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On the Lamb problem. Forced vibrations of an isotropic elastic half-space

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Abstract

The problem of vibrations generated in a homogeneous isotropic elastic half-space by spatially concentrated forces, known in Seismology as (part of) the Lamb problem, is formulated here in terms of Helmholtz potentials of the elastic displacement. The method is based on time Fourier transforms, spatial Fourier transforms with respect to the coordinates parallel to the surface (in-plane Fourier transforms) and generalized wave equations, which include the surface values of the functions and their derivatives. This formulation provides a formal general solution to the problem of forced elastic vibrations of the isotropic half-space. Explicit results are given for forces derived from a gradient, localized at an inner point in the half-space, which correspond to a scalar seismic moment in the double-couple representation of seismic sources. Similarly, explicit results are given for a surface force perpendicular to the surface and localized at a point on the surface. Both harmonic time dependence and time δ-pulses are considered. It is shown that a δ-like time dependence of the forces generates perturbations which are vanishing in time, which, consequently, cannot be viewed properly as vibrations. The distinction between the generation and propagation of the seismic waves and the vibrations problem with inclusion of the boundary conditions is emphasized, as well as the role played by the eigenmodes of the isotropic elastic half-space. Similarly, the distinction is highlighted between the transient regime of wave propagation prior to the establishment of the elastic vibrations and the stationary waves regime.

Introduction. The generation and propagation of the seismic waves is a basic problem in Seismology. It gives information about the processes occurring in an earthquake focus, about the inner structure of the Earth and the effects the seismic waves have on the Earth's surface. The focal region of an "elementary" earthquake is localized beneath the Earth's surface, such that, for long distances, we may consider point seismic sources, *i.e.* sources represented by δ-functions, or derivatives of the δ-functions. A similar representation holds for the time dependence of such "elementary" seismic sources. As regards the effects of the seismic waves on the Earth's surface, it is convenient to approximate the Earth as a half-space bounded by a plane surface; by another useful simplification the Earth is viewed as a homogeneous isotropic elastic solid. Within such circumstances, the generation and propagation of the seismic waves, as well as the vibrations of the isotropic elastic half-space are known as defining the so-called Lamb problem.[1, 2]

In the complex of physical phenomena involved in the Lamb problem (generation and propagation of seismic waves, vibrations generated by seismic sources, eignemodes) there exists a certain difficulty, related to the requirement of satisfying the boundary conditions at the Earth's surface, usually considered a free surface. With boundary conditions the Lamb problem is not a wave propagation problem, but a vibrations problem. The problem of vibrations of an elastic sphere was solved as early as 1882.[3]-[5] In contrast, the problem of vibrations of an elastic half-space is not solved, probably because the interest was focused on wave propagation, although the formulation of the problem is one of vibrations. We give here a formal solution of the vibrations of an isotropic elastic half-space, as well as a few explicit results for some particular cases.

The difficulties related to Lamb problem are even more complicated by the interpretation of the seismic records. The general structure of any seismic record exhibits a preliminary feeble tremor, consisting of the primary P- and S-waves (which at large distances are separated), followed by a main shock with a long tail.[6, 7] It is almost generally accepted that the main shock and its long tail are due to the surface waves,[8, 9] although these waves are free waves, *i.e.* solution of the homogeneous elastic waves equation, whose amplitude is not determined. In addition, the solution is often approximated by incident and reflected plane waves (which satisfy the boundary conditions), although, similarly, these plane waves are free waves. A series of other similar approximations are currently made in this context, related to various types of waves, like head (or "lateral") waves, cylindrical or conical waves, leaking waves, inhomogeneous, damped waves, etc.[10]-[27]

The seismic waves suffer multiple reflections on the Earth's surface (or on the interfaces of the internal Earth's layers), such that the stationary regime of oscillations sets in a finite time interval. The relevant magnitude of this amount of time is of the order R/c, where R is the radius of the Earth and c is the wave velocity. For $R = 6370km$ and a mean velocity $c = 5km/s$ of the elastic waves we get $R/c \simeq 1274s$; this time interval is much longer than the time taken by the seismic waves to propagate from the source to the Earth's surface. We can see that the effects of the seismic waves on the Earth's surface are produced in a time much shorter than the time needed for attaining the stationary regime of vibrations. It follows that, as regards the effect of the earthquakes, we are interested primarily in the transient regime of the seismic waves, where the boundary conditions are practically radiation conditions. An "elementary" earthquake is produced by sources localized both in space and time. In these circumstances, in a first approximation, the solution consists of primary P- and S- spherical waves generated by temporal and spatial δ-pulses from the seismic source (or derivatives of the δ function). For sources with a finite temporal or spatial extension (or for multiple sources) a structure factor of the focal region is necessary. The interaction of the primary waves with the surface generate additional wave sources placed on the surface, which may be termed secondary waves; they are responsible for the main shock and the long tail recorded in seismograms.

