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On the measurement process in Quantum Mechanics

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Abstract

The uncertainty relations are revisited, the measurement limitations of the Quantum Mechanics are reanalyzed, and a measurement procedure is defined and illustrated.

One of the most general form of motion in space and time is described by the variation

$$\mathbf{k}d\mathbf{r} - \omega dt = d\Phi \ , \tag{1}$$

where $\mathbf{r}$ is the position vector, t denotes the time and $\mathbf{k}$ and ω are some constants; what is relevant here is $\mathbf{k}$ along $d\mathbf{r}$; if this $\mathbf{k}$ is such that $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, then $i\Phi$ is the phase of a wavefunction; λ is the wavelength and T is the period, $\mathbf{k}$ is the wavevector, ω is the frequency of a wave; any wavefunction can be expanded in a Fourier series of such "plane waves"; multiplying by Planck's constant $\hbar$, we get the momentum $\mathbf{p} = \hbar\mathbf{k}$, the energy $E = \hbar\omega$, the mechanical action $S = \hbar\Phi$ and write the wavefunction as

$$\psi = e^{\frac{i}{\hbar}S} \ . \tag{2}$$

we can see that a wavefunction can get a mechanical interpretation and may serve to describe a general motion; moreover, the Φ is relativistically invariant. This sort of motion description is the Quantum Mechanics. We may see that we get the momentum and energy by applying the differential operators $-i\hbar\partial/\partial\mathbf{r}$ and, respectively, $i\hbar\partial/\partial t$ to the wavefunction. We leave aside here the machinery of the Quantum Mechanics (including the probability, the superposition principle, the operators, matrices, eigenstates, commutation relations, etc) and focus on the fundamental restrictions implied by the wavefunction.

First, we note that a reproductible result of a measurement of the position x at a fixed moment of time is meaningless inside the wavelength; similarly, a reproductible measurement of the time at a given position is meaningless inside a period; or, equally well, a measurement of the momentum can only be made with the maximal accuracy h/λ and a measurement of the energy can only be made with the maxiumal accuracy h/T. This follows more formally from the indeterminacy $\simeq \pi$ in the phase of any wavefunction, *i.e.*

$$\Delta p \Delta x \simeq h/2 \ , \ \Delta E \Delta t \simeq h/2 \ . \tag{3}$$

More formally, let us form

$$\left| -i\hbar\frac{\partial}{\partial x} - \lambda x \right|^2 \geq 0 \ , \tag{4}$$

or

$$-\hbar^2\frac{\partial^2}{\partial x^2}+i\hbar\lambda+\lambda^2x^2\geq 0 \ , \tag{5}$$

where λ is a parameter; this trinomial in λ is positive for

$$\Delta p\Delta x\geq \hbar/2 \tag{6}$$

(what we always measure is difference in positions, momenta, etc). A similar relation is obtained similarly for energy and time; the difference between h and $\hbar$ in equations (3) and (6) arises from the fact that the constant $\hbar$ is not determined in equation (4). These are the Heisenberg uncertainty relations. Moreover, since

$$h/2\leq\Delta p\Delta x=h\frac{\Delta\lambda}{\lambda^2}\Delta x\leq h\Delta x/\lambda \ , \tag{7}$$

we get

$$\Delta x\geq\lambda/2 \ ; \tag{8}$$

and, since

$$h/2\leq\Delta E\Delta t=h\frac{\Delta T}{T^2}\Delta t\leq h\Delta t/T \ , \tag{9}$$

we get

$$\Delta t\geq T/2 \ . \tag{10}$$

We can see that the maximal accuracy of any measurement in a wavefunction is $\lambda/2$ in position and $T/2$ in the time. Hence, by using the uncertainty relations, the maximal accuracy in the measurement of the momentum is h/λ and in energy is h/T. It follows that in using wavefunctions in Quantum Mechanics we cannot have information for distances shorter than the wavelength (at a given moment of time), neither for durations of time shorter than the wave period (at a given position).

These restrictions have important consequences for quantum relativity. The uncertainty by which we can measure the velocity of the interaction is of the order

$$\Delta v=\frac{\Delta x}{\Delta t}=\frac{\lambda}{T}=\frac{\omega}{k}=\frac{E}{p}=c\sqrt{1+\frac{m^2c^2}{p^2}} \ , \tag{11}$$

where m is the mass of the particle and c is the speed of light. We can see that the measurements in quantum relativity imply velocity higher than the speed of light, which means that the quantum relativity and the quantum theory of the relativistic fields are, basically, meaninglesss; since they operate with entities which are not measurable, according to the principle of relativity. Since it is difficult to give up the principle of relativity, we have no other choice than to accept that the quantum treatment of the relativistic fields is, essentially, inappropriate.

