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Normal modes and coupled harmonic oscillators in structural engineering, the model of the embedded bar and amplification factors

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Abstract

The effects of the earthquakes on buildings and the concept of seismic base isolation are investigated by using the model of the vibrating bar embedded at one end. The normal modes and the eigenfrequencies of the bar are highlighted and the response to oscillating shocks is computed for several typical shock structures. A special atention is devoted to the oscillating shock, especially with a sharp wavefront, deemed as a suitable model for the seismic main shock with its long tail. It is shown that in all cases the response of the bar is governed by an amplification factor, which includes cumulative information about the shock duration, height of the bar above the ground surface and the velocity of the elastic waves in the bar. The amplification of the response is due to the excitation of the normal modes (eigenmodes). The efect is much enhanced at resonance, for oscillating shocks which contain eigenfrequencies of the bar. Also, the response of two linearly joined bars with one end embedded is calculated. It is shown that for very different elastic properties the eigenfrequencies are due mainly to the "softer" bar. The model of the embedded bar provides a way of understanding the well-known amplification site effects of the earthquakes, as arising from the excitation of the normal modes in local inhomogeneities. The effect of the base isolation in seismic structural engineering is assessed by formulating the model of coupled harmonic oscillators, as a simplified model for the structure building-foundation viewed as two coupled vibrating bars. The coupling decreases the lower eigenfrequencies of the structure and increases the higher ones. Similar amplification factors are derived for coupled oscillators at resonance with an oscillating shock.

Introduction. Embedded bar. The concept of seismic base isolation aims at formulating solutions for protecting buildings against earthquakes, by designing special couplings betwen buildings and their foundations. As a key element in earthquake engineering it aims at designing means of achieving to some extent a building-foundation decoupling, such that the response of the building to vibrations be not (too) damaging. Usually, the dynamics of the structure building-foudation is approached by means of the model of two coupled oscillators. We examine here the formulation of this model starting with coupled elastic bars.[1]-[7] To this end, it is useful to assess first the response to the ground excitation of a vibrating bar with one end embedded in the ground. For ground excitation we focus mainly on oscillating shocks with a sharp wavefront, as corresponding to seismic excitation, especially the seismic main shock with its long tail. At resonance it is shown that the response of the bar exhibits amplification factors which may attain large values. The amplification factors are given by a combination of the shock duration, the heigth of the bar above the ground surface and the velocity of the elastic waves in the bar. The amplification factors arise as a consequence of the excitation of the normal modes in the bar. Two coupled bars are also studied, excited at their lower end; for bars with very different elastic properties it is shown that the eigenfrequencies of this system are given mainly by the "softer" bar. Such an information may throw light upon the elasticity of composite structures, like, for instance, those including voids. A bar buried completely in the ground may serve as a model for local inhomogeneities in the soil; it is shown that such an inhomogeneity may exhibit large spectral amplification factors. Making use of the information gained from the study of the vibrating bars we fromulate the model of coupled harmonic oscillators and investigate its response to an oscillating shock. It is shown that the lower frequency of the system is lowered by the coupling, while the higher frequency is raised. At resonance the coupled oscillators exhibit amplification factors, similar with the vibrating bar.