General solution. The equation of the elastic waves in an isotropic body reads[28]

$$\ddot{\mathbf{u}} - c_2^2 \Delta \mathbf{u} - (c_1^2 - c_2^2) grad \cdot div \mathbf{u} = \mathbf{F} \ , \tag{1}$$

where $\mathbf{u}$ is the displacement, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$ are the wave velocities, λ, μ are the Lame elastic moduli, ρ is the density and $\mathbf{F}$ is the force (per unit mass). We consider this equation in the half-space occupying the region $z < 0$ and bounded by the plane surface $z = 0$; the force $\mathbf{F}$ is placed inside the half-space.

As it is well known, in the absence of the force $\mathbf{F}$ ($\mathbf{F} = 0$) the homogeneous equation (1) (the "free" equation) extended to the whole space exhibits two types of ("free") waves: longitudinal waves, propagating with velocity c_1, and transverse waves, propagating with velocity c_2.[29, 30, 31] In the half-space the free equation (1) exhibits a combination of incident and reflected longitudinal and transverse waves which satisfy the boundary conditions; insofar as the amplitude of the incident wave is a free parameter and the waves satisfy the boundary conditions these combinations of incident and reflected waves may be viewed as a special type of eigenmodes for the vibrations problem; the specificity originates in the fact that the half-space is only partially finite. This feature is more obvious in the damped surface waves (Rayleigh waves),[8, 28] which propagate along the surface and are damped along the direction perpendicular to the surface. They are solutions of the homogeneous equation (1), satisfy the boundary conditions, have a free (undetermined) amplitude and their dependence on the coordinate perpendicular to the surface is separated from the dependence of the combined other two coordinates and time; this later feature is typical for vibrations; the surface waves exhibit this feature only partially; they are eigenmodes of the elastic vibrations of the isotropic half-space. A similar character may be attributed to other similar waves which propagate at interfaces, like the well-known Love waves, or Stonely waves.[6, 10, 32, 33] In addition, we shall show below that other special eigenmodes are present in the isotropic elastic half-space, represented by plane waves which propagate along directions parallel with the surface $z=0$; we may call them lateral waves, though the term "lateral" used here has a different meaning than the same term used in the current seismological literature (see, for instance, Ref. [27]). We emphasize the distinction, as it is usual, between eigenmodes of vibrations ("free waves"), with undetermined amplitudes, and forced vibrations, well-determined by known by forces.

We consider now the solutions determined by the force $\mathbf{F}$ in the inhomogeneous equation (1) for the half-space with boundary conditions; these solutions are "forced" vibrations. The force $\mathbf{P}$ on the surface (the pressure), with the components P_i is given by $\sigma_{iz}\mid_{z=0}=-P_i$, where $\sigma_{ij}=\rho\left[2c_2^2u_{ij}+(c_1^2-2c_2^2)div\mathbf{u}\delta_{ij}\right]$ is the stress tensor and u_{ij} is the strain tensor.[28] We use labels $i,j,k...=1,2,3$ for the coordinates $x=x_1$, $y=x_2$, $z=x_3$, as well as labels $\alpha,\beta,\gamma...=1,2$ for the coordinates $x=x_1$, $y=x_2$. The boundary conditions read

$$\partial_\alpha u_3+\partial_3 u_\alpha\mid_{z=0}=-\frac{P_\alpha}{\rho c_2^2}\ ,\ 2\partial_3 u_3+\frac{c_1^2-2c_2^2}{c_2^2}div\mathbf{u}\mid_{z=0}=-\frac{P_3}{\rho c_2^2}\ ; \tag{2}$$

we introduce the notations $p_i=P_i/\rho c_2^2$.

We use the Helmholtz decomposition for the displacement $\mathbf{u}$ and force $\mathbf{F}$, by introducing

$$\begin{gathered}\mathbf{u}=grad\Phi+curl\mathbf{A}\ ,\ div\mathbf{A}=0\ ,\\ \mathbf{F}=grad\varphi+curl\mathbf{H}\ ,\ div\mathbf{H}=0\ ;\end{gathered} \tag{3}$$

equation (1) is transformed in two standard wave equations

$$\ddot{\Phi}-c_1^2\Delta\Phi=\varphi\ ,\ \ddot{\mathbf{A}}-c_2^2\Delta\mathbf{A}=\mathbf{H}\ , \tag{4}$$

where the potentials φ and $\mathbf{H}$ are given by $\Delta\varphi=div\mathbf{F}$, $\Delta\mathbf{H}=-curl\mathbf{F}$; for the moment, we consider these potentials as known quantities. Since the time is uniform and the in-plane coordinates (x_1,x_2) are also uniform, it is convenient to use time and in-plane Fourier transforms of the form

$$\mathbf{u}(\mathbf{r},z,t)=\frac{1}{2\pi}\int d\omega e^{-i\omega t}\frac{1}{(2\pi)^2}\int d\mathbf{k}\mathbf{u}(\mathbf{k},z,\omega)e^{i\mathbf{kr}}\ , \tag{5}$$

where $\mathbf{r}=(x_1,x_2)$ is the in-plane position vector. For simplicity, we use the same symbol for the mutual Fourier transforms, without any risk of confusion; similarly, we drop the arguments, which can easily be read from the context. Equations (4) become

$$\Phi''+\kappa_1^2\Phi=-\varphi/c_1^2\ ,\ \mathbf{A}''+\kappa_2^2\mathbf{A}=-\mathbf{H}/c_2^2\ , \tag{6}$$

where

$$\kappa_{1,2}^2=\omega^2/c_{1,2}^2-k^2\ ; \tag{7}$$

the prime means the derivation with respect to z; we note that $\kappa_{1,2}$ may be either real or imaginary, with either sign.