In the rest frame the maximal accuracy of measurement of the position is $\Delta x\simeq\lambda/2=h/2mc$, where $\lambda=h/mc$ is the Compton wavelength; the maximal accuracy of the measurement of the momentum is mc; the maximal accuracy of the measurement of the energy is mc^2; and the maximal accuracy of the measurement of the time is $\Delta t\simeq h/2mc^2$. It follows that we have no reliable method of predicting (describing) the production of particle-antiparticle pairs from vacuum, the vacuum polarization and breakdown, radiative corrections, mass and charge renormalization, etc, since we have no means of reliable measuring these phenomena. On the other hand, it is true that for some particular effects in quantum electrodynamics, correct results can be obtained (in accord with experience), by accidental coincidences; the mass and charge renormalization is among these.[1]-[5] The wavefunction for a relativistic particle with mc^2 for energy and mc for momentum, $e^{-\frac{i}{\hbar}mc^2t+\frac{i}{\hbar}mcx} = e^{-\frac{i}{\hbar}mc(ct-x)}$, is meaningless in Quantum Mechanics for $c\Delta t \simeq \Delta x \simeq h/mc$; in any reference frame; on the other hand, for $c\Delta t$, $\Delta x \gg h/mc$ it describes a (quasi-) classical particle; the relativistic particles are (quasi-) classical particles, since all the durations of time and all the distances are much larger than the Compton period h/mc^2 and Compton wavelength h/mc. Similarly, a radiation (photon) wavefunction $e^{-\frac{i}{\hbar}p(ct-x)} = e^{-ik(ct-x)} = e^{-i2\pi(ct-x)/\lambda}$ describes a (quasi-) classical particle for times and distances much larger than λ/c and, respectively, λ; and a quantum particle for times and distances larger, but comparable, with λ/c and, respectively, λ.

Let

$$\psi = \sum_n a_n \psi_n \tag{12}$$

be an expansion of ψ in a series of (orthogonal) wavefunctions ψ_n; then

$$\int d\mathbf{r}\psi_n^* \psi = a_n \ ; \tag{13}$$

on the other hand,

$$\int d\mathbf{r} \left|\psi\right|^2 = \sum_n \left|a_n\right|^2 \int d\mathbf{r} \left|\psi_n\right|^2 \ ; \tag{14}$$

if $\int d\mathbf{r} \left|\psi\right|^2$ is the probability of being in the state ψ in the volume of integration (with proper normalization), and $\int d\mathbf{r} \left|\psi_n\right|^2$ is the probability of being in the state ψ_n in the same volume, then $\left|a_n\right|^2$ is the probability that state ψ_n be included in state ψ in the volume of integration. It follows that we can take the a_n as the amplitude of probability that state ψ_n be in state ψ; or, the amplitude of the extent to which the state ψ_n is included in state ψ. Therefore, the quantity

$$a = \int d\mathbf{r}\psi_1^* \psi_2 \tag{15}$$

is an expression of a measurement process: its squared modulus indicates the amount to which the state ψ_1 is measured by state ψ_2. In a more descriptive sense, ψ_2 is the instrument (the apparatus) by which we measure the presence, and the extent, of ψ_1, and the process of measurement is the integration in equation (15). Obviously, these statements are valid for any generalized coordinate.

Let us assume that we try to gauge the state ψ_1 by its momentum; then we use

$$\psi_2 = \frac{1}{\sqrt{\Delta V}} e^{\frac{i}{\hbar}\mathbf{p}'\mathbf{r}} \ ; \tag{16}$$

it is conceivable that our meter has several values of $\mathbf{p}'$ distributed within the range $\Delta\mathbf{p}'$ (around some value $\mathbf{p}'$), so we may use

$$\psi_2 = \frac{1}{\Delta\mathbf{p}'\sqrt{\Delta V}} \int d\mathbf{p}' e^{\frac{i}{\hbar}\mathbf{p}'\mathbf{r}} \ ; \tag{17}$$

we get

$$a = \frac{1}{\Delta\mathbf{p}'\sqrt{\Delta V}} \int d\mathbf{p}' \int d\mathbf{r}\psi_1^* e^{\frac{i}{\hbar}\mathbf{p}'\mathbf{r}} \ ; \tag{18}$$

let us assume that ψ_1 has a certain $\mathbf{p}$ to some extent $a_{\mathbf{p}}$,

$$\psi_1 = \frac{1}{\sqrt{\Delta V}} a_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\mathbf{r}} \ ; \tag{19}$$

then,

$$a = \frac{h^3}{\Delta\mathbf{p}'\Delta V}\int d\mathbf{p}' a^*_{\mathbf{p}}\delta(\mathbf{p}-\mathbf{p}') = \frac{h^3}{\Delta\mathbf{p}\Delta V}a^*_{\mathbf{p}} \ . \tag{20}$$

If $\Delta\mathbf{p}\Delta V$ is very large, *i.e.* the accuracy of the measurement is very low, and our apparatus ψ_2 is a quantum-mechanical apparatus, then a is extremely small, and we have no definite result; our measurement process fails. On the contrary, if the apparatus is classical, *i.e.* $\Delta\mathbf{p}\Delta V \simeq h^3$, then $a \simeq a_{\mathbf{p}}$, which means that, indeed, we succeeded to measure the state ψ_1 defined by its momentum. However, it is worth noting that the quantum-mechanical character of the product $\Delta\mathbf{p}\Delta V$ is valid for finite $\Delta\mathbf{p}$ and ΔV and large values of the product $\Delta\mathbf{p}\Delta V$.