The most convenient model for investigating the response of a building to ground vibrations is the bar embedded at one end. Let us assume that a vertical elastic bar with uniform cross-section is fixed in the ground at one end, having a length l above the ground surface; the bar end above the ground is free. Under the action of the seismic waves the buried end of the bar is set in motion. We assume the cross-sectional dimensions of the bar being much smaller than the bar length, so we may limit ourselves only to the z-dependence of the displacement, where z is the vertical coordinate (along the bar). At the same time, we consider the length of the bar and the excitation sufficiently small, such that the bar does not enter the regime of flexural elasticity (bending). The strain tensor reduces to $u_{xz} = u_{zx} = \frac{1}{2}\partial u_x/\partial z$, $u_{yz} = u_{zy} = \frac{1}{2}\partial u_x/\partial z$ and $u_{zz} = \partial u_z/\partial z$; the stress tensor is $\sigma_{xx} = \sigma_{yy} = \lambda\partial u_z/\partial z$, $\sigma_{zz} = (\lambda + 2\mu)\partial u_z/\partial z$, $\sigma_{xz} = \sigma_{zx} = \mu\partial u_x/\partial z$ and $\sigma_{yz} = \sigma_{zy} = \mu\partial u_y/\partial z$, where λ and μ are the Lame coefficients. It follows an elastic force density $f_x = \partial\sigma_{xz}/\partial z = \mu\partial^2 u_x/\partial z^2$, $f_y = \partial\sigma_{yz}/\partial z = \mu\partial^2 u_y/\partial z^2$ and $f_z = (2\mu + \lambda)\partial^2 u_z/\partial z^2$. The equations of the elastic motion are[8]

$$\rho\ddot{u}_x - \mu\frac{\partial^2 u_x}{\partial z^2} = 0\ ,\ \rho\ddot{u}_y - \mu\frac{\partial^2 u_y}{\partial z^2} = 0\ ,$$
$$\rho\ddot{u}_z - (\lambda + 2\mu)\frac{\partial^2 u_x}{\partial z^2} = 0\ , \tag{1}$$

where ρ is the density of the bar. We may limit ourselves to only one equation of motion, which we write in the generic form

$$\ddot{u} - c^2\frac{\partial^2 u}{\partial z^2} = 0\ , \tag{2}$$

where u and c stand for $u_{x,y,z}$ and, respectively, $c_l = \sqrt{(\lambda + 2\mu)/\rho}$, $c_t = \sqrt{\mu/\rho}$; $c_{l,t}$ are the velocities of the longitudinal and, respectively, transverse waves in the bar.

Shock-type excitation. We solve equation (2) for a free upper end of the bar, while the lower end has the prescribed motion $u_0(t)$ of the ground; therefore, we impose the boundary conditions

$$\frac{\partial u}{\partial z}\mid_{z=l}= 0\ ,\ u\mid_{z=0}= u_0(t)\ ; \tag{3}$$

the motion is limited to $t > 0$ and $0 < z < l$. Using time Fourier transform, equation (2) and the boundary conditions (3) read

$$u'' + \kappa^2 u = 0\ ,\ u_l' = 0\ ,\ u_0 = u_0(\omega)\ , \tag{4}$$

where $\kappa^2 = \omega^2/c^2$ and the prime denotes the derivation with respect to z.

We note that the solution for the limited interval $0 < z < l$ can also be obtained by extending the equation to the whole space and limiting ourselves to the restriction of the solution to the interval $0 < z < l$; as it is well known, this is achieved by multiplying the equation by $\theta(z)\theta(l - z)$ and absorbing the step functions θ in the derivatives (the method of generalized functions). Unfortunately, we should use in this case the Green function which implies wave propagation in both directions, in order to satisfy the boundary conditions at infinity; this complicates the technical procedure (in contrast with the half-line, where one-direction Green function is needed).

The natural "initial" condition which requires the vanishing of the solution for past times ($t < 0$) is treated most conveniently by integrating over frequency ω in the lower half-plane (the causality condition).

The solution of the equation (4) has the form

$$u = A\cos\kappa z + B\sin\kappa z \ , \tag{5}$$

where the constants A and B are determined by the boundary conditions; we get

$$A = u_0 \ , \ B = u_0 \tan\kappa l \tag{6}$$

and

$$u(z,\omega) = u_0(\omega)\left(\cos\kappa z + \tan\kappa l \cdot \sin\kappa z\right) \ . \tag{7}$$