In order to approach in a convenient manner the boundary conditions (2) we introduce the surface values of the functions and their derivatives in equations, *i.e.* we write

$$\Phi^{''}+\kappa_1^2\Phi=-\varphi/c_1^2-\Phi^1\delta(z)-\Phi^0\delta^{'}(z)\ ,$$
$$\mathbf{A}^{''}+\kappa_2^2\mathbf{A}=-\mathbf{H}/c_2^2-\mathbf{A}^1\delta(z)-\mathbf{A}^0\delta^{'}(z)\ , \tag{8}$$

where $\mathbf{\Phi}^0=\mathbf{\Phi}\mid_{z=0}$, $\mathbf{A}^0=\mathbf{A}\mid_{z=0}$, $\Phi^1=d\Phi/dz\mid_{z=0}$ and $\mathbf{A}^1=d\mathbf{A}/dz\mid_{z=0}$; integrating these equations along a perpendicular of infinitesimal length across the surface $z=0$, we check immediately the terms $\Phi^1\delta(z)$, $\mathbf{A}^1\delta(z)$; multiplying the equations by z and repeating the procedure we check the terms $\Phi^0\delta^{'}(z)$, $\mathbf{A}^0\delta^{'}(z)$. More formally, we can justify the presence of these singular terms by using the generalized functions (distributions) $\Phi\theta(-z)$ and $\mathbf{A}\theta(-z)$, where $\theta(z)=1$ for $z>0$, $\theta(z)=0$ for $z<0$ is the step function.[34] The parameters Φ^1, $\mathbf{A}^1$ are not independent of parameters Φ^0, $\mathbf{A}^0$ (due to the boundary conditions at infinity).

Making use of the Green function $G=e^{i\kappa|z|}/2i\kappa$ of the one-dimensional Helmholtz equation $G^{''}+\kappa^2G=\delta(z)$, we can write immediately the solutions of the equations (5):

$$\Phi=-\frac{1}{2i\kappa_1c_1^2}\int_{-\infty}^0dz^{'}\varphi(z^{'})\left[e^{i\kappa_1\left|z-z^{'}\right|}-e^{i\kappa_1\left|z+z^{'}\right|}\right]+\Phi^0e^{i\kappa_1|z|}\ ,$$
$$\mathbf{A}=-\frac{1}{2i\kappa_2c_2^2}\int_{-\infty}^0dz^{'}\mathbf{H}(z^{'})\left[e^{i\kappa_2\left|z-z^{'}\right|}-e^{i\kappa_2\left|z+z^{'}\right|}\right]+\mathbf{A}^0e^{i\kappa_2|z|}\ ; \tag{9}$$

We note the occurrence of the "reflected" ("image") Green function $e^{i\kappa_{1,2}\left|z+z^{'}\right|}/2i\kappa_{1,2}$ in these formulae. We can check the transversality condition $div\mathbf{A}=0$ in equations (9) (due to $div\mathbf{H}=0$), which in Fourier transforms reads $ik_\alpha A_\alpha+A_3^{'}=0$; we assume $ik_\alpha A_\alpha^0+A_3^1=0$. It is also worth noting in equations (9) that $\kappa_{1,2}$ may have either sign. The derivatives on the surface of these functions are given by

$$\Phi^1=-\frac{1}{c_1^2}\int_{-\infty}^0dz^{'}\varphi(z^{'})e^{i\kappa_1\left|z^{'}\right|}-i\kappa_1\Phi^0\ ,$$
$$\mathbf{A}^1=-\frac{1}{c_2^2}\int_{-\infty}^0dz^{'}\mathbf{H}(z^{'})e^{i\kappa_2\left|z^{'}\right|}-i\kappa_2\mathbf{A}^0\ . \tag{10}$$

We write now the boundary conditions given by equations (2) by using the Fourier transforms; to this end we need the second derivative $\Phi^{(2)}=d^2\Phi/dz^2\mid_{z=0}$ on the surface, which can be derived immediately from equation (8): $\Phi^{(2)}=-\kappa_1^2\Phi^0-\varphi^0/c_1^2$, where $\varphi^0=\varphi\mid_{z=0}$ (a similar notation will be used for $\mathbf{H}$). The boundary conditions can now be written as

$$2\kappa_1k_1\Phi^0+2k_1k_2A_1^0+\left(\kappa_2^2+k_2^2-k_1^2\right)A_2^0=q_1\ ,$$
$$2\kappa_1k_2\Phi^0-\left(\kappa_2^2+k_1^2-k_2^2\right)A_1^0-2k_1k_2A_2^0=q_2\ , \tag{11}$$
$$\left(k^2-\kappa_2^2\right)\Phi^0-2\kappa_2k_2A_1^0+2\kappa_2k_1A_2^0=q_3\ ,$$

where

$$q_1 = -p_1 - H_2^0/c_2^2 + \tfrac{2ik_1}{c_1^2} \int_{-\infty}^0 dz^{'} \varphi(z^{'}) e^{i\kappa_1 \left| z^{'} \right|} \ ,$$

