrequencies of the Earth (an estimation of the longest periods of the Earth's eigenmodes gives an order of magnitude $2\pi R/c\beta \simeq 37$ minutes, for the wave velocity $c = 5km/s$ and $\beta = 3.6$ where R is the Earth's radius).

In the absence of the Coriolis force in equation (51) we decompose the force $\mathbf{F}$ and the displacement $\mathbf{u}$ in normal modes by using the spheroidal and toroidal functions. Let us focus on one normal mode, for instance a toroidal mode $\mathbf{u}_{lm}^{(n)} = h_l^{(n)}\mathbf{T}_{lm}$, corresponding to the eigenfrequency $\omega_{ln} = (c_2/R)\beta_{ln}$, where β_{ln} is, approximately, a zero of the function $j_l(k^{(n)}R)$; the eigenfunctions $h_l^{(n)}$ are given by the spherical Bessel functions $j_l(k^{(n)}r)$; it is preferrable to multiply these functions by constants and fix these constants such as

$$\int dr \cdot r^2 h_l^{(n)}(r) h_l^{(n')}(r) = \delta_{nn'} \ ; \tag{52}$$

we recall that the toroidal functions are orthogonal, $i.e.$

$$\int do \mathbf{T}_{lm}\mathbf{T}^*_{l'm'} = \delta_{ll'}\delta_{mm'} \ . \tag{53}$$

Since $\Omega/\omega_{ln} \ll 1$ we solve equation (51) by a perturbation-theory method. First, we drop the labels l, m and n and use the notations $\mathbf{u}_{lm}^{(n)} = \mathbf{u}_0$, $\omega_{lm} = \omega_0$; we seek the solution as a series in powers of Ω/ω_0

$$\mathbf{u} = \mathbf{u}_0 + \frac{\Omega}{\omega_0}\mathbf{u}_1 + ... \ , \tag{54}$$

where $\mathbf{u}_1$, to be determined, is asumed orthogonal on $\mathbf{u}_0$,[20] with respect to the scalar product defined as the integration over the whole space, $i.e.$

$$\int d\mathbf{r}\mathbf{u}_1\mathbf{u}_0 = 0 \ . \tag{55}$$

A similar series is valid for the frequency

$$\omega = \omega_0 + \frac{\Omega}{\omega_0}\omega_1 + ... \ . \tag{56}$$

Introducing these series in equation (51), with time Fourier transforms, we get

$$\begin{gathered} -\rho\omega_0^2\mathbf{u}_0 = \mathbf{F} \ , \\ -\rho\omega_0\Omega\mathbf{u}_1 - 2\rho\Omega\omega_1\mathbf{u}_0 - 2i\rho\omega_0\boldsymbol{\Omega} \times \mathbf{u}_0 = 0 \ ; \end{gathered} \tag{57}$$

the first equation (57) defines the function $\mathbf{u}_0$; in the second equation (57) we take the scalar product with $\mathbf{u}_0$ and use the orthogonality of $\mathbf{u}_0$ with $\mathbf{u}_1$; we get

$$\omega_1 = -\frac{i\omega_0}{l(l+1)}\int d\mathbf{r}\mathbf{e}_z(\mathbf{u}_0 \times \mathbf{u}_0^*) \ , \tag{58}$$

where we put $\mathbf{\Omega}=\Omega\mathbf{e}_z$, $\mathbf{e}_z$ being the unit vector along the z-axis. Here we use $\mathbf{e_z}=\cos\theta\mathbf{e}_r-\sin\theta\mathbf{e}_\theta$, $\mathbf{u}_0=h_l^{(n)}\mathbf{T}_{lm}$ and $\mathbf{T}_{lm}$ from equations (3); we get immediately

$$\omega_1=\omega_0\frac{m}{l(l+1)} \ , \tag{59}$$

where m denotes all integers from $-l$ to l. It follows that the frequencies ω_{ln}, which are degenerate with respect to m, are splitted into $2l+1$ branches

$$\omega_{ln}\rightarrow\omega_{ln}+\Omega\frac{m}{l(l+1)} \ ; \tag{60}$$

using ω_1 thus determined, we can get $\mathbf{u}_1$ from the second equation (57). Higher-order contributions can be obtained in a similar manner. An m-band occurs for each ω_{ln}, of width $2\Omega/(l+1)$, with the separation frequency $\Omega/l(l+1)$. For a typical eigenperiod 60 minutes the ratio Ω/ω_0 is approximately $1/20\ll 1$.