The reverse Fourier transform gives

$$u(z,t) = \frac{1}{2\pi}\int d\omega u_0(\omega)e^{-i\omega t}\cos\kappa z + \frac{1}{2\pi}\int d\omega u_0(\omega)e^{-i\omega t}\tan\kappa l \cdot \sin\kappa z \ , \tag{8}$$

or

$$u(z,t) = \frac{1}{2}\left[u_0(t - z/c) + u_0(t + z/c)\right] + \frac{1}{2\pi}\int d\omega u_0(\omega)e^{-i\omega t}\tan\kappa l \cdot \sin\kappa z \ , \tag{9}$$

where we take the real part. We can see that half of the displacement $u_0(t)$ applied to the grounded end propagates along the bar with velocity c, while the other half propagates in the opposite direction (it "comes from the future").

We consider seismic excitations which have a general aspect of shocks, *i.e.* they are concentrated in at the initial moment of time. This is valid for both the primary P and S waves, as well as for the main shock produced by the so-called surface waves. Consequently, we assume first a shock-like ground motion $u_0(t) = u_0 T\delta(t)$, $u_0(\omega) = u_0 T$, where T is a measure for the duration of the shock. The second integral in equation (9) implies the contribution of the poles arising from the zeroes of the denominator of $\tan\kappa l$: $\cos\kappa l = \cos\omega_n l/c = 0$, $\omega_n = (2n+1)\pi c/2l$, where n is any integer. We get

$$u(z,t) = \frac{1}{2}u_0 T\left[\delta(t - z/c) + \delta(t + z/c)\right] + u_0\frac{cT}{l}\sum_n \sin\omega_n t \cdot \sin\omega_n z/c \ . \tag{10}$$

We can see that half of the δ-pulse applied at the fixed end at the initial moment propagates along the bar up to $z = l$, while the other half, "coming from the future", brings no contribution to the motion of the bar ($0 < z < l$), except for $z = 0$; in addition, vibrations given by the normal modes with the eigenfrequencies ω_n are excited in the bar.

The amplitude of the pulse is of the order u_0, while the amplitude of the normal modes is of the order $u_0 cT/l$; we introduce the parameter

$$g = \frac{cT}{l} \tag{11}$$

and denote by $u_n(t,z)$ the contribution to the displacement of the n-th normal mode, *i.e.*

$$u_n = g u_0 \sin\omega_n t \cdot \sin\omega_n z/c \ ; \tag{12}$$

the corresponding velocity and acceleration are given by

$$\dot{u}_n = gu_0\omega_n \cos\omega_n t \cdot \sin\omega_n z/c = g\frac{u_0}{T}(\omega_n T)\cos\omega_n t \cdot \sin\omega_n z/c \tag{13}$$

and, respectively,

$$\ddot{u}_n = -gu_0\omega_n^2 \sin\omega_n t \cdot \sin\omega_n z/c = -g\frac{u_0}{T^2}(\omega_n T)^2 \sin\omega_n t \cdot \sin\omega_n z/c \ . \tag{14}$$

A δ-shock of the form $u(t) = u_0 T\delta(t)$ includes a superpositions of oscillations with equal weights for all frequencies. We can see from the above equations that the response of the bar is affected by the factor g and powers of $\omega_n T$; beside the ground displacement u_0, the quantities u_0/T and u_0/T^2 may be viewed as ground velocity and, respectively, acceleration. Typical values of the velocity of the elastic waves in the bar are $c \simeq 3 \times 10^3 m/s$; for a short duration $T = 0.1s$ we get $g = 10$ for a length $l = 30m$. We can see that the displacement, velocity and acceleration amplitudes in the bar could be enhanced in comparison with their ground counterparts. This is why we may call the parameter g the amplification factor.