$$q_2 = -p_2 + H_1^0/c_2^2 + \tfrac{2ik_2}{c_1^2} \int_{-\infty}^0 dz^{'} \varphi(z^{'}) e^{i\kappa_1 \left| z^{'} \right|} \ , \tag{12}$$

$$q_3 = -p_3 + \varphi^0/c_2^2 + \tfrac{2i}{c_2^2} \int_{-\infty}^0 dz^{'} \left[k_1 H_2(z^{'}) - k_2 H_1(z^{'}) \right] e^{i\kappa_2 \left| z^{'} \right|} \ ;$$

in these equations $k_{1,2}$ are te components of the vector $\mathbf{k} = (k_1, k_2)$. Equations (11) represent a system of three equations with the unknowns Φ^0,A_1^0 and A_2^0; A_3^0 is eliminated by the transversality condition $div\mathbf{A} = 0$ $(ik_\alpha A_\alpha + A_3^{'} = 0)$; from equations (10) it is given by

$$\kappa_2 A_3^0 = k_\alpha A_\alpha^0 + \frac{i}{c_2^2} \int_{-\infty}^0 dz^{'} H_3(z^{'}) e^{i\kappa_2 \left| z^{'} \right|} \ . \tag{13}$$

The solutions of the system of equations (11) are

$$\Phi^0 = \tfrac{1}{\Delta} \left[-2\kappa_2(\kappa_2^2 + k^2)(k_1 q_1 + k_2 q_2) + (\kappa_2^4 - k^4) q_3 \right] \ ,$$

$$A_1^0 = \tfrac{1}{\Delta} \{ -2k_1 k_2 (k^2 - \kappa_2^2 + 2\kappa_1\kappa_2) q_1 + [4\kappa_1\kappa_2 k_1^2 -$$

$$-(k^2 - \kappa_2^2)(\kappa_2^2 + k_2^2 - k_1^2)] q_2 + 2\kappa_1 k_2 (\kappa_2^2 + k^2) q_3 \} \ , \tag{14}$$

$$A_2^0 = \tfrac{1}{\Delta} \{ [-4\kappa_1\kappa_2 k_2^2 + (k^2 - \kappa_2^2)(\kappa_2^2 + k_1^2 - k_2^2] q_1 +$$

$$+2k_1 k_2 (k^2 - \kappa_2^2 + 2\kappa_1\kappa_2) q_2 - 2\kappa_1 k_1 (\kappa_2^2 + k^2) q_3 \} \ ,$$

where

$$\Delta = -(\kappa_2^2 + k^2) \left[(\kappa_2^2 - k^2)^2 + 4\kappa_1\kappa_2 k^2 \right] = -\frac{\omega^2}{c_2^2} \left[(\kappa_2^2 - k^2)^2 + 4\kappa_1\kappa_2 k^2 \right] \ . \tag{15}$$

Having known the parameters Φ^0 and $\mathbf{A}^0$ the potentials Φ and $\mathbf{A}$ given by equations (9) and the displacement $\mathbf{u} = grad\Phi + curl\mathbf{A}$ are completely determined; it remains to perform the reverse time and spatial Fourier transforms.

General time dependence. Eigenmodes. The displacement $\mathbf{u}$ computed from the above formulae includes terms of the general form

$$f(\omega, \mathbf{k}) F(\omega^2, \mathbf{k}) \ , \ \frac{f(\omega, \mathbf{k}) F(\omega^2, \mathbf{k})}{\kappa_{1,2}} \ , \ \frac{f(\omega, \mathbf{k}) F(\omega^2, \mathbf{k})}{\Delta} \ , \ \frac{f(\omega, \mathbf{k}) F(\omega^2, \mathbf{k})}{\kappa_2 \Delta} \ , \tag{16}$$

where the function f comes both from the volume force (via φ and $\mathbf{H}$) and surface force and the function F arises from the structure of the wave equations. The κ_1 in the denominator arises from the potential Φ, the κ_2 arises from A_3^0 and Δ originates in the parameters Φ^0, $\mathbf{A}^0$ (equations (14)).

A proper force which generates vibrations includes time harmonic oscillations which lead to a general form

$$f = f_1 \delta(\omega - \Omega) + f_1^* \delta(\omega + \Omega) \tag{17}$$

for the function f, where Ω is the frequency of the force. It is easy to see that, by multiplying the functions in equation (16) by $e^{-i\omega t}$ and integrating over ω, in order to get the time reversed

Fourier transform, we get a time dependence of the form $\sim \cos\Omega t$, as expected; harmonic time oscillations of the force generate harmonic vibrations.