Centrifugal force. The equation of the elastic motion for a body in rotation with a (constant) angular velocity $\mathbf{\Omega}$ reads

$$\rho\ddot{\mathbf{u}}+2\rho\mathbf{\Omega}\times\dot{\mathbf{u}}+\rho\mathbf{\Omega}\times[\mathbf{\Omega}\times(\boldsymbol{r}+\mathbf{u})]=\mu\Delta\mathbf{u}+(\lambda+\mu)grad\cdot div\mathbf{u}+\mathbf{F} \ , \tag{61}$$

where $\mathbf{F}$ is an external force. We note that the centrifugal term $\rho\mathbf{\Omega}\times(\mathbf{\Omega}\times\boldsymbol{r})$ is static, so we can write it as

$$\mathbf{F}_c=\rho\mathbf{\Omega}(\mathbf{\Omega}\boldsymbol{r})-\rho\Omega^2\mathbf{r}=\mu\Delta\mathbf{u}+(\lambda+\mu)grad\cdot div\mathbf{u} \ , \tag{62}$$

where we denoted by $\mathbf{F}_c$ the centrifugal force and dropped any other external force ($\mathbf{F}=0$); we may neglect $\mathbf{u}$ in the centrifugal force, since it is very small in comparison with $\mathbf{r}$. The angular velocity is oriented along the z-axis, $\mathbf{\Omega}=\Omega\mathbf{e}_z$. Making use of $\mathbf{e}_z=\cos\theta\mathbf{e}_r-\sin\theta\mathbf{e}_\theta$ and the spherical harmonics

$$Y_{00}=\frac{1}{\sqrt{4\pi}} \ , \ Y_{20}=\sqrt{\frac{5}{16\pi}}(1-3\cos^2\theta) \ , \tag{63}$$

it is easy to see that we can write F_c as a series expansion

$$\mathbf{F}_c=-\rho\Omega^2 r\left(\alpha\mathbf{R}_{00}+2\beta\mathbf{R}_{20}-\beta\mathbf{S}_{20}\right) \tag{64}$$

in spheroidal functions, where $\alpha=2\sqrt{4\pi}/3$ and $\beta=\sqrt{16\pi/5}$. We seek a similar expansion for the displacement $\mathbf{u}$,

$$\mathbf{u}=f_1\mathbf{R}_{00}+f_2\mathbf{R}_{20}+g\mathbf{S}_{20} \ ; \tag{65}$$

equations (11) lead to

$$\begin{gathered}
f_1^{''}+\tfrac{2}{r}f^{'}-\tfrac{2}{r^2}f_1=-\tfrac{\rho\Omega^2\alpha}{\lambda+2\mu}r \ , \\
f_2^{''}+\tfrac{2}{r}f_2^{'}-\tfrac{2(\lambda+5\mu)}{\lambda+2\mu}\tfrac{1}{r^2}f_2+\tfrac{6(\lambda+3\mu)}{\lambda+2\mu}\tfrac{1}{r^2}g-\tfrac{6(\lambda+\mu)}{\lambda+2\mu}\tfrac{1}{r}g^{'}=-\tfrac{2\rho\Omega^2\beta}{\lambda+2\mu}r \ , \\
g^{''}+\tfrac{2}{r}g^{'}-\tfrac{6(\lambda+2\mu)}{\mu}\tfrac{1}{r^2}g+\tfrac{2(\lambda+2\mu)}{\mu}\tfrac{1}{r^2}f_2+\tfrac{\lambda+\mu}{\mu}\tfrac{1}{r}f_2^{'}=-\tfrac{\rho\Omega^2\beta}{\mu}r \ .
\end{gathered} \tag{66}$$

We seek solutions of these equations of the form $f_{1,2}$, $g=Ar^n$; the solution of the homogeneous equations (regular in the origin) corresponds to $n=1$; we get

$$f_1=-\frac{\rho\Omega^2\alpha}{10(\lambda+2\mu)}r^3+C_1 r \tag{67}$$

and

$$f_2 = C_2 r \ , \ g = \frac{\rho\Omega^2\beta}{6\lambda} r^3 + C_3 r \ , \tag{68}$$

where $C_{1,2,3}$ are constants of integration. These constants are determined from the boundary conditions given by equations (16) for a free surface. Finally, we get the displacement

$$\begin{aligned}\mathbf{u} = -\tfrac{\rho\Omega^2}{3\lambda}\left[\tfrac{\lambda}{5(\lambda+2\mu)} r\left(r^2 - \tfrac{5\lambda+2\mu}{3\lambda+2\mu}R^2\right) - R^2 r(1-3\cos^2\theta)\right]\mathbf{e}_r + \\ + \tfrac{\rho\Omega^2}{3\lambda} r\left[r^2 - \tfrac{2(3\lambda+\mu)}{3\lambda}R^2\right]\sin\theta\cos\theta\mathbf{e}_\theta \ .\end{aligned} \tag{69}$$

