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On the role of the boundary conditions in the wave motion propagated in semi-infinite fluids

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Abstract

It is shown that in some cases of physical interest the surface forces (boundary conditions) may have a major effect on the waves propagation in fluids. This point is illustrated, first, for waves propagating in a semi-infinite (half-space) fluid with a surface force normal to the free surface (pressure), where the gravitational field is neglected. If a constant, localized surface force is applied suddenly, it generates a cylindrical wave with a (quasi-) singular wavefront and a long tail. The displacement field in the singular wave is parallel to the surface and is directed outwards from the wave source for a surface force acting downwards. A small but finite spatial spread of the surface force, or a small but finite duration of the process of application of the force, eliminates the wavefront discontinuity but preserves its high magnitude. Another illustration of the boundary conditions effect is provided by the gravity waves propagating along the plane surface of a fluid in gravitational field. A constant, localized surface force, suddenly applied downwards, generates a packet of gravity-waves on the fluid surface, whose net effect is a motion directed outwards from the source of waves. Both the singular wave and the gravity waves on fluid surface can carry along a casual observer away from the wave source, to regions where the wave amplitude is smaller. Such a picture evokes the alleged effect oil films spread over sea surface may have upon stilling surface waves (storm breakers).

This paper is motivated by the alleged effect a thin oil film spread over the surface of water may have upon calming down the waves propagating over the fluid surface (storm breakers). There are rumors that such an effect would be real, being discovered by chance by ancient Greek sailors,[1] scarcely documented on tempests on the sea, and demonstrated by Franklin in London in 1773.[2] A convincing proof of its existence is lacking, though some disparate information is available;[3] like, for instance, that the effect would occur only with (polar) vegetable or animal oil, or the oil film should be thin and spread over a large area to windward of the ships.[4] The spreading of (polar) oil films on water surface was extensively studied and the damping of capillary, short and even moderately-long surface waves was attributed to the viscosity of the surface films.[5]-[10] This could be part of the explanation of this strange effect; we describe here another relevant aspect, which is caused by external forces applied to the surface.

The present approach is based on the assumption that the effect is related to the boundary conditions imposed on the plane surface of an elastic medium propagating longitudinal waves.[10] We choose here two instances of investigating the role of the boundary conditions upon wave propagation in elastic media. The first case is that of waves propagating in a homogeneous fluid which occupies a half-space and is bounded by a plane surface, where the gravitational field is neglected. It is shown that a pressure applied downwards instantaneously on the surface of the fluid and localized just above the wave source generates a cylindrical wave, with a (quasi-) singular wavefront, propagating away from the wave source on the fluid surface; it may carry along an observer away from the wave source, to regions where the waves are diminished in amplitude; this could be the meaning of "calming down" the waves. (The singularity of the wavefront is smoothed out by a finite spatial spread, or a finite duration of the pressure application process). The second illustrative case is a fluid in gravitational field bounded by a plane surface, which, as it is well known, may propagate gravity waves. It is shown that a constant pressure suddenly applied downwards to the surface just above the wave source produces a packet of gravity waves, whose tangential components can carry the observer away from the wave source, with the same effect in the absence of the gravitational field. Such major effects the boundary conditions may have upon wave propagation arise from the excitation of specific eigenmodes, which are "lateral" waves in the absence of the gravitational field (tangential to the surface) and damped surface waves (gravity waves) for fluids in gravitational field. In addition, the net motion in the surface waves is related to the transient regime generated by suddenly applied surface forces.

Let us consider the motion of a local displacement field $\mathbf{u}$ in a homogeneous fluid; such a displacement field generates density variations $\delta n = -ndiv\mathbf{u}$, where n is the constant (uniform) density (particle concentration); we assume $\delta n/n \ll 1$, which means a slow spatial variation of the field $\mathbf{u}$; the density change arises from a corresponding change in the local volume and the pressure, such that the equation of motion for the field $\mathbf{u}$ is

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = \mathbf{e} \ , \tag{1}$$

where c is the wave velocity and $\mathbf{e}$ is the external field (per unit mass); taking the *div* in equation (1) we get the wave equation

$$\frac{\partial^2 \delta n}{\partial t^2} - c^2 \Delta \delta n = -ndiv\mathbf{e} \tag{2}$$

for density waves; in the limit of infinite wavelengths c is the sound velocity. We recognize in equation (1) the equation of motion of the elastic deformation of a homogeneous, isotropic solid with only one Lame coefficient (λ, with the notations from Elasticity, corresponding to dilatational motion).[11] The boundary condition relevant for the motion described by equation (1) is

$$nc^2 \partial_f u_f \mid_F = -P_f \ , \tag{3}$$

where f denotes the coordinate along the direction normal to the surface F bounding the volume V of the fluid, u_f is the component of the displacement normal to the surface F and P_f (directed downwards along the normal to the surface F) is the force per unit mass and unit area of the cross-section of the surface (pressure per unit mass); ∂_f stands for the derivative with respect to the coordinate f perpendicular to the surface. For convenience we introduce the notation $p_f = P_f/nc^2$ and write the boundary condition as