From a technical standpoint it is interesting the rather unphysical case where the time dependence of the force is a δ-impulse and f does not depend on ω. In this case the denominator Δ in equation (16) brought a double pole $\omega = 0$; this pole does not contribute to the Fourier transform, since the functions F, F/κ_2 are even functions of ω. It is convenient to change $\kappa_{1,2} \rightarrow i\kappa_{1,2}$ in equation (16); then we can see that $\Delta = 0$ implies

$$(\kappa_2^2 - k^2)^2 - 4\kappa_1\kappa_2 k^2 = 0 \tag{18}$$

which is the well-known dispersion relation of the damped surface waves (Rayleigh waves);[28] the solution of this equation is $\omega_0 = c_2\xi k$, where ξ $(0 < \xi < 1)$ is the solution of the Rayleigh equation

$$\xi^6 - 8\xi^4 + 8(3 - 2c_2^2/c_1^2)\xi^2 - 16(1 - c_2^2/c_1^2) = 0 \ . \tag{19}$$

Indeed, the Rayleigh waves are eigenmodes of the isotropic elastic half-space, and their dispersion relation is expected to occur in the denominator of the forced vibrations. As it is well known,28 the ratio c_2/c_1 varies from $1/\sqrt{2}$ to 0, and the root ξ of the equation (19) varies approximately from 0.87 to 0.95. We expand the determinant Δ in powers of $\omega - \omega_0$ and get

$$\Delta = -\frac{4\omega_0^2}{c_2^3}\Delta_0(\xi)k^3(\omega - \omega_0) + ... \ , \tag{20}$$

where

$$\Delta_0(\xi) = \xi\left[\xi^2 - 2 + \frac{4\left(1 - 2c_2^2\xi^2/c_1^2 + c_2^2/c_1^2\right)}{\sqrt{(1-\xi^2)(1 - c_2^2\xi^2/c_1^2)}}\right] \ ; \tag{21}$$

we can see that terms with Δ in the denominator in equation (16) have a pole $\omega = \omega_0$. The reverse time Fourier tranforms of such terms gives contributions of the form

$$g(\mathbf{k})e^{-ic_2\xi kt} + c.c. \ , \tag{22}$$

where g is a function of the wavevector $\mathbf{k}$ (we note that there exists another pole at $\omega = -\omega_0$); these contributions are harmonic oscillations of the form $\cos c_2\xi kt$, $\sin c_2\xi kt$.

The spatial dependence is more difficult to be computed, in general. On effecting integrals of the form

$$u = \int d\mathbf{k} f(\mathbf{k}, \kappa)e^{i\mathbf{kr}}e^{i\kappa|z|} \ , \tag{23}$$

which appear in the reverse spatial Fourier transform, we should be aware of the presence of κ in the integrand $f(\mathbf{k}, \kappa)$, which must obey the symmetry condition $f^*(-\mathbf{k}, -\kappa) = \mathbf{f}(\mathbf{k}, \kappa)$ in order the function u be real. It is more convenient to use the formula

$$u = \frac{1}{2}\int d\mathbf{k}e^{i\mathbf{kr}}\left[f(\mathbf{k}, \kappa)e^{i\kappa|z|} + f^*(-\mathbf{k}, \kappa)e^{-i\kappa|z|}\right] \tag{24}$$

for such integrals, which does not imply the change of the sign of κ.

A typical spatial dependence is provided by the Sommerfeld-Weyl integral[35]

$$\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{\kappa}e^{i\kappa|z|} \ , \tag{25}$$

which in this context leads to terms of the form

$$\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{i\alpha k}e^{-\alpha k|z|}e^{-ic_2\xi kt} \ , \tag{26}$$

where $\alpha=\sqrt{1-c_2^2\xi^2/c_1^2}$ for $\kappa=\kappa_1$ and $\alpha=\sqrt{1-\xi^2}$ for $\kappa=\kappa_2$. The analytic continuation of the Sommerfeld-Weyl integral gives for equation (26)

$$\begin{gathered}\tfrac{1}{\sqrt{S}}e^{-i\chi/2}\ ,\ S=\left[\left(r^2+\alpha^2z^2-c_2^2\xi^2t^2\right)^2+4\alpha^2z^2c_2^2\xi^2t^2\right]^{1/2}\ ,\\ \tan\chi=\tfrac{2\alpha c_2\xi|z|t}{r^2+\alpha^2z^2-c_2^2\xi^2t^2}\ .\end{gathered}\tag{27}$$

We can see that in the limit of long times $S\rightarrow\infty$ and $\chi\rightarrow 0$, *i.e.* forces with a pulse-like time dependence do not contribute to vibrations, as expected. It is the transient regime, prior to the establishment of the stationary vibrations regime, which is relevant for time δ-pulses of perturbations, and it is this transient regime which is associated with the earthquakes.

The vanishing of the denominators $\kappa_{1,2}$ in equation (16) means $\omega=c_{1,2}k$; in this case there does not exist a z-dependence; the contribution of these terms corresponds to lateral waves, *i.e.* waves which propagate along directions which are parallel with the surface $z=0$; these waves are also eigenmodes of the isotropic elastic half-space. In this context it is worth recalling the incident and reflected plane waves, which are well-known eigenmodes of the isotropic elastic half-space. These modes originate in special linear combinations of the matrix of the system of equations of the boundary conditions, realized with reflection coefficients, which lead to a matrix with a vanishing determinant. There does not exist a special dispersion relation in this case (actually, the relevant dispersion relations $\omega^2=c_{1,2}^2(\kappa_{1,2}^2+k^2)$ are already included in the boundary conditions). Such linear combinations are possible by the special symmetry of the problem to reversing the signs of the transverse components $\kappa_{1,2}$ of the wavevectors.