It is worth estimating the equatorial displacement ($\theta = \pi/2$) for the Earth radius $R = 6370km$; with $\rho = 5g/cm^3$ and λ, $\mu = 10^{11}dyn/cm^2$ we get $u = u_r \simeq 10km$.

The temperature of an earthquake. Let us multiply by $\dot{\mathbf{u}}$ the equation of the elastic motion,

$$\rho\ddot{\mathbf{u}} + \mu curl \cdot curl\mathbf{u} - (\lambda + 2\mu) grad \cdot div\mathbf{u} = \mathbf{F} \ ; \tag{70}$$

integrating by parts, we get the law of energy conservation

$$\frac{\partial \mathcal{E}}{\partial t} = -div\mathbf{S} + w \ , \tag{71}$$

where

$$\mathcal{E} = \frac{1}{2}\rho\dot{\mathbf{u}}^2 + \frac{1}{2}\mu(curl\mathbf{u})^2 + \frac{1}{2}(\lambda+2\mu)(div\mathbf{u})^2 \tag{72}$$

is the energy density,

$$S_i = \mu(\dot{u}_j\partial_j u_i - \dot{u}_j\partial_i u_j) - (\lambda+2\mu)\dot{u}_i\partial_j u_j \tag{73}$$

are the components of the energy flux density and $w = \dot{\mathbf{u}}\mathbf{F}$ is the density of mechanical work done by the external force per unit time. It is worth noting that the energy density given by equation (72) differs from the energy density derived from the other form of the equation of motion, *e.g.*,

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+2\mu) grad \cdot div\mathbf{u} = \mathbf{F} \ , \tag{74}$$

by the divergence of a vector; it follows that the energy density and the energy flux density are not unique (well defined).

Making use of equations (6), (7) and (10) we can write symbolically

$$\begin{aligned} curl\mathbf{u} = \tfrac{h}{r} l(l+1)\mathbf{R} + \tfrac{1}{r}\tfrac{d}{dr}(rh)\mathbf{S} + \left[\tfrac{f}{r} - \tfrac{1}{r}\tfrac{d}{dr}(rg)\right]\mathbf{T} \ , \\ div\mathbf{u} = \tfrac{1}{r^2}\tfrac{d}{dr}(r^2 f) - \tfrac{g}{r} l(l+1) \ . \end{aligned} \tag{75}$$

We compute the total energy E by introducing these expressions for $curl\mathbf{u}$ and $div\mathbf{u}$ in equation (72), integrating over the solid angle and integrating by parts over the radius r; for large values of R the boundary conditions given by equations (16) for free vibrations ensure the vanishing of the "surface terms" in the r-integration by parts; in addition, for large values of R we may neglect the f-term in $curl\mathbf{u}$ and the g-term in $div\mathbf{u}$; making use of the equations of motion (11), we get finally

$$E \simeq \frac{2l+1}{8\pi}\int d\mathbf{r}\left[\rho\omega^2 f^2 + l(l+1)\rho\omega^2 g^2 + l(l+1)\rho\omega^2 h^2\right] \ , \tag{76}$$

where the summation over l is omitted (the factor $2l+1$ arises from the summation over m). The functions ωf, ωg and ωh in equation (76) are superpositions of their own normal modes (labelled by n); for large values of R all these eigenmodes may be taken as the spherical Bessel functions and the eigenfrequencies are given by the zeroes of the derivatives of the spherical Bessel functions; we note that these eigenmodes are orthogonal with respect to the r-integration; the f-part in equation (76) is related to the velocity c_1 (the combination of $\lambda + 2\mu$ of the elastic moduli), while the g- and h-parts are related to the velocity c_2(modulus μ).