Similarly, we can use an apparatus

$$\psi_2 = \frac{1}{\Delta E'}\int dE' e^{-\frac{i}{\hbar}E't}\phi_2(\mathbf{r}) \tag{21}$$

for measuring a state

$$\psi_1 = e^{-\frac{i}{\hbar}Et}\phi_1(\mathbf{r}) \tag{22}$$

defined by its energy E; it is conceivable that the measuring process takes a finite time Δt, during which we collect the results; we get

$$a = \frac{1}{\Delta E'\Delta t}\int dt \int dE' e^{\frac{i}{\hbar}(E-E')t}\int d\mathbf{r}\phi^*_1(\mathbf{r})\phi_2(\mathbf{r}) \ \ , \tag{23}$$

or, leaving aside the spatial factor,

$$a = \frac{h}{\Delta E'\Delta t}\int dE'\delta(E-E') = \frac{h}{\Delta E\Delta t} \ ; \tag{24}$$

if the apparatus is quantum-mechanical, there will be large uncertainties in energy and time, and the measurement process fails (it gives a vanishingly small, indefinite result); if the apparatus is classical, the uncertainties are of the order $\Delta E\Delta t \simeq h$, and we get a definite result a; which shows to what extent a state defined by its energy is present in the measured state ψ_1.

Let us go back to the typical energy, or momentum wavefunctions of the form of the plane wave, $e^{-\frac{i}{\hbar}Et}$ or $e^{\frac{i}{\hbar}\mathbf{pr}}$. If we try to consider such wavefunctions as basic elements of reality, they are useless, as partly useless are their consequences like the quantum-mechanical energy levels, for instance. This is so because energy or momentum are not determined for finite intervals of time or finite regions of space, in the sense of the uncertainty relations. Consequently, the energy and the momentum are not conserved, which means the loss of scientific objectivity (reproductibility, control of errors, etc). This can also be seen in the process of measurement, where factors like $e^{\frac{i}{\hbar}(E-E')t}$, or integrals over finite durations of time of such factors, are far away of $\delta(E-E')$ (and similarly for the momentum). A typical measurement, for instance, is to send a monochromatic light wave upon a collection of atoms; we shall see that the absorption law$\hbar\omega = \Delta E$ (for absorbing one photon), where ω is the radiation frequency and ΔE the difference between two atomic energy levels, is very scattered; and only after a long time ($t \gg 1/\hbar\omega$), we get a well-defined spectral peak. In the uncertainty relation $\Delta E\Delta t = nh$, for some large n, we may have $\Delta t = nh/\delta E$, such that $\Delta E = \delta E$, whereδE is arbitrarily small; then, we can have a sharp measurement of the energy. It follows that we can still use a quantum-mechanical apparatus (an atom) for the measurement process, at the price of waiting a long time. However, for a long time, the wavefunction $\sim e^{-\frac{i}{\hbar}Et}$ is in its quasi-classical condition, since we have many oscillations. It follows that the quantum-mechanical concept only have a meaning in relation with the classical concepts; of course, as long as we accept as positive knowlege only what we are able to measure with a controllable accuracy; which is unavoidable for the scientific meaning.

We note that in the above considerations the position and the time are considerd as independent variables. So are the energy and the momentum, inasmuch as the energy is in fact a difference of energies. This is specific to the non-relativist Quantum Mechanics. In Relativity the situation is quite different. The energy and momentum are related by a definite rest energy mc^2, through $E = \sqrt{m^2c^4 + p^2c^2}$ for a free particle (bound states are inappropriate in full Relativity, and particles dressed with interaction preserve the relation). Similarly, the position and the time are related by the finite, maximal speed of light. Then, we have the absolute uncertainty relations $\Delta x > h/mc$, $\Delta p > mc$, $\Delta E > mc^2$, $\Delta t > h/mc^2$. For an elecron the Compton wavelength is $h/mc = 2 \times 10^{-10} cm$, the rest momentum is $mc = 3 \times 10^{-17} g \cdot cm/s$, the rest energy is $mc^2 = 9 \times 10^{-7} erg = 0.5 MeV$ and the minimal time is $h/mc^2 = 6 \times 10^{-21} s$. For more massive particles the Compton wavelength and the minimal time are eevn smaller; it follows that for all distances and times of interest the relativistic particles are in the quasi-classical limit.

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