A gradient force. Harmonic oscillations. The seismic sources are currently associated with the so-called double-couple representation of the forces, given by

$$F_i=m_{ij}(t)\partial_j\delta(\mathbf{r})\delta(z-z_0)\ ,\tag{28}$$

where $m_{ij}(t)$ is the seismic moment and the source is placed at $\mathbf{r}=0$, $z=z_0$, $z_0<0$ (see, for instance, Ref. [27], 2nd edition, p. 60, Exercise 3.6). We consider a particular case where the tensor of the seismic moment reduces to a scalar $m(t)$; it is easy to see that such an expression for the force may mimick an explosion source (for a time dependence proportional to $\delta(t)$). We consider also a free surface, *i.e.* we set $p_i=0$. In this case it is easy to see that the force derives from a potential, $\mathbf{F}=grad\varphi$, where the Fourier transform of the potential φ is $\varphi=m\delta(z-z_0)$; the potential $\mathbf{H}$ is zero and

$$q_\alpha=\frac{2ik_\alpha m}{c_1^2}e^{i\kappa_1|z_0|}\ ,\ q_3=0\ ;\tag{29}$$

similarly, the boundary parameters A_3^0 and A_α^0 are not zero. Making use of equations (9) and (14) we get immediately Φ and Φ^0, A_α^0; it is convenient to limit ourselves to the surface displacement only, given by

$$\begin{gathered}u_\alpha^0=-\tfrac{2m\kappa_2k_\alpha}{c_1^2\overline{\Delta}}e^{i\kappa_1|z_0|}\ ,\\ u_3^0=v_3^0+w_3^0\ ,\ v_3^0=-\tfrac{m}{c_1^2}e^{i\kappa_1|z_0|}\ ,\\ w_3^0=-\tfrac{2mk^2(\kappa_2^2-k^2-2\kappa_1\kappa_2)}{c_1^2\overline{\Delta}}e^{i\kappa_1|z_0|}\ ,\end{gathered}\tag{30}$$

where

$$\overline{\Delta}=(\kappa_2^2-k^2)^2+4\kappa_1\kappa_2k^2\ .\tag{31}$$

If the source is a harmonic oscillation with frequency Ω, of the form $m = m_0 \cos \Omega t$ (as for an isotropic pulsed source concentrated at an inner point in the half-space), the surface displacement has the same time dependence $u^0(\mathbf{k}, t) \sim m_0 \cos \Omega t$, where ω in equations (30) is replaced by Ω. The Fourier transforms of the term v_3^0 can be computed by means of the Sommerfeld-Weyl integral[35]

$$\frac{i}{2\pi} \int d\mathbf{k} \frac{e^{i\mathbf{kr}}}{\kappa} e^{i\kappa|z|} = \frac{e^{i\omega R/c}}{R} \ , \tag{32}$$

where $\kappa = \sqrt{\omega^2/c^2 - k^2}$ and $R = \sqrt{r^2 + z^2}$; we get

$$v_3^0(\mathbf{r}, t) = -\frac{m_0}{\pi c_1^2} \frac{\partial}{\partial |z_0|} \frac{\sin \Omega R_0/c_1}{R_0} \cos \Omega t \ , \tag{33}$$

where $R_0 = \sqrt{r^2 + z_0^2}$; we recognize in equation (33) a spherical-wave vibration.

In order to estimate the spatial dependence of u_α^0 and w_3^0 we note that we are often interested in distances much longer than the wavelengths $c_{1,2}/\Omega$, such that we may assume $k < k_c \ll \Omega/c_{1,2}$ in equations (30), where k_c is a cutoff wavevector. Within this approximation we get

$$u_\alpha^0(\mathbf{r}, t) \simeq -\frac{4m_0 c_2^3}{(2\pi)^2 c_1^2 \Omega^3} \partial_\alpha \int d\mathbf{k} e^{i\mathbf{kr}} \sin \frac{\Omega}{c_1} |z_0| \cos \Omega t \ ; \tag{34}$$

the integration over a finite range of $\mathbf{k}$ in equation (34) leads to a function localized over a range of the order $(\Delta r)^2 \simeq 1/k_c^2$ (if we extend the integration to infinity we get the function $\delta(\mathbf{r})$).

Within the same short-wavelengths approximation the term w_3^0 in equations (30) is

$$w_3^0(\mathbf{r}, t) \simeq -\frac{m_0 c_2^2 (c_1 - 2c_2)}{4\pi c_1^3 \Omega^2} k_c^4 \cos \frac{\Omega}{c_1} |z_0| \cos \Omega t \ . \tag{35}$$

We can see that for large distances the main contribution to the displacement is

$$v_3^0(\mathbf{r}, t) \simeq -\frac{m_0 \Omega}{\pi c_1^3} \frac{|z_0|}{R_0^2} \cos \frac{\Omega R_0}{c_1} \cos \Omega t \ , \tag{36}$$

arising from equation (33).