However, a pulse with a finite duration T excites mainly frequencies ω_n up to $\simeq \pi/T$; in the above formulae, a weight factor $f(\omega_n)$ should be inserted, which decreases appreciably for frequencies $\omega_n > \pi/T$; therefore, the amplification parameter is subject to the condition

$$\omega_n T = \frac{(2n+1)\pi}{2} g \leq \pi \ , \tag{15}$$

which implies values for g of the order as high as unity, corresponding to the fundamental frequency $\omega_0 = \pi c/2l$ ($n = 0$). In addition, it is well knwon that the seismic spectrum includes a range of frequencies extending up to $\simeq 10s^{-1}$, which is far below a fundamental frequency of the order $c/l \simeq 100s^{-1}$ for $c \simeq 3 \times 10^3 m/s$ and $l = 30m$. Therefore, it is unlikely that a short pulse can excite normal modes which might lead to appreciable amplification factors in reasonable conditions. However, the situation is different if the pulse includes resonance frequencies.

As a technical point, we note that if the pulse is applied at some point on the bar, different from the bar ends, then we deal in fact with two bars; the solution has four constants of the type A and B in equation (5) and the boundary conditions are the continuity of the displacement at the point of application of the excitation, the equality of the displacement with the excitation at that point and the conditions at the two ends; the resulting four equations determine the four constants of the solution.

Oscillating shock. If the ground displacement has the form of a harmonic oscillation $u_0(t) = u_0 \cos\omega_0 t$, $u_0(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$, the displacement in the bar (equation (9)) is

$$u(z,t) = u_0 \cos\omega_0 t\,(\cos\omega_0 z/c + \tan\omega_0 l/c \cdot \sin\omega_0 z/c) = u_0 \cos\omega_0 t \frac{\cos\omega_0 (z-l)/c}{\cos\omega_0 l/c} \ ; \tag{16}$$

if ω_0 happens to be an eigenfrequency of the bar ($\omega_0 = \omega_n$), then the amplitude increases indefinitely, and the dangerous resonance phenomenon occurs (we recall that the eigenfrequencies ω_n are the roots of the equation $\cos\omega_n l/c = 0$).

Let us assume a ground motion given by

$$u_0(t) = u_0\theta(t)e^{-\alpha t}\cos\omega_0 t \ , \tag{17}$$

where $\theta(t) = 1$ for $t > 0$, $\theta(t) = 0$ for $t < 0$ is the step function and $0 < \alpha \ll \omega_0$; it represents an oscillating shock with a sharp wavefront, attenuated in time with the rate α, which is deemed to model the seismic main shock with its long tail (produced by the so-called surface waves).[9] The Fourier transform of $u_0(t)$ is

$$u_0(\omega) = \frac{i}{2}u_0\left(\frac{1}{\omega - \omega_0 + i\alpha} + \frac{1}{\omega + \omega_0 + i\alpha}\right) \ . \tag{18}$$

The displacement given by equation (9) includes the propagating shocks $\theta(t\pm z/c)e^{-\alpha(t\pm z/c)}\cos\omega_0(t\pm z/c)$, which we leave aside here. The integral in equation (9) includes contributions from the poles $\pm\omega_0 - i\alpha$ of the shock and contributions from the normal modes with eigenfrequencies ω_n (the poles of $\tan\kappa l$). For ω_0 different from all ω_n we get

$$\begin{gathered} u(z,t) \simeq u_0 e^{-\alpha t}\cos\omega_0 t\cdot\tan\omega_0 l/c\cdot\sin\omega_0 z/c- \\ -\tfrac{c}{2l}u_0\sum_n\left[\tfrac{(\omega_n-\omega_0)\cos\omega_n t-\alpha\sin\omega_n t}{(\omega_n-\omega_0)^2+\alpha^2}+\tfrac{(\omega_n+\omega_0)\cos\omega_n t-\alpha\sin\omega_n t}{(\omega_n+\omega_0)^2+\alpha^2}\right]\sin\omega_n z/c \ ; \end{gathered} \tag{19}$$

for $\omega_0 = \omega_n$ (at resonance)

$$u(z,t) = u_0\frac{c}{l}\frac{1-e^{-\alpha t}}{\alpha}\sin\omega_0 t\cdot\sin\omega_0 z/c \ . \tag{20}$$