Let us write the energy given by equation (76) for the normal modes as

$$E \simeq \frac{1}{8\pi} \sum_{lmn} \int d\mathbf{r} \left[\rho\omega_{ln}^{(r)2} f_{ln}^2 + \rho\omega_{ln}^{(s)2} g_{ln}^2 + \rho\omega_{ln}^{(t)2} h_{ln}^2 \right] \ , \tag{77}$$

where the summation over m is restored and the coefficients $l(l+1)$ are included in g_{ln} and h_{ln}. We may use approximately the asymptotic expressions for the functions f_{ln}, g_{ln}, h_{ln} of the form $f_{ln} = a_{ln} \cos[kr - (l+1)\pi/2]/kr$ (spherical Bessel functions), with amplitudes a_{ln}; and, similarly, for g_{ln} and h_{ln} with amplitudes b_{ln} and, respectively, c_{ln}. Effecting the integral, we get

$$E \simeq \frac{1}{4} R \sum_{lmn} \left[\rho c_1^2 a_{ln}^2 + \rho c_2^2 b_{ln}^2 + \rho c_2^2 c_{ln}^2 \right] \ , \tag{78}$$

where R is the radius of the sphere and $c_{1,2}$ are the wave velocities. This is a simple expression, of the form

$$E = \sum_s \rho R c^2 a_s^2 \ , \tag{79}$$

where s is a generic notation for the normal modes.

Let us assume that energy E is given to the vibrating sphere; we ask how it is distributed among the normal modes. It is reasonable to assume that, after many reflections from the surface, the distribution of energy reaches an equilibrium state, in the sense that it does not depend anymore on the time. This state is characterized by a probability density w, which is multiplicative for different spheres; $\ln w$ is additive and function

$$S = -w \ln w \tag{80}$$

should have a maximum value in the equilibrium state, corresponding to a maximal "disorder"; this represents our idea of equilibrium. Obviously, the function S given by eqauation (80) is the entropy. Its maximum value for constant energy is reached for the extremum of the function $S - \beta w E$, where β is a Lagrange multiplier; we get the Boltzmann distribution

$$w = const \cdot e^{-\beta E} \ , \tag{81}$$

or, for one mode,

$$w = \sqrt{\beta \rho R c^2/\pi} e^{-\beta \rho R c^2 a^2} \ . \tag{82}$$

The mean energy per mode is

$$\overline{\varepsilon} = \frac{1}{2} \sqrt{\rho R c^2 T} \tag{83}$$

and the mean value of the square amplitude is

$$\overline{a^2} = \frac{1}{2} \sqrt{\frac{T}{\rho R c^2}} \ , \tag{84}$$

where we introduced the temperature $T = 1/\beta$. The total mean energy is $\overline{E} = N\overline{\varepsilon} = N\sqrt{\rho R c^2 T}/2$, where N is the total number of modes; this equality gives the parameter temperature.

Making use of the asymptotic expressions of the spherical Bessel functions (for the radial functions) we get the normal modes given by $k_{ln}R = (2n+l+1)\pi/2$; hence, we see that the normal modes are equidistant; the corresponding wavelengths are $\lambda_{ln} = 4R/(2n+l+1)$. We may take, tentatively, a cutoff of short wavelengths of the order $1cm$ (corresponding to a frequency $\simeq 500kHz$, for velocity $5km/s$); it is reasonable to admit that below this distance the homogeneous elastic qualities of the Earth do no hold anymore. For this cutoff, we get a maximum number $2n+l+1$ of the order $N_c = 10^9$ and a number of modes of the order $N = N_c^3 = 10^{27}$. For an energy $\overline{E} = 10^{26} dyn \cdot cm$ (corresponding to an earthquake of magnitude $M_w = 7$) we get, from equation (83), a temperature $T = 10^{-22} erg$ ($i.e., \simeq 10^{-5}K$, since $1.38 \times 10^{17}K = 1erg = 1dyn \cdot cm$); the quantity ρRc^2 in equation (83) is $\rho Rc^2 \simeq 10^{20} g/s^2$ (for $\rho = 5g/cm^3$, $R \simeq 6 \times 10^8 cm$ and $c = 5km/s$). The estimation of the temperature is very sensitive to the number of eigenmodes N; for instance, for a cutoff wavelength $10cm$ we get a temperature $T \simeq 10K$. Part of the energy released in an earthquake is spent in mechanical work associated with the motion of the rocks, soil and the damage produced at the Earth's surface; the remaining is dissipated as heat, after a long while; we may see that a big earthquake ($M_w = 7$) may raise the Earth's temperature by as much as cca $10^{-5}K - 10K$ (the inner Earth's temperature is $\simeq 6000K$). We note that the cutoff wavelength, which affects essentially the numerical estimation of the temperature, corresponds to the mean inter-atomic distance in the Debye estimation of the statistical equilibrium of the elastic vibrations (phonons) in a crsytal.