$$\partial_f u_f \mid_F = -p_f \ . \tag{4}$$

The dynamics governed by equation (1) implies the derivation of the diplacement field $\mathbf{u}$ from the gradient of a scalar potential Φ and, similarly, the derivation of the external field $\mathbf{e}$ from the gradient of a potential φ; making use of $\mathbf{u} = grad\Phi$ and $\mathbf{e} = grad\varphi$, equation (1) and boundary condition (4) become

$$\ddot{\Phi} - c^2 \Delta \Phi = \varphi \ , \ \partial_f^2 \Phi \mid_F = -p_f \ . \tag{5}$$

We apply these equations to the motion of the density of a semi-infinite fluid which occupies the half-space $z<0$, bounded by a plane surface $z=0$; f in the above equations denotes the coordinate z, which is taken along the vertical direction. We use the time Fourier transform (with frequency ω) and decomposition in Fourier plane waves with wavevector $\mathbf{k}$ parallel with the surface $z=0$; we use the same notation for functions and their Fourier transforms, whenever there not exists a risk of confusion. Equations (5) become

$$\frac{d^2\Phi}{dz^2}+\kappa^2\Phi=-\varphi/c^2\ ,\ \partial_z^2\Phi\mid_{z=0}=-p_z\ , \tag{6}$$

where $\kappa^2=\omega^2/c^2-k^2$. In order to account conveniently for the boundary condition it is useful to extend the equation (6) to the whole space and to limit ourselves to the restriction of the solution to the lower half-space. To this end, we multiply the wave equation (6) by $\theta(-z)$, where $\theta(z)=1$ for $z>0$ and $\theta(z)=0$ for $z<0$ is the step function, and absorb the function $\theta(-z)$ in the derivatives; limiting ourselves to the restriction of the solution to the lower half-space, equations (6) can be written as

$$\frac{d^2\Phi}{dz^2}+\kappa^2\Phi=-\varphi/c^2-\Phi^{(1)}\delta(z)-\Phi^{(0)}\delta^{'}(z)\ ,\ \Phi^{(2)}=-p_z\ , \tag{7}$$

where $\Phi^{(0)}=\Phi_{z=0}$, $\Phi^{(1)}=\frac{d\Phi}{dz}\mid_{z=0}$ and $\Phi^{(2)}=\frac{d^2\Phi}{dz^2}\mid_{z=0}$. Making use of the Green function $e^{i\kappa|z|}/2i\kappa$ for the equation (7), we get the solution

$$\Phi=-\frac{1}{2i\kappa c^2}\int_{-\infty}^{0}dz^{'}\left(e^{i\kappa|z-z^{'}|}-e^{i\kappa|z+z^{'}|}\right)\varphi(z^{'})+\Phi^{(0)}e^{i\kappa|z|} \tag{8}$$

and

$$\Phi^{(1)}=-\frac{1}{c^2}\int_{-\infty}^{0}dz^{'}e^{i\kappa|z^{'}|}\varphi(z^{'})-i\kappa\Phi^{(0)}\ , \tag{9}$$

where $\varphi^{(0)}=\varphi_{z=0}$; using equation (7), the boundary condition becomes

$$\kappa^2\Phi^{(0)}=-\varphi^{(0)}/c^2+p_z\ . \tag{10}$$

We can see the occurrence of the image Green function $e^{i\kappa\left|z+z^{'}\right|}/2i\kappa$ in equation (8), as expected; also, we can see that $\Phi^{(0)}$ plays the role of an integration constant, which is determined by the boundary condition. In addition, for free waves ($\varphi=0$, $p_z=0$), we may see the eigenfrequencies $\omega=ck$, given by $\kappa^2=0$ in equation (10); the corresponding eigenmodes are free "lateral" waves of the form $e^{-i\omega t-i\mathbf{kr}}$, $\omega=ck$, propagating along directions parallel with the surface $z=0$, where $\mathbf{r}$ is the in-plane position vector. For $\kappa^2\neq 0$ we determine $\Phi^{(0)}$ from equation (10) and, using equation (8), get the full solution

$$\Phi=-\frac{1}{2i\kappa c^2}\int_{-\infty}^{0}dz^{'}\left(e^{i\kappa|z-z^{'}|}-e^{i\kappa|z+z^{'}|}\right)\varphi(z^{'})-\frac{\varphi^{(0)}}{c^2\kappa^2}e^{i\kappa|z|}+\frac{p_z}{\kappa^2}e^{i\kappa|z|}\ . \tag{11}$$