We can see that the displacement amplitudes at resonance are $(c/l\alpha)u_0$, *i.e.* in the amplification factor $g = cT/l$ the duration T is replaced by $1/\alpha$, as expected. We note that for $\omega_0 = 0$ the amplitude is reduced to $(c/l\omega_n)u_0$. Similarly, the response velocity and aceleration include factors $u_0\omega_0$ and $u_0\omega_0^2$, respectively, which now can be viewed as corresponding to the ground velocity and acceleration; the amplification factor for these quantities is $g = c/l\alpha$, as for the displacement.

A similar conclusion is reached for a ground motion given by

$$u(t) = u_0 e^{-\alpha^2(t-T/2)^2}\cos\omega_0 t \ , \tag{21}$$

which is a harmonic oscillation with frequency ω_0 attenuated in time by a gaussian with the rate $\alpha \ll \omega_0$, centered at the moment T; its Fourier transform is

$$u(\omega) = \frac{1}{2}u_0\frac{\sqrt{\pi}}{\alpha}e^{\frac{1}{2}iT(\omega+\omega_0)-(\omega+\omega_0)^2/4\alpha^2} + (\omega_0 \to -\omega_0) \ ; \tag{22}$$

the contribution of the normal modes is

$$u_n(z,t) = \frac{cu_0}{2l\alpha}\sqrt{\pi}\sin\left[\omega_n(t-T/2)+\omega_0 T/2\right]\sin\omega_n z/c\cdot e^{-\frac{(\omega_n-\omega_0)^2}{4\alpha^2}} + (\omega_0 \to -\omega_0) \ . \tag{23}$$

We can see from equation (23) that at resonance ($\omega_0 = \omega_n$) the amplitude of the oscillations is of the order $(c/l\alpha)u_0$, where $1/\alpha$ is a measure for the pulse duration T. For a superposition of harmonic oscillations weight factors $f(\omega_n)$ should be inserted in the amplification factor $c/l\alpha$. It is worth emphasizing that amplification factors of the type $g = c/l\alpha$ may attain high values.

Coupled bars. Let us assume a bar with length l fixed at $z = 0$ to another long bar with length l_0; we denote the former bar by 1 and the latter bar by 2; bar 1 extends above the ground surface, while bar 2 is buried in the ground. The equations of elastic motion in the two bars are

$$\ddot{u}_1 - c_1^2 u_1^{''} = 0 \ , \ \ddot{u}_2 - c_2^2 u_2^{''} = 0 \ , \tag{24}$$

where $u_{1,2}$ are the displacement in the two bars; the boundary conditions are

$$\begin{gathered} u_2\mid_{z=-l_0} = u_0(t) \ , \ u_1\mid_{z=0} = u_2\mid_{z=0} \ , \\ \mu_1 u_1^{'}\mid_{z=0} = \mu_2 u_2^{'}\mid_{z=0} \ , \ u_1^{'}\mid_{z=l} = 0 \ , \end{gathered} \tag{25}$$

which signify, respectively, a ground motion applied to the lower end $z=-l_0$ of bar 2, the continuity of the displacement at the joining point (the bars are rigidly connected to each other), the absence of the force at the interface $z=0$ and the free upper end $z=l$; the force is written for a shear displacement; for compression (dilatation) the rigidity moduli $\mu_{1,2}$ should be replaced by $\lambda_{1,2}+2\mu_{1,2}$. The solutions of equations (24) and (25) are

$$u_1(z,\omega)=u_0(\omega)\frac{\cos\kappa_1(z-l)}{\cos\kappa_1 l\cos\kappa_2 l_0-\frac{\mu_1\kappa_1}{\mu_2\kappa_2}\sin\kappa_1 l\sin\kappa_2 l_0}$$
$$u_2(z,\omega)=u_0(\omega)\frac{\cos\kappa_1 l\cos\kappa_2 z+\frac{\mu_1\kappa_1}{\mu_2\kappa_2}\sin\kappa_1 l\sin\kappa_2 z}{\cos\kappa_1 l\cos\kappa_2 l_0-\frac{\mu_1\kappa_1}{\mu_2\kappa_2}\sin\kappa_1 l\sin\kappa_2 l_0}\ . \tag{26}$$