Concluding remarks. Apart from self-gravitation and rotation, the inhomogeneities may bring an important effect upon the vibrations of the solid sphere. For instance, from equation (1), a (uniform) change $\delta\rho$ in density cause a change $\delta\omega/\omega = -\delta\rho/2\rho$ in frequency. The effect of similar changes in the elastic moduli λ and μ can be estimated by using the changes in the wave velocities c in the relation $\omega_{ln} \simeq (c/R)\beta_{ln}$.[21]

An approximate procedure is given in this paper for estimating the spectrum of eigenfrequencies (and eigenfunctions) of the vibrations of a solid sphere, with application to the Earth's vibrations, as those produced by an earthquake. The procedure is sufficiently convenient to be applicable to other, more complex situations involving the vibrations of a solid sphere, as, for instance, the corrections brought about by self-gravitation, rotation and inhomogeneities. The distribution of the energy among the vibrations eigenmodes is also estimated here and the concept of the temperature of an earthquake is tentatively introduced, as another means of characterizing earthquakes and estimating the earthquake's effects.

Acknowledgments. The author is indebted to his colleagues in the Department of Engineering Seismology, Institute of Earth's Physics, Magurele-Bucharest, for many enlightening discussions, and to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many useful discussions and a throughout checking of this work. This work was partially supported by the Romanian Government Research Grant #PN16-35-01-07/11.03.2016.

References

[1] H. Lamb, "On the vibrations of an elastic sphere", Proc. London Math. Soc. **XIII** 189-212 (1882).n

[2] H. Lamb, "On the oscillations of a viscous spheroid", Proc. London Math. Soc. **XIII** 51-66 (1881).

[3] T. J. I'A. Bromwich, "On the influence of gravity on elastic waves, and, in particular, on the vibrations of an elastic globe", Proc. London Math. Soc. **30** 98-120 (1898).

[4] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, USA Government Printing Office, Washington (1964).

[6] J. Mendiguren, "Identification of free oscillation spectral peaks for 1970 July 31 Colombian deep shock using the excitation criterion", Geophys. J. Roy. Astron. Soc. **33** 281-321 (1973.

[7] J. Mendiguren, "High resolution spectroscopy of the Earth's free oscillations, knowing the earthquake source mechanism", Science **179** 179-180 (1973).

[8] F. Gilbert and A. M. Dziewonski, "An application of the normal mode theory to the retrieval of the structural parameters and source mechanisms from seismic spectra", Phil. Trans. Roy. Soc. (London) **A278** 187-269 (1975).

[9] N. A. Haskell, "The dispersion of surface waves in multilayered media", Bul. Seism. Soc. Am. **43** 17-34 (1953).

[10] M. Ewing, W. Jardetzky and F.Press, *Elastic Waves in Layered Media,* McGraw-Hill, NY (1957).

[11] H.Benioff, B.Gutenberg and C. F. Richter, *Progr. Report*, Trans. Am. Geophys. Union **35** 979-987 (1954).

[12] H. Benioff, F.Press and S. W. Smith, "Excitation of the free oscillations of the Earth by earthquakes", J. Geophys. Res. **66** 605-620 (1961).

[13] N. F. Ness, J. C. Harrison and L. B. Slichter, "Observations of the free oscillations of the Earth", J. Geophys. Res. **66** 621-629 (1961).

[14] L. E. Alsop, G. H. Sutton and M. Ewing,"Free oscillations of the Earth observed on strain and pendulum seismographs", J. Geophys. Res. **66** 631-641 (1961).

[15] H.Kanamori, "The energy release in great earthquakes", J. Geophys. Res. **82** 2981-2987 (1977).

[16] B. Block, J. Dratler and R. D. Moore, "Earth and normal modes from a 6.5 magnitude earthquake", Nature **226** 343-344 (1970).

[17] I. Lehmann, "P'", Publications du Bureau Central Seismologique International, Serie A, Travaux Scientifiques, **14** 87-115 (1936).

[18] A. M. Dziewonski and F. Gilbert, "Solidity of the inner core of the Earth inferred from normal mode observations", Nature **234** 465-466 (1971).

[19] C. L. Pekeris and Y. Accad, "Dynamics of the liquid core of the Earth", Phil. Trans. Roy. Soc. London **A273** 237-260 (1972).

[20] F. A. Dahlen and M. L. Smith, "The influence of rotation on the free oscillations of the Earth", Phil. Tras. Roy. Soc. London **A279** 583-629 (1975).

[21] K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009).