We consider a body-force potential $\varphi(\mathbf{r},z;t)=a^3\varphi_0(t)\delta(\mathbf{r})\delta(z-z_0)$, localized at $\mathbf{r}=0$, $z=z_0<0$ in a small volume with linear dimension a; we assume a periodic time dependence $\varphi_0(t)=\varphi_0\cos\Omega t$ with frequency Ω. Equation (11) gives the volume contribution to the potential

$$\Phi_v(\mathbf{k},z;\omega)=-\frac{\pi a^3\varphi_0}{2i\kappa c^2}\left(e^{i\kappa|z-z_0|}-e^{i\kappa|z+z_0|}\right)\left[\delta(\omega-\Omega)+\delta(\omega+\Omega)\right] \tag{12}$$

(without the contribution of the surface p_z-term; we note that $\varphi^{(0)}$ is zero). The inverse Fourier transform of this function is

$$\Phi_v(\mathbf{r},z;t)=-\frac{\pi a^3\varphi_0}{2(2\pi)^3ic^2}\int d\omega d\mathbf{k}e^{-i\omega t}\frac{e^{i\mathbf{kr}}}{\kappa}\left(e^{i\kappa|z-z_0|}-e^{i\kappa|z+z_0|}\right)\left[\delta(\omega-\Omega)+\delta(\omega+\Omega)\right]\ ; \quad (13)$$

making use of the Sommerfeld integral[12]

$$\frac{i}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{\kappa}e^{i\kappa|z|}=\frac{e^{i\omega R/c}}{R}\ , \quad (14)$$

where $R=\sqrt{r^2+z^2}$, we get

$$\Phi_v(\mathbf{r},z;t)=\frac{a^3\varphi_0}{4\pi c^2}\left[\frac{\cos\Omega(t-R_1/c)}{R_1}-\frac{\cos\Omega(t-R_2/c)}{R_2}\right]\ , \quad (15)$$

where $R_1=\sqrt{r^2+(z-z_0)^2}$ and $R_2=\sqrt{r^2+(z+z_0)^2}$; these are (periodic) spherical vawes produced by a localized volume source. On the surface ($z=0$) the tangential displacement is zero, while the normal displacement is

$$u_z=-\frac{a^3\varphi_0\Omega z_0}{2\pi c^3R_0^2}\left[\sin\Omega(t-R_0/c)-\frac{c}{R_0\Omega}\cos\Omega(t-R_0/c)\right]\ , \quad (16)$$

where $R_0=\sqrt{r^2+z_0^2}$. We view the waves given by equation (16) as "regular" waves propagating over the surface of the solid; their amplitude decreases with increasing distance (as $1/R_0^2$ and $1/R_0^3$) and for any given position $\mathbf{R}_0$ the motion in the "regular" waves is an oscillating motion. We describe below the effect of the surface forces upon these "regular" waves.

We denote by

$$\Phi_b=\frac{p_z}{\kappa^2}e^{i\kappa|z|} \quad (17)$$

the contribution of the surface force p_z to equation (11); the inverse Fourier transform of this contribution is

$$\Phi_b(\mathbf{r},z;t)=\frac{1}{(2\pi)^3}\int d\omega d\mathbf{k}p_z(\mathbf{k},\omega)e^{-i\omega t}\frac{e^{i\mathbf{kr}}}{\kappa^2}e^{i\kappa|z|}\ . \quad (18)$$

We consider a surface force given by $p_z=p_0d^2\theta(t)\delta(\mathbf{r})$, *i.e.* a constant force with magnitude p_0, localized at the origin over a small region with linear dimension d (just above the source of the "regular" waves), which is applied suddenly at the initial moment of time $t=0$, lasting thereafter, with the same magnitude p_0, for a long time; its Fourier transform is $ip_0d^2/(\omega+i\varepsilon)$, $\varepsilon\rightarrow 0^+$. The most convenient way of effecting the integrals in equation (18) is to perform first the integration with respect to frequency; enclosing the integration path in the lower half-plane (in order to have $\Phi_b=0$ for past times $t<0$), we get

$$\Phi_b(\mathbf{r},z;t)=\frac{p_0d^2}{(2\pi)^2}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k^2}\left(\cos ckt-e^{-k|z|}\right)\ . \quad (19)$$

The integral in equation (19) is singular, but the displacement $\mathbf{u}_b=grad\Phi_b$ is finite; using the Sommerfeld integral[12]

$$\frac{1}{2\pi}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k^2}e^{-k|z|}=\frac{1}{R}\ , \quad (20)$$

we get immediately the z-component of the displacement caused by the surface force

$$u_{bz}(\mathbf{r},z)=-\frac{p_0d^2}{2\pi R}\ ; \quad (21)$$

we can see that it is a static displacement. The tangential components of the displacement are computed from equation (19) by using

$$\mathbf{u}_{b\mathbf{r}}(\mathbf{r},z;t)=\frac{p_0d^2}{(2\pi)^2}\frac{\mathbf{r}}{r}\frac{\partial}{\partial r}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{k^2}\left(\cos ckt-e^{-k|z|}\right)\ \ , \tag{22}$$

