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Preliminary tremor, main shock and the "tail" produced by earthquakes on Earth's surface

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Abstract

It is shown that the interaction of the primary seismic waves with Earth's surface gives rise to wave sources on the surface, which generate secondary seismic waves. Using a simple model of interaction with the surface, we compute the secondary waves and show that they produce a main shock followed by a long "tail" of seismic movement. Since the intersection points of the primary seismic spherical waves with the surface move faster than the elastic waves, the main shock arrives later than the primary wave (there is a time lag) and subsides slowly, with a long "tail". All the components of the secondary seismic waves are of the same order of magnitude.

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Introduction. It is well known that any seismic record on the surface of the Earth exhibits, as leading features, a preliminary tremor of feeble movement, followed by a main shock which subsides with a long "tail".[1, 2] It is widely accepted that the preliminary tremor consists of dilatational (P) and distortional (S) waves, while the main shock and its long tail are attributed to the surface waves.[3, 4] In a homogeneous isotropic elastic body a point seismic source localized in the earthquake focus generates, at long distances from the focus, two spherical waves of finite extension, propagating with constant velocities $c_l = \sqrt{(\lambda + 2\mu)/\rho}$ (P wave) and, respectively, $c_t = \sqrt{\mu/\rho}$, where λ and μ are the Lame elastic moduli and ρ is the density of the body. At the point $\mathbf{R}$ on the radius drawn from the focus the P wave arrives at the time $t_P = R/c_l$, while the S wave arrives at $t_S = R/c_t$. If the distance R is sufficiently large these two waves are separated at the moment t by distance $(c_l - c_t)t$. The primary P and S waves interact with the Earth's surface and generate elastic forces in their points of contact with the surface.[5, 6] These forces generate additional waves, which we call secondary waves, propagating in the whole Earth, and on the surface, of course. If we view the interaction of the primary waves with the Earth's surface as producing an undefinite disturbance, we may treat the resulting secondary waves as normal modes, satisfying definite boundary conditions, corresponding to a free surface. The damped surface waves are among such modes. However, the effects of the interaction of the primary waves with the surface occur much before the time needed to set the normal-mode regime; in general, the normal-mode regime is established by multiple reflections on the Earth's surface, which takes a long time. It is worth noting that the damped surface waves have, partially, the form of the stationary waves, as far as their static dependence on the coordinate perpendicular to the surface is disentangled from the time dependence. Such a damped regime is reached in a time of the order of the time needed by the wave to propagate over its characteristic damping length. As normal modes, we may assimilate the secondary waves with damped surface waves, among others; their amplitudes are free parameters.[7]-[9] However, for the effects the secondary waves produce on the Earth's surface it seems more suitable to view them as waves generated by external forces acting on the surface in the transient regime prior to the normal-mode regime. This is the standpoint taken in the present paper. We show below that within this approach we are able to derive the main shock and its long tail produced by earthquakes on the Earth's surface. It is easy to see that the secondary waves interact in their turn with the Earth's surface and generate subsequent waves, and so on, until the whole surface movement subsides to infinity. For the effects the earthquakes produce, the Earth's surface is both the recipient of seismic waves and the siege of wave sources.

Primary seismic waves. A common representation of the force density (per unit mass) generated by a point faulting source (the so-called "double-couple" representation) is given by[10]

$$F_i(\mathbf{R},t)=m_{ij}(t)\partial_j\delta(\mathbf{R}-\mathbf{R}_0)\ , \tag{1}$$

where F_i is the i-th component of the force, $m_{ij}(t)$ is the tensor of the seismic moment $M_{ij}(t)$ divided by density ($m_{ij}=M_{ij}/\rho$) and $\mathbf{R}_0$ is the position vector of the source. For an "elementary" earthquake we may take the time-dependence of the seismic moment as being the Dirac delta function, $m_{ij}(t)=m_{ij}T\delta(t)$, where T is the duration of the earthquake. The spatial Dirac function $\delta(\mathbf{R}-\mathbf{R}_0)$ should be viewed as a function localized over a small distence l, of the order $l=cT$, where c is a generic notation for the velocity of the seismic waves. The seismic moment M_{ij} has the dimension of an energy; if we denote by M its characteristic magnitude, it is natural to assume that this energy is spent to destroy the elastic consistency of the material which is ruptured in the focal volume V during the earthquake; therefore, the equality $M/V\simeq\rho c^2$ may hold. For $M=10^{26}dyn\cdot cm$ (corresponding to an earthquake magnitude $M_w=7$), $\rho=5g/cm^3$ for the average Earth's density and $c=5km/s$ for a mean value of the velocity of the elastic waves we get a volume $V=8\times10^{13}cm^3$ of the focal region and a localization length $l=V^{1/3}\simeq 1km$. This spatial uncertainty leads to a time uncertainty of the order $T=l/c=0.2s$ (for a mean velocity $c=5km/s$).