A gradient force. δ-pulse time dependence. If the time dependence of the seismic moment is of the form $m(t) = m_0 \delta(t)$, the reverse Fourier transform of the term v_3^0 given by equations (30) can be calculated by using the integral in equation (32); it leads to

$$v_3^0(\mathbf{r}, t) = \frac{m_0}{2\pi c_1^2} \frac{\partial}{\partial |z_0|} \frac{\delta(t - R_0/c_1)}{R_0} \ , \tag{37}$$

which is the derivative of a propagating spherical wave; since the support of this function is zero, its contribution to the boundary conditions is zero.

For u_3^0 and w_3^0 in equations (30) the poles associated with the surface waves are active. With $\kappa_{1,2} \rightarrow i\kappa_{1,2}$ the denominator $\overline{\Delta}$ in equations (30) has poles at $\omega = \pm\omega_0$, where $\omega_0 = c_2 \xi k$ is the frequency of the Rayleigh surface waves. The expansion in powers of $\omega \pm \omega_0$ gives

$$(\kappa_2^2 - k^2)^2 - 4\kappa_1 \kappa_2 k^2 = \pm 4 \frac{\Delta_0(\xi)}{c_2} k^3 (\omega \mp \omega_0) + \ldots \ , \tag{38}$$

where $\Delta_0(\xi)$ is given by equation (21); taking the reverse time Fourier transforms in equations (30) we get

$$u_\alpha(\mathbf{k},t)=\tfrac{m_0c_2\sqrt{1-\xi^2}}{c_1^2\Delta_0(\xi)}\tfrac{k_\alpha}{k^2}e^{-\alpha k|z_0|}\sin c_2\xi kt \ ,$$
$$w_3^0(\mathbf{k},t)=\frac{m_0c_2\left(2-\xi^2-2\alpha\sqrt{1-\xi^2}\right)}{c_1^2\Delta_0(\xi)}\tfrac{1}{k}e^{-\alpha k|z_0|}\sin c_2\xi kt \ , \tag{39}$$

where $\alpha=\sqrt{1-c_2^2\xi^2/c_1^2}$. The reverse spatial Fourier transform of $w_3^0(\mathbf{k},t)$ in equations (39) implies integrals given in equation (26); we get $w_3^0(\mathbf{r},t)\sim\frac{1}{\sqrt{S}}\sin\chi/2$, where S and χ are given in equations (27) with z replaced by z_0; in the limit of large t the function $w_3^0(\mathbf{r},t)$ vanishes.

The reverse spatial Fourier transform of $u_\alpha^0(\mathbf{k},t)$ given in equations (39) can be effected by using the identities

$$J_\alpha=\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k^2}k_\alpha e^{-k|z|}=-i\partial_\alpha J \ , \ J=\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k^2}e^{-k|z|} \tag{40}$$

and

$$\frac{\partial J}{\partial\,|z|}=-\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k}e^{-k|z|}=-\frac{1}{R} \ , \tag{41}$$

where $R=\sqrt{r^2+z^2}$; we get

$$J=-\ln\left(|z|+R\right) \ , \ J_\alpha=\frac{ix_\alpha}{R(|z|+R)} \ ; \tag{42}$$

making use of $R=\sqrt{r^2+(\alpha\,|z_0|\mp ic_2\xi t)^{\,2}}$ and replacing $|z|$ by $\sqrt{\alpha^2z_0^2+c_2^2\xi^2t^2}$ in equation (42) we can see that $u_\alpha^0(\mathbf{r},t)\rightarrow 0$ for $t\rightarrow\infty$, as expected. It is worth noting that in this formulation of the problem the surface displacement given by $u_\alpha^0(\mathbf{r},t)$ and $w_3^0(\mathbf{r},t)$ is different from zero immediately after the initial moment $t=0$, when the $\delta(t)$-perturbation occurrs at the point $\mathbf{r}=0$, $z=z_0$, $z_0<0$ (as expected for a vibrations approach); this is not so for the spherical wave $v_3^0(\mathbf{r},t)$; hence, we can see that applying the vibrations approach to δ-like point forces concentrated both in time and space is not an adequate formulation of the problem.

Force on the surface. Let us assume that the volume force is zero ($\mathbf{F}=0$, $\varphi=0$, $\mathbf{H}=0$) and only the component $p_3=p(t)\delta(\mathbf{r})$ of a surface force localized at $\mathbf{r}=0$ on the surface $z=0$ is non-vanishing. Making use of equations (9), (12) and (14) we get immediately the components of the displacement

$$u_\alpha(\mathbf{k},\omega)=\tfrac{ik_\alpha p}{\Delta}\left[(\kappa_2^2-k^2)e^{i\kappa_1|z|}+2\kappa_1\kappa_2e^{i\kappa_2|z|}\right] \ ,$$
$$u_3(\mathbf{k},\omega)=-\tfrac{i\kappa_1p}{\Delta}\left[(\kappa_2^2-k^2)e^{i\kappa_1|z|}-k^2e^{i\kappa_2|z|}\right] \ . \tag{43}$$