the Bessel function $J_0(kr)$, an integration by parts and the analytic continuation of the Sommerfeld integral; we get

$$\mathbf{u}_{b\mathbf{r}}(\mathbf{r},z;t)=\frac{p_0d^2}{2\pi}\cdot\frac{\mathbf{r}}{r^2}\left[\frac{ct}{\sqrt{c^2t^2-r^2}}\theta(ct-r)+\frac{z}{R}\right]\ \ , \tag{23}$$

where we used the integrals13

$$\int_0^\infty dxJ_0(x)\cos\lambda x=\frac{1}{\sqrt{1-\lambda^2}}\theta(1-\lambda)\ ,\ \int_0^\infty dxJ_0(x)\sin\lambda x=\frac{1}{\sqrt{\lambda^2-1}}\theta(\lambda-1)\ . \tag{24}$$

Leaving aside the static contributions, we can see that the surface force generates a cylindrical wave

$$\mathbf{u}_{b\mathbf{r}}^{cyl}(\mathbf{r},z;t)=\frac{p_0d^2}{2\pi}\cdot\frac{\mathbf{r}}{r^2}\frac{ct}{\sqrt{c^2t^2-r^2}}\theta(ct-r)\ \ , \tag{25}$$

with a discontinuous, singular wavefront, which propagates with the wave velocity c; it arises from the excitation of the "lateral" eigenmodes with frequency $\omega=ck$.

For $p_0>0$ (surface force acting downwards) the displacement produced by the surface force is positive; an observer placed on the surface $z=0$ may be carried away from the origin (the source of the "regular" waves) by this singular wave, to larger values of r, where the "regular" waves produced by the body-force $\mathbf{e}$ diminish in amplitude; it is difficult to refrain from not associating such a circumstance to the alleged effect the oil films spread over a water surface may have in "calming down" the waves. If we take into account the small, but finite, extension d of the surface force, or the small, but finite, interval of time T during which the force is applied (*i.e.*, it increases from zero to p_0), the singular denominator in equation (25) can be replaced approximately by $\left[(c^2t^2-r^2)^2+d^2c^2t^2\right]^{1/4}$ or $\left[(c^2t^2-r^2)^2+c^2T^2r^2\right]^{1/4}$ in the neighbourhood of the wavefront; we can see that the cylindrical wave is preserved, with a finite, smooth wavefront and a long tail (for large times the displacement decreases in time as $1/t^2$). We note that a similar result is obtained for a surface force spreading over the surface with velocity $v\ll c$, given by $p_z=p_0(r)\theta(vt-r)$, providing we take care of the decreasing of p_0 with increasing radius r. It is also worth noting the local velocity

$$\mathbf{v}_{b\mathbf{r}}^{cyl}\simeq\frac{\partial\mathbf{u}_{b\mathbf{r}}^{cyl}}{\partial t}=\frac{p_0cd}{2\pi}\cdot\frac{\mathbf{r}}{r}\delta(r-ct) \tag{26}$$

in the cylindrical wave (where $ct-r\simeq d$). Typical numerical estimations give $v_{b\mathbf{r}}^{cyl}\ll v_z\ll c$, where $v_z=\dot{u}_z$ is the velocity in the "regular" waves produced by volume forces (equation (16)). We emphasize the net motion of the cylindrical wave, as implied by equation (25), in comparison with the local oscillating motion in the "regular" waves (equation (16)).

Let us consider the same problem for a fluid, confined to the half-space $z<0$, with a free surface at $z=0$ in gravitational field. Usually, the internal stress $\rho c^2div\mathbf{u}$ in equation (1) is greater than the gravitational pressure ρgu_z, where g is the gravitational acceleration, such that we may neglect the effect of the gravitational field. Nonetheless, we include here a discussion of this effect. The Euler equation for an ideal fluid is

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}grad)\mathbf{v}=-\frac{1}{\rho}gradP+\mathbf{e}+\mathbf{g}\ \ , \tag{27}$$

where $\mathbf{v}$ is the flow velocity, P is the pressure, ρ is the mass density, $\mathbf{e}$ and $\mathbf{g}$ are the external field and, respectively, the gravitational field per unit mass.[14] For small velocities we neglect the term quadratic in velocity and write $\mathbf{v} \simeq \partial\mathbf{u}/\partial t$, where $\mathbf{u}$ is the displacement field. We consider a small change P_1 in pressure, determined by the displacement field, and write $P = P_0 + P_1$, where P_0 is the constant, uniform, equilibrium pressure; equation (27) becomes

$$\ddot{\mathbf{u}} = -\frac{1}{\rho} gradP_1 + \mathbf{e} + \mathbf{g} \ , \tag{28}$$