The equation

$$\ddot{\mathbf{u}}-c_t^2\Delta\mathbf{u}-(c_l^2-c_t^2)grad\cdot div\mathbf{u}=\mathbf{F} \tag{2}$$

of the elastic waves, where $\mathbf{u}$ is the displacement, can be solved with the tensorial force given by equation (1). In the far-field region the solution is given by

$$u_i\simeq\frac{Tm_{ij}x_j}{4\pi c_tR^2}\delta^{'}(R-c_tt)+\frac{Tm_{jk}x_ix_jx_k}{4\pi R^4}\left[\frac{1}{c_l}\delta^{'}(R-c_lt)-\frac{1}{c_t}\delta^{'}(R-c_tt)\right]\ , \tag{3}$$

where $\mathbf{R}$ is taken with respect to $\mathbf{R}_0$; we can see that there are two distinct, localized ("double-shock", proportional to the function $\delta^{'}$) spherical waves. The waves propagating with velocity c_l are the primary P waves (compressional waves), while the waves propagating with velocity c_t are the primary S-waves (they include the shear contribution). The second term on the right in equation (3) is longitudinal ($\sim\mathbf{R}$), while the polarization of the first term depends on the moment tensor. The magnitude of these waves is of the order $u\simeq MT/\rho cRl^2$, where c is a mean wave velocity and $l=cT$ is the linear dimension of the localization of the δ-function (linear dimension of the earthquake's focus). Making use of a seismic moment $M=10^{26}dyn\cdot cm$ (earthquake's magnitude 7), density $\rho=5g/cm^3$, a mean velocity $c=5km/s$, $l=1km$, for an earthquake's duration $T=0.2s$, we get at distance $R=100km$ a far-field wave u of the order $1m$. The result given by equation (3) is very similar with the classical result derived by Stokes.[10, 11] The solution given by equation (3) can be obtained straightforwardly by using the well-known Helmholtz decomposition $\mathbf{u} = grad\Phi + curl\mathbf{A}$ in potentials Φ and $\mathbf{A}$ ($div\mathbf{A} = 0$) and $\mathbf{F} = grad\phi + curl\mathbf{H}$ ($div\mathbf{H} = 0$), the latter being given by $\Delta\phi = div\mathbf{F}$, $\Delta\mathbf{H} = -curl\mathbf{F}$; then, the potentials Φ and $\mathbf{A}$ are given by the wave equations

$$\ddot{\Phi} - c_l^2\Delta\Phi = \phi \ , \ \ddot{\mathbf{A}} - c_t^2\Delta\mathbf{A} = \mathbf{H} \ . \tag{4}$$

Interaction with the surface. We consider a homogeneous isotropic elastic half-space extending in the region $z < 0$ and bounded by the flat surface $z = 0$. The faulting source, which generates the force $\mathbf{F}$, is placed at $\mathbf{R}_0 = (0, 0, z_0)$, $z_0 < 0$. The coordinates of the position vector $\mathbf{R}$ are denoted by (x, y, z).

The wavefront of the spherical waves given by equation (3) intersects the surface $z = 0$ along a circular line defined by $R = (r^2 + z_0^2)^{1/2}$, where $\mathbf{r} = (x, y)$, $r = (x^2 + y^2)^{1/2}$ is the distance from the origin (placed on the surface) to the intersection points (the epicentre). The radius R moves with velocity c, $R = ct$, $t > |z_0|/c$, and the in-plane radius r moves according to the law $r = \sqrt{R^2 - z_0^2} = \sqrt{c^2t^2 - z_0^2}$, where c stands for the velocities $c_{l,t}$; its velocity $v = dr/dt = cR/r = c^2t/r$ is infinite for $r = 0$ ($R = ct = |z_0|$) and tends to c for large distances.