For a harmonic force $p(t)=p_0\cos\Omega t$, within the short wavelength approximation described above we get

$$u_\alpha(\mathbf{r},z,t)=\tfrac{c_2^2(c_1++2c_2)}{(2\pi)^2c_1\Omega^2}p_0\partial_\alpha\int d\mathbf{k}e^{i\mathbf{kr}}\left(\cos\tfrac{\Omega}{c_1}z+\cos\tfrac{\Omega}{c_2}z\right)\cos\Omega t \ ,$$
$$u_3(\mathbf{r},z,t)=\tfrac{c_2^2p_0}{2\pi^2c_1\Omega}\left[\int d\mathbf{k}e^{i\mathbf{kr}}\sin\tfrac{\Omega}{c_1}\,|z|-\tfrac{2c_2^2}{\Omega^2}\int d\mathbf{k}k^2e^{i\mathbf{kr}}\sin\tfrac{\Omega}{c_2}\,|z|\right]\cos\Omega t \ , \tag{44}$$

where the integration is performed over a finite range $0<k<k_c\ll\Omega/c_{1,2}$. For a time impulse $p(t)=p_0\delta(t)$ the contribution to the reverse time Fourier transform comes from the poles of $\overline{\Delta}$; in this case we reach the same conclusion as above, *viz.* in the limit $t\rightarrow\infty$ the displacement is vanishing.

Static limit. It is worth noting that we are not allowed to take the static limit $\omega \rightarrow 0$ in the formulation given here for the vibrations problem, as expected. Indeed, both volume and surface static forces determine a deformation of the surface $z = 0$, such that the boundary conditions should be imposed on the deformed surface; it follows that the boundary conditions imposed here on the surface $z = 0$ become inadequate in this case. This can also be seen from the boundary-conditions system of equations (11), whose solutions given by equations (14) become meaningless in the static limit, since they include terms like $\delta(\omega)/\omega^2$, or $\omega^2\delta(\omega)/\omega^2$, arising from $\Delta \sim \omega^2$, $\kappa_2^2 + k^2 = \omega^2/c_2^2$ and time Fourier transforms of static forces, which are proportional to $\delta(\omega)$.

The static limit exhibits a special problem. We can start, in the present formulation, with equations (8) for potentials and equations (11) for the boundary conditions in the static limit, *i.e.* for $\omega = 0$. Then we see immediately that the boundary-conditions system of equations (11) is incompatible (its determinant is vanishing). This is due to the condition $div\mathbf{A} = 0$ which is too restrictive in this case. This particularity of the static limit is related to the fact that the contributions associated with Δ and $grad \cdot div$ in the equation of static equilibrum are entangled in the static limit. If we give up this condition, then the boundary-conditions system of equations (11) is compatible, and we may set $A_3^0 = 0$, for instance (or any other convenient relationship between the four unknowns Φ^0 and $\mathbf{A}^0$). Such special features of the static limit can be seen in the well-known Grodskii-Neuber-Papkovitch approach.[36]-[38] (see also Ref. [39]).

Concluding remarks. The problem of elastic vibrations of anisotropic half-space, known usually as part of the Lamb problem in Seismology, is formulated here in terms of the Helmholtz potentials of the elastic displacement. The formulation is based on time Fourier tranforms, spatial Fourier transforms with respect to the coordinates parallel to the surface of the half-space and wave equations for generalized functions (distributions), which include the surface values of the functions and their derivatives. This formulation allows a formal general solution for the vibrations of the isotropic half-space. Explicit results are given for forced vibrations generated in the half-space by forces derived from a gradient and concentrated at an inner point in the half-space; these forces correspond to a scalar seismic moment in the double-couple representation of the seismic sources. Similarly, explicit resuls are given for forces concentrated at a point on the surface of the half-space. Both harmonic oscillations and δ-like time pulses are considered for these forces. It is shown that time pulses of the δ-type generate perturbations which are vanishing in time; consequently, such perturbations cannot be properly regarded as vibrations, as expected. For harmonic oscillations the vibrations of the half-space are driven by forces, while for a δ-like time dependence of the forces the results are governed by surface (Rayleigh) and lateral eigenmodes. The combinations of incident and reflected waves do not contribute, in virtue of their special eigenmode character. It is emphasized that the vibrations formulation of the problem is meaningful only for long times, such that the waves have sufficient time to reach the surface, establish the stationary vibrations regime by multiple reflections and the surface get a chance to be active in this process.

It is well known that forces concentrated both in time and space (like δ-functions, or derivatives of δ-functions) generate elastic spherical waves; the boundary conditions are irrelevant for such propagating waves, due to their vanishing support. Their interaction with the surface generates additional wave sources, which produce secondary waves; the secondary waves should obey the boundary conditions, as they last long on the surface, but their contribution, although lasting for long times, is small over extended spatial regions. It is this transient regime of propagating waves which is relevant in the "elementary" earthquakes, *i.e.* earthquakes which are produced by forces concentrated both in space and time. The original spherical waves are the primary waves associated with the "preliminary feeble tremor";[6, 7] the secondary waves generate the main shock and the long tail, documented by the seismic records. The surface waves, or the lateral waves, as eigenmodes, have no direct bearing on the vibrations, other than contributing through their dispersion relations to the perturbations produced by time δ-pulses; while the other eigenmodes - the combination of incident and reflected waves - have no influence on vibrations. As regards the propagating regime, all the eigenmodes of the isotropic elastic half-space have no relevance, as they are undetermined waves.

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