where ρ can be considered here as the constant, uniform, equilibrium density. In the volume of the fluid the change in pressure is given by $P_1 = (\partial P/\partial\rho)_S \delta\rho = -\rho(\partial P/\partial\rho)_S div\mathbf{u}$, where we introduce the adiabatic coefficient of compressibility (S denotes the entropy); in the fluid volume $\mathbf{g}$ is constant, such that it may produce only static deformations, which we neglect; equation (28) becomes the wave equation (1)

$$\ddot{\mathbf{u}} - c^2 grad \cdot div\mathbf{u} = \mathbf{e} \ , \tag{29}$$

where $c = \sqrt{(\partial P/\partial\rho)_s}$ is the sound velocity; using $\mathbf{u} = grad\Phi$ and $\mathbf{e} = grad\varphi$, equation (28) becomes the wave equation (5) with the solution given by equation (8). At this point it is useful to introduce the velocity potential ψ, which gives the velocity field $\mathbf{v} = grad\psi$; it is related by the displacement potential Φ by $\psi = \dot{\Phi}$. Making use of equation (15) we can see that the tangential velocities produced by the volume force on the surface $z = 0$ are zero, while the component v_z for $z = 0$ is

$$v_z = -\frac{a^3\varphi_0\Omega^2 z_0}{2\pi c^3 R_0^2}\left[\cos\Omega(t - R_0/c) + \frac{c}{R_0\Omega}\sin\Omega(t - R_0/c)\right] \ ; \tag{30}$$

as expected, v_z is the time derivative of the displacement component u_z given by equation (16); we consider v_z given by equation (30) as the "regular" velocity waves, caused by volume forces and propagating over the fluid surface.

The boundary conditions for a fluid in gravitational field are different from the boundary conditions given by equations (6); the difference arises from the mobility of the fluid surface $z = 0$. Taking the limit $z \to 0$ in equation (28) we see that P_1 is the additional pressure imposed on the surface (with minus sign), the external field $\mathbf{e}$ disappears and the vertical component of the gravitational field can be written as $-g\partial u_z/\partial z \mid_{z=0}$; we consider a uniform additional pressure $-P_1$, such that the tangential components of equation (28) are vanishing; we are left with the boundary condition

$$\ddot{\Phi} \mid_{z=0} = \frac{1}{\rho}P_1 - g\frac{\partial\Phi}{\partial z} \mid_{z=0} \ . \tag{31}$$

Making use of the Fourier transforms, this boundary condition can be written as

$$\omega^2\Phi^{(0)} - g\Phi^{(1)} = -\frac{1}{\rho}P_1 \ , \tag{32}$$

or, using equation (9),

$$(\omega^2 + i\kappa g)\Phi^{(0)} = -\frac{g}{c^2}\int_{-\infty}^{0} dz' e^{i\kappa|z'|}\varphi(z') - \frac{1}{\rho}P_1 \ . \tag{33}$$

We can see from equation (33) that there exist approximate eigenmodes with eigenfrequency given by $\omega_g^2 = gk$, as long as $g \ll c^2 k$ (*i.e.*, wavelengths $\lambda \ll c^2/g$); these are the well-known gravity, damped waves of the form $e^{-i\omega_g t+i\mathbf{kr}-k|z|}$ (surface waves); their wavelengths can be very long, since $c^2/g \simeq 10^5 m$ for $c = 10^3 m/s$. In equation (33) we may set $\kappa \simeq ik$ and get

$$\Phi^{(0)} = -\frac{g}{c^2(\omega^2-\omega_g^2)}\int_{-\infty}^{0} dz' e^{-k|z'|}\varphi(z') - \frac{1}{\rho(\omega^2-\omega_g^2)}P_1 \ . \tag{34}$$

From equation (8) we get a gravity-induced volume potential

$$\Phi_v^g = -\frac{g}{c^2(\omega^2-\omega_g^2)}\int_{-\infty}^{0} dz' e^{-k|z+z'|}\varphi(z') \tag{35}$$

and a surface contribution

$$\Phi_b = -\frac{1}{\rho(\omega^2-\omega_g^2)}P_1 e^{-k|z|} \ . \tag{36}$$

The inverse Fourier trasform of the gravity-induced volume contribution is given by

$$\Phi_v^g(\mathbf{r}, z; t) = \frac{ga^3\varphi_0}{(2\pi)^2 c^2}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{gk-\Omega^2}e^{-k|z+z_0|}\cos\Omega t \ , \tag{37}$$

where the integral is taken as the principal value; in contrast with the volume force, which generates propagating waves (equation (30)), the gravity-induced volume force generates stationary oscillations (vibrations). For $r\Omega^2/g \gg 1$ the gravity-induced stationary waves do not represent an appreciable contribution.