The finite duration T of the source makes the δ'-functions in equation (3) to be viewed as functions with a finite spread $l = \Delta R = cT \ll R$; consequently, the intersection line of the waves with the surface has a finite spread Δr, which can be calculated from

$$R^2 = r^2 + z_0^2 \ , \ (R + l)^2 = (r + \Delta r)^2 + z_0^2 \ ; \tag{5}$$

hence,

$$\Delta r \simeq \frac{2Rl}{r + \sqrt{r^2 + 2Rl}} \ . \tag{6}$$

we can see that for $r \to 0$ the width $\Delta r \simeq \sqrt{2|z_0| l}$ of the seismic spot on the surface is much larger than the width of the spot for large distances $\Delta r \simeq l$ ($2|z_0| \gg l$). For values of r not too close to the origin (epicentre) we may use the approximation $\Delta r \simeq Rl/r$. As long as the spherical wave is fully included in the half-space its total energy is conserved, distributed over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy decreases in the proportion of the spherical sector which subtends the solid angle $2\pi(1 + \cos\theta)$, where $\cos\theta = |z_0|/R$. This amount of energy is transferred to the surface, which generates secondary waves (according to Huygens principle).

In the seismic spot of width Δr generated on the surface by the far-field primary P- and S-waves given by equation (3) we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves scattered from the surface. Strictly speaking, if the reaction force is limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer of thickness Δz ($\Delta z \ll |z_0|$), where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave may depend on R (and r); for simplicity, we take it as a constant.

The volume force per unit mass is given by $\partial_j\sigma_{ij}/\rho$, where $\sigma_{ij} = \rho\left[2c_t^2u_{ij} + (c_l^2 - 2c_t^2)u_{kk}\delta_{ij}\right]$ is the stress tensor and u_{ij} is the strain tensor. The reaction force which compensates the elastic force is

$$f_i = -\partial_j\sigma_{ij}/\rho = -\partial_j\left[2c_t^2u_{ij} + (c_l^2 - 2c_t^2)u_{kk}\delta_{ij}\right] \ . \tag{7}$$

We can calculate the strain tensor from the displacement given by equation (3); in order to compute the secondary waves we use the decomposition in Helmholtz potentials. We denote by $\mathbf{u}'$ the displacement in the secondary waves, and introduce the Helmholtz potentials Φ and $\mathbf{A}$ ($div\mathbf{A} = 0$) by $\mathbf{u}' = grad\Phi + curl\mathbf{A}$; then, we decompose the force $\mathbf{f}$ according to $\mathbf{f} = grad\phi + curl\mathbf{H}$ ($div\mathbf{H} = 0$), where $\Delta\phi = div\mathbf{f}$ and $\Delta\mathbf{H} = -curl\mathbf{f}$; by the equation of the elastic waves, the Helmholtz potentials satisfy the wave equations (4); by straightforward calculations we get $\phi = -c_l^2 u_{ii}$ and $\mathbf{H} = c_t^2 curl\mathbf{u}$.

We can calculate the displacement in the secondary waves $\mathbf{u}' = grad\Phi + curl\mathbf{A}$, by solving equations (4) with $\phi = -c_l^2 u_{ii}$ and $\mathbf{H} = c_t^2 curl\mathbf{u}$ restricted to the superficial layer of thickness Δz, the primary displacement $\mathbf{u}$ being given by equation (3). Apart from appreciable complications, this procedure brings many superfluos features which obscure the relevant physical picture. This is why we prefer to use a simplified model of the form

$$\phi = \phi_0(r)\delta(z)\delta(r - v_l t) \ , \ \mathbf{H} = \mathbf{H}_0(r)\delta(z)\delta(r - v_t t) \ , \tag{8}$$

where $div\mathbf{H}_0 = 0$; equations (8) describe wave sources, distributed circularly on the surface, propagating on the surface with constant velocities $v_{l,t}$ and limited to a superficial layer of zero thickness; their magnitude decreases with increasing r, approximately as $1/r$ (for large r), according to equations (3); the velocities $v_{l,t}$ in equation (8) correspond to the velocities $v_{l,t} = dr/dt = c_{l,t}^2 t/r$ calculated above, which are greater than $c_{l,t}$, depend on r and tends to $c_{l,t}$ for large values of the distance r. We make a further simplification and consider them as constant velocities slightly greater than $c_{l,t}$. Also, we consider the origin of the time at $r = 0$ (the epicentre).

Secondary waves. The solutions of equations (4) are given by

$$\begin{gathered}\Phi = \frac{1}{4\pi c_l^2}\int_0^\infty dt' \int_V d\mathbf{R}' \frac{\phi_0(r')\delta(z')\delta(r'-v_l t')}{\left|\mathbf{R}-\mathbf{R}'\right|}\delta\left(t - t' - \left|\mathbf{R}-\mathbf{R}'\right|/c_l\right) \ , \\ \mathbf{A} = \frac{1}{4\pi c_t^2}\int_0^\infty dt' \int_V d\mathbf{R}' \frac{\mathbf{H}_0(r')\delta(z')\delta(r'-v_t t')}{\left|\mathbf{R}-\mathbf{R}'\right|}\delta\left(t - t' - \left|\mathbf{R}-\mathbf{R}'\right|/c_t\right) \ ,\end{gathered} \tag{9}$$

where V denotes the integration volume of the half-space. We focus first on the potential Φ, which can be written as

$$\Phi = \frac{1}{4\pi vc^2}\int_V d\mathbf{r}' \frac{\phi_0(r')}{\left(r^2 + r'^2 - 2rr'\cos\varphi + z^2\right)^{1/2}}\delta\left[t - r'/v - \left(r^2 + r'^2 - 2rr'\cos\varphi + z^2\right)^{1/2}/c\right] \ , \tag{10}$$

where φ is the angle between the vectors $\mathbf{r}$ and $\mathbf{r}'$ and we use c and v for c_l and, respectively, v_l, for simplicity. In order to calculate the integral with respect to the angle φ in equation (10) we introduce the function

$$F(\cos\varphi) = t - r'/v - \left(r^2 + r'^2 - 2rr'\cos\varphi + z^2\right)^{1/2}/c \tag{11}$$

and look for its zeroes, $F_0 = F(\cos\varphi_0) = 0$; we note that, if there exists one root of this equation, there exists another one at least, in view of the symmetry $\cos\varphi = \cos(2\pi - \varphi)$. Then, we expand in a Taylor series the function F in the vicinity of its zero, according to

$$F = F_0 + (\cos\varphi - \cos\varphi_0)\, F_0' + ... = (\cos\varphi - \cos\varphi_0)\, F_0' + ... \ , \tag{12}$$

where F_0' is the derivative of the function F with respect to $\cos\varphi$ for $\cos\varphi = \cos\varphi_0$. It is easy to see that the integral reduces to

$$\Phi = \frac{1}{2\pi cvr}\int_0^\infty dr' \frac{\phi_0(r')}{\sin\varphi_0} \ , \tag{13}$$

where φ_0 is the root of the equation $F(\cos\varphi_0) = 0$, lying between 0 and π.

The root $\cos\varphi_0$ is given by

$$F(\cos\varphi_0) = t - r^{'}/v - \left(r^2 + r^{'2} - 2rr^{'}\cos\varphi_0 + z^2\right)^{1/2}/c = 0 \ , \tag{14}$$

or

$$\left(1 - c^2/v^2\right)r^{'2} - 2\left(r\cos\varphi_0 - c^2t/v\right)r^{'} - \left(c^2t^2 - r^2 - z^2\right) = 0 \tag{15}$$

for $r^{'} < vt$. The important feature brought by the diference between the two velocities c and v can be accounted for conveniently by assuming that the two velocities are close to one another; we set $v = c(1+\varepsilon)$, $0 < \varepsilon \ll 1$. In this circumstance we may neglect the quadratic term $\sim r^{'2}$ in equation (15) and replace t by the "advanced" time $t^{'} = t(1-\varepsilon)$ (*i.e.*, $t^{'}_{l,t} = t(1-\varepsilon_{l,t})$); we get