For the contribution to the surface term given by equation (36) we use a surface force $P_1 = P_1^0 d^2\theta(t)\delta(\mathbf{r})$ (placed just above the source of the "regular" waves), whose Fourier transform is $iP_1^0 d^2/(\omega + i\varepsilon)$, $\varepsilon \to 0^+$; we get

$$\psi_b(\mathbf{k}, z; \omega) = \frac{i\omega}{\rho(\omega^2-\omega_g^2)}P_1 e^{-k|z|} = -\frac{P_1^0 d^2}{\rho(\omega^2-\omega_g^2)}e^{-k|z|} \ ; \tag{38}$$

the reverse time Fourier transform of this potential is

$$\psi_b(\mathbf{k}, z; t) = -\frac{P_1^0 d^2}{2\pi\rho}e^{-k|z|}\int d\omega\frac{1}{\omega^2-\omega_g^2}e^{-i\omega t} \ , \tag{39}$$

where the integral in equation (36) can be readily computed; it yields

$$\psi_b(\mathbf{k}, z; t) = \frac{P_1^0 d^2}{\rho\omega_g}e^{-k|z|}\sin\omega_g t \ . \tag{40}$$

Finally, it remains to compute the spatial reverse Fourier transform

$$\psi_b(\mathbf{r}, z; t) = \frac{P_1^0 d^2}{(2\pi)^2\rho\sqrt{g}}\int d\mathbf{k}\frac{e^{i\mathbf{kr}}}{\sqrt{k}}e^{-k|z|}\sin\sqrt{gk}t \ . \tag{41}$$

This integral can also be written as

$$\psi_b(\mathbf{r}, z; t) = \frac{P_1^0 d^2}{(2\pi)^2\rho\sqrt{g}}Im\int d\mathbf{k}\frac{e^{i(\sqrt{gk}t-\mathbf{kr})}}{\sqrt{k}}e^{-k|z|} \ ; \tag{42}$$

it gives the surface contribution to velocity

$$\begin{aligned} v_{bz} &= \frac{P_1^0 d^2}{(2\pi)^2\rho\sqrt{g}}Im\int d\mathbf{k}\sqrt{k}e^{i(\sqrt{gk}t-\mathbf{kr})}e^{-k|z|} \ , \\ \mathbf{v}_{br} &= \frac{P_1^0 d^2}{(2\pi)^2\rho\sqrt{g}}\frac{\mathbf{r}}{r}\frac{\partial}{\partial r}Im\int d\mathbf{k}\frac{e^{i(\sqrt{gk}t-\mathbf{kr})}}{\sqrt{k}}e^{-k|z|} \ ; \end{aligned} \tag{43}$$

it is worth noting that these velocities are propagating (and dispersive) waves.

The integrals in equations (42) and (43) are superpositions of waves propagating along the surface and damped down the depth; near the surface $z=0$ we may neglect the damping exponential in equations (42) and (43). At a given position $\mathbf{r}$ the relevant contribution to the wave superposition comes from waves with wavevectors $\mathbf{k}$ close to some wavevector $\mathbf{k}_0$, specific to each integral; its magnitude is related to the distance r by $k_0 r=\alpha$, where the initial phase α is an undetermined parameter. Indeed, the long wavelengths have not an appreciable contribution to the motion, while the short wavelengths are expected to give interference effects. We introduce the notation $\mathbf{k}=\mathbf{k}_0+\mathbf{q}$ and integrate over $\mathbf{q}$ in equations (42) and (43). At the same time, we expand the frequency $\omega_g=\sqrt{gk}$ in powers of the components of $\mathbf{q}$ and introduce the group velocity $\mathbf{v}_g=\partial\omega_g/\partial\mathbf{k}\mid_{\mathbf{k}_0}=\frac{1}{2}\sqrt{g/k_0}(\mathbf{k}_0/k_0)$. We assume that the second order derivatives in the expansion of ω_g are sufficiently small, such that their contribution may be neglected; the integration over $\mathbf{q}$ is extended to the whole space. This wave-packet approximation is valid as long as the inequality $2(d\omega_g/dk)/(d^2\omega_g/dk^2)\mid_{k_0}\gg k_0$ holds (it amounts to $4\gg 1$!). The validity extends over a distance $\Delta r\simeq 1/\Delta k$, where Δk is given by $\Delta k\simeq 2(d\omega_g/dk)/(d^2\omega_g/dk^2)\mid_{k_0}=4k_0$, *i.e.* $\Delta r\simeq r/4\alpha$ (this is the width of the wave-packet). Similarly, the validity of the wave-packet approximation extends over time intervals of the order $\Delta t\simeq\Delta r/v_g=\sqrt{r/4\alpha g}$ (this is the "life-time" of the wave-packet). It is easy to see that the integral in equation (42) becomes