$$\cos\varphi_0 \simeq \frac{2ct^{'}r^{'} - B}{2rr^{'}} \ , \ B = c^2t^{'2} - r^2 - z^2 \tag{16}$$

for $r^{'} < vt = ct^{'}(1+2\varepsilon)$. It is easy to see that this equation has no solution for $ct^{'} < r$ $(B < 0)$; for $ct^{'} > r$ and z close to zero (near the surface) it has two solutions

$$r^{'}_1 = \frac{B}{2(ct^{'} + r)} \ , \ r^{'}_2 = \frac{B}{2(ct^{'} - r)} \ , \tag{17}$$

corresponding to $\cos\varphi_0 = -1$ $(\varphi_0 = \pi)$ and, respectively, $\cos\varphi_0 = 1$ $(\varphi_0 = 0)$. In the integral given by equation (13) we pass from the variable $r^{'}$ to the variable φ_0; the potential $\phi_0(r^{'})$ can be approximated by a decreasing function of $R = (r^2 + z_0^2)^{1/2}$ $(\phi_0 \simeq 1/R)$ and it may be taken out of the integral sign; we get

$$\Phi \simeq \frac{B\phi_0}{4\pi c^2}\int_0^{\pi} d\varphi_0 \frac{1}{(r\cos\varphi_0 - ct^{'})^2} \ , \tag{18}$$

or

$$\Phi \simeq \frac{B\phi_0}{4\pi c^2 r^2}\frac{\partial}{\partial x}\int_0^{\pi/2} d\varphi_0 \left(\frac{1}{\cos\varphi_0 - x} - \frac{1}{\cos\varphi_0 + x}\right) \ , \ x = ct^{'}/r > 1. \tag{19}$$

the integrals in equation (19) can be effected immediately; we get the potential

$$\Phi \simeq \frac{\phi_0}{4c_l^2}\frac{(c_l^2t_l^{'2} - r^2 - z^2)c_lt_l^{'}}{\left(c_l^2t_l^{'2} - r^2\right)^{3/2}} \tag{20}$$

(for z close to $z = 0$), where the velocity c_l is restored. Similarly, we get from equations (9) the vector potential

$$\mathbf{A} \simeq \frac{\mathbf{H}_0}{4c_t^2}\frac{(c_t^2t_t^{'2} - r^2 - z^2)c_tt_t^{'}}{\left(c_t^2t_t^{'2} - r^2\right)^{3/2}} \ . \tag{21}$$

The qualitative singular behaviour of these waves resembles the algebraic singularity of the waves in two dimensions produced by localized sources.[12]

Discussion and conclusion. Making use of $\mathbf{u} = grad\Phi + curl\mathbf{A}$ we can compute the displacement $\mathbf{u}$. First, we note that the displacement is singular at $c_{l,t}t^{'} = r$; this indicates the existence of two main shocks, occcurring after the arrival of the primary waves. Indeed, the primary waves arrive at the observation point $\mathbf{r}$ at the time $t_p = r/v_{l,t} = (r/c_{l,t})(1 - \varepsilon_{l,t})$, while the main shocks occur at $t_m = t^{'}_{l,t}/(1 - \varepsilon_{l,t}) \simeq (r/c_{l,t})(1 + \varepsilon_{l,t})$; we can see that there exists a time delay

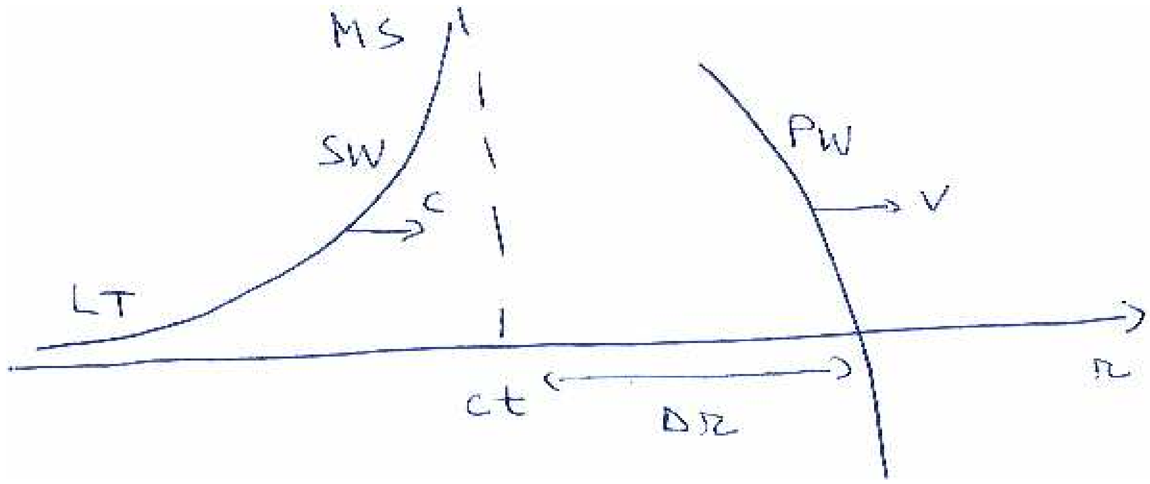

Figure 1: Primary wave (PW), moving with velocity v on the Earth's surface, secondary wave (SW), moving with velocity $c < v$, the main shock (MS) and the long tail (LT); the separation between the two wavefronts is $\Delta r = 2(v - c)t$ and the time delay is $\Delta t = (2r/c)(v/c - 1)$, where r is the distance on the surface from the epicentre.

$\Delta t \simeq t_m - t_p \simeq 2(r/c_{lt})\varepsilon_{lt}$ between the primary waves and the wavefronts of the secondary waves (the main shocks). The singularity in equations (20) and (21) originates in using constant velocities $v_{l,t}$; actually, an undeterminacy of the form $\Delta v \simeq c\varepsilon$ exists in these velocities, which entails an undeterminacy $t^{'}\varepsilon$ in the time $t^{'}$, such that the smallest value of the denominator in equations (20) and (21) is of the order $c^2 t^{'2}\varepsilon$. In the vicinity of the two main shocks the leading contributions to the components of the surface displacement ($z = 0$, in polar cylindrical coordinates) are given by

$$u_r \simeq \frac{\phi_0 t_l^{'}}{4c_l} \cdot \frac{r}{\left(c_l^2 t_l^{'2} - r^2\right)^{3/2}} \ ,$$

$$u_\varphi \simeq -\frac{H_{0z} t_t^{'}}{4c_t} \cdot \frac{r}{\left(c_t^2 t_t^{'2} - r^2\right)^{3/2}} \ , \qquad (22)$$

$$u_z \simeq \frac{H_{0\varphi} t_t^{'}}{4c_t} \cdot \frac{r}{\left(c_t^2 t_t^{'2} - r^2\right)^{3/2}} \ ;$$

we can see that there exists a horizontal component of the displacement perpendicular to the propagation direction (u_φ) and both the r-component and the φ, z-components, which make right angles with the propagation direction, are of the same order of magnitude.[1] For long times ($c_{l,t}t_{l,t}^{'} \gg r$) the displacement (from equations (20) and (21)) goes like

$$u_r \simeq \frac{\phi_0 r}{4c_l^4 t_l^{'2}} \ , \ u_\varphi \simeq -\frac{H_{0z} r}{4c_t^4 t_t^{'2}} \ , \ u_z \simeq \frac{H_{0\varphi}}{4c_t^2 r} \ , \qquad (23)$$

which show that the displacement exhibits a long tail, especially the z-component; it subsides as a consequence of the time-dependence induced in the potential $\mathbf{H}_0$ by the integration variable $r^{'}$, a circumstance which is neglected in the calculations presented here. Primary and secondary waves, the main shock and the long tail are shown in Fig.1.

In conclusion, we may say that the interaction of the primary P- and S-seismic waves with the Earth's surface gives rise to sources of secondary waves. Since the intersection points of the primary waves with the surface move faster on the Earth's surface than the elastic waves, the secondary waves arrive at the observation point with a time lag. At the observation point the secondary waves produce main shocks, followed by long tails, in accordance with the recorded seismic observations. The main shocks and the tails of the seismic secondary waves are calculated in this paper, by using a simple model of interaction of the primary waves with the Earth's surface.

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