$$\begin{aligned}&\int d\mathbf{k}\frac{e^{i(\sqrt{gk}t-\mathbf{kr})}}{\sqrt{k}}\simeq\frac{1}{\sqrt{k_0}}e^{i(\sqrt{gk_0}t-\mathbf{k}_0\mathbf{r})}\int d\mathbf{q}e^{i\mathbf{q}(\mathbf{v}_g t-\mathbf{r})}=\\&=\frac{(2\pi)^2}{\sqrt{k_0}}e^{i(\sqrt{gk_0}t-\mathbf{k}_0\mathbf{r})}\delta(\mathbf{r}-\mathbf{v}_g t)=\frac{(2\pi)^2}{\sqrt{k_0}}e^{i\alpha}\delta(\mathbf{r}-\mathbf{v}_g t)\ ,\end{aligned}\tag{44}$$

which is a circular wave on the surface, propagating with the velocity $v_g=\frac{1}{2}\sqrt{g/k_0}\simeq\sqrt{gr/4\alpha}$, directed along the radius $\mathbf{r}$. Inserting equation (44) into equations (42) and (43) we get

$$\psi_b(\mathbf{r},z=0;t)\simeq\frac{P_1^0 d^2}{\rho\sqrt{g}}\frac{\sin\alpha}{\sqrt{\alpha}}\sqrt{r}\delta(\mathbf{r}-\mathbf{v}_g t)\tag{45}$$

and the leading contributions to the surface velocities (for $z=0$)

$$\begin{aligned}v_{bz}&\simeq\frac{P_1^0 d^2}{\rho\sqrt{g}}\sqrt{\alpha}\sin\alpha\cdot\frac{1}{\sqrt{r}}\delta(\mathbf{r}-\mathbf{v}_g t)\ ,\\\mathbf{v}_{b\mathbf{r}}&\simeq\frac{P_1^0 d^2}{2\rho\sqrt{g}}\frac{\sin\alpha}{\sqrt{\alpha}}\frac{\mathbf{r}}{r^{3/2}}\delta(\mathbf{r}-\mathbf{v}_g t)\ .\end{aligned}\tag{46}$$

The initial phase α remains undetermined; since, from equations (43), v_{bz} and $v_{b\mathbf{r}}$ are positive for $r\rightarrow 0$ and $P_1^0>0$, we may infer that $\sin\alpha>0$ for these integrals. Equations (46) give the distribution of velocities on the fluid surface, according to the wave-packet picture. We can see that these velocities "move" on the surface with the group velocity $\mathbf{v}_g$, their magnitude decreasing with increasing distance from the origin. They describe a wave packet moving with velocity $\mathbf{v}_g$ from the origin (where the source of the "regular" waves is placed) to infinity. In this motion the distance r increases as $r=v_g t=\sqrt{gr/4\alpha}t$, *i.e.* $r=gt^2/4\alpha$, which indicates an accelerated motion. For $P_1^0>0$ we can see that a casual observer placed on the fluid surface is carried away from the source of the "regular" waves by the wave-packet of the gravity waves generated by the surface force, to surface regions where the amplitude of the "regular" waves is smaller. As in the case of the absence of the gravity discussed above, this circumstance resembles the effect the oil layers spread over the fluid surface may have on "stilling the waters".

The wave-packet picture described above is supported by an estimation of the integrals in equations (43), which, basically, we own to Refs. [15, 16]). Equations (43) can be written as

$$\begin{aligned} v_{bz} &= \frac{P_1^0 d^2}{2\pi\rho\sqrt{g}} \int dk k^{3/2} J_0(kr) e^{-k|z|} \sin\sqrt{gk}t \ , \\ \mathbf{v}_{b\mathbf{r}} &= \frac{P_1^0 d^2}{2\pi\rho\sqrt{g}} \frac{\mathbf{r}}{r^2} \int dk k^{3/2} \frac{d}{dk} J_0(kr) e^{-k|z|} \sin\sqrt{gk}t \ ; \end{aligned} \tag{47}$$

we introduce the new variable $x = \sqrt{gk}t$ and get integrals of the form

$$\int_0^\infty dx \cdot x^n J_0(rx^2/gt^2) e^{-\frac{|z|}{gt^2}x^2} \cos x \ , \tag{48}$$

where $n = 1, 2, 3....$ These integrals are controlled by the parameter r/gt^2; we can give results in two limiting cases:

$$v_{bz} = \frac{24P_1^0 d^2}{\pi\rho g^3 t^5} \left(1 - \frac{420r^2}{g^2t^4} + ...\right) \ , \ \mathbf{v}_{b\mathbf{r}} = \frac{360P_1^0 d^2}{\pi\rho g^4 t^7} \left(1 - \frac{630r^2}{g^2t^4} + ...\right) \mathbf{r} \ , \ r/gt^2 \ll 1 \ , \tag{49}$$

and

$$v_{bz} \simeq -\frac{3P_1^0 d^2}{4\sqrt{2}\pi\rho} \cdot \frac{t}{r^3} \cos\frac{gt^2}{4r} \ , \ \mathbf{v}_{b\mathbf{r}} \simeq -\frac{3P_1^0 d^2}{4\sqrt{2}\pi\rho} \cdot \frac{t\mathbf{r}}{r^4} \sin\frac{gt^2}{4r} \ , \ r/gt^2 \gg 1 \tag{50}$$

for $z = 0$. We can see that these waves oscillate slowly for large distances ($r \gg gt^2$) and have a wavefront around r of the order gt^2, which moves according to the acelerated-motion law $r = gt^2$; for $P_1^0 > 0$ the waves behind the wavefront are positive. The velocity waves given by equations (49) and (50) for $r \simeq gt^2$ correspond to the wave-packets derived above. This result supports the conclusion that a surface force generates packets of gravity waves that can carry an observer away from the origin (source of the "regular" wave).

In conclusions, we have investigated here the role played by some boundary conditions of physical interest upon the waves propagating in a semi-infinite (half-space) fluid. It is shown that, neglecting the gravity, a pressure applied instantaneously downwards the plane surface of a fluid in a small region localized above the source of the "regular" waves, and lasting an indefinite time, produces a cylindrical wave with a singular wavefront which propagates away from the wave source (the wavefront singularity is smoothed out by a finite spatial spread or a finite duration of the process of applying the pressure). Similarly, a constant pressure applied suddenly downwards on the plane surface of a fluid in gravitational field in a small region localized above the source of the "regular" waves and lasting an indefinite time may generate wave-packets of gravity waves which move away from the wave source. In both case a casual observer on the surface can be moved away from the source of the "regular" waves, toward regions where the wave amplitude is smaller. This could be the meaning of the notion of "calming down" the surface waves by oil layers spread over the fluids surface. This paper was motivated by the alleged effect such oil films may have upon the surface waves. The major effects illustrated here for the boundary conditions upon the wave propagation in semi-infinite fluids arise from the excitation of specific eigenmodes: "lateral" waves when the gravity is neglected (tangential to the surface) and damped surface waves (gravity waves) in the gravitational field. Basically, the effect described here can be viewed as the transfer of the applied pressure to directions tangential to the surface. In gravitational field this transfer is mediated by the gravitational acceleration. Also, it is worth noting an important difference between the "regular" waves and the waves generated by the surface forces investigated here. In the "regular" waves the local motion at any given position is an oscillating motion (equations (16), (30)), while the transient regime generated by surface forces suddenly applied produces a net motion directed outwards from the application point (equations (25), (46)).

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References

[1] J. Beresford, "Oil on troubled waters; a reappraisal of the storm tactics decribed in Acts 27", Int. J. Maritime History **26** 752-761 (2014).

[2] B. Franklin, "On the stilling of waves by means of oil", Phil. Trans. Roy. Soc. London **64** 445-460 (1774).

[3] C. Cox, "Suppression of breakers in stormy seas by an oil film", Int. J. Maritime History **27** 528-536 (2015).

[4] A. B. Wyckoff, "The use of oil in storms at sea", Proc. Am. Phil. Soc. **23** 383-388 (1886).

[5] D. W. Camp and J. C. Berg, "The spreading of oil on water in the surface tension regime", J. Fluid Mech. **184** 445-462 (1987).

[6] A. E. Korinenko and V. V. Malinovsky, "Field study of film spreading on a sea surface", Oceanologia **56** 461-475 (2014).

[7] W. Alpers and H. Huhnerfuss, "The damping of ocean waves by surface films: a new look at an old problem", J. Geophys. Res. **94** 6251-6265 (1989).

[8] M. Gade, H. Huhnerfuss and G. Korenowski, eds., *Marine Surface Films*, Springer (2006).

[9] H. Lamb, *Hydrodynamics*, Cambridge University Press, Cambridge (1916).

[10] V. G. Levich, *Physicochemical Hydrodynamics*, Prentice-Hall, Englewood Cliffs (1962).

[11] L. Landau and E. Lifshitz, *Theory of Elasticity, Course of Theoretical Physics*, vol. 7, Elsevier, Oxford (1986).

[12] A. Sommerfeld, *Partielle Differentialgleichungen der Physik, Vorlesungen uber Theoretische Physik*, Bd. VI, Akad. Verlag, Leipzig (1966).

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th ed., eds. A. Jeffrey and D. Zwillinger, p. 709, 6.671 (1,2), Academic Press (2000).

[14] L. Landau and E. Lifshitz, *Fluids Mechanics, Course of Theoretical Physics*, vol. 6, Elsevier, Oxford (2004).

[15] S. D. Poisson, "Memoire sur la theorie des ondes", Mem. de l'Acad. Roy. des Sciences **1** 71-186 (1816).

[16] A. L. Cauchy, "Memoire sur la theorie de la propagation des ondes a la surface d'un fluide pesant d'une profondeur indefinie", Mem. Sav. Etrang. **1** 5-318 (1827) (*Oeuvres Completes*, 1er Serie, vol. 1, Gauthier-Villars, Paris (1882)).