The eigenfrequencies are given now by

$$\tan\omega_n l/c_1\cdot\tan\omega_n l_0/c_2=\frac{\mu_2\kappa_2}{\mu_1\kappa_1}=\sqrt{\frac{\rho_2\mu_2}{\rho_1\mu_1}} \tag{27}$$

and amplification factors appear, similarly with a single bar. If bar 2 is much "softer" than bar 1 ($\mu_2/\rho_2\ll\mu_1/\rho_1$) the (lowest) eigenfrequencies are given by $\omega_n=(c_2/l_0)\alpha_n$, where α_n are the roots of the equation $\alpha_n\tan\alpha_n=\rho_2 l_0/\rho_1 l$. If bar 1 is "softer", the eigenfrequencies are $\omega_n=(c_1/l)\beta_n$, where $\beta_n\tan\beta_n=\mu_2 l/\mu_1 l_0$. We can see that the eigenfrequencies are controlled by the elastic properties of the "softer" bar. This result gives an indication regarding the vibration properties of bars with a composite structure (*e.g.*, including voids).

Buried bar. We consider now a bar completely buried in the ground, with both ends free (its orientation is immaterial); we assume that the bar moves freely in the ground, with a displacement

$$u=A\cos\kappa z+B\sin\kappa z \tag{28}$$

superposed upon the displacement u_0 of the ground. We assume a ground excitation

$$u_0(z,t)=u_0\theta(t)e^{-\alpha t}\cos\omega_0 t\cdot\cos\kappa_0 z\ \ , \tag{29}$$

where $\kappa_0=\omega_0/c_0$, c_0 being the wave velocity in the soil; the Fourier transform of this excitation is

$$u_0(z,\omega)=\frac{i}{2}u_0\cos\kappa_0 z\left(\frac{1}{\omega-\omega_0+i\alpha}+\frac{1}{\omega+\omega_0+i\alpha}\right)\ . \tag{30}$$

The condition of free ends (with the displacement u_0+u) gives

$$u(z,\omega)=-\frac{i}{2}u_0\frac{\kappa_0\sin\kappa_0 l}{\kappa\sin\kappa l}\left(\frac{1}{\omega-\omega_0+i\alpha}+\frac{1}{\omega+\omega_0+i\alpha}\right)\cos\kappa z\ . \tag{31}$$

The reverse Fourier transform of the displacement u includes contributions from the excitation poles $\omega=\pm\omega_0$ and from the eigenfrequencies $\omega_n=n\pi c/l$, for n any non-vanishing integer (the roots of the equation $\sin\omega_n l/c=0$). For ω_0 different from all ω_n the displacement is

$$u(z,t)=-u_0\frac{c\sin\omega_0 l/c_0}{c_0\sin\omega_0 l/c}e^{-\alpha t}\cos\omega_0 t\cdot\cos\omega_0 z/c-$$
$$-u_0\sum_n(-1)^n\frac{\omega_0 c^2}{2c_0 l\omega_n}\sin\omega_0 l/c_0\left(\frac{e^{i\omega_n t}}{\omega-\omega_0+i\alpha}+\frac{e^{-i\omega_n t}}{\omega+\omega_0+i\alpha}\right)\cos\omega_n z/c\ . \tag{32}$$

For $\omega_0=\omega_n$ (resonance) the displacement is given by

$$u(z,t)=(-1)^n u_0\frac{c^2}{c_0 l}\sin\omega_0 l/c_0\frac{1-e^{-\alpha t}}{\alpha}\sin\omega_0 t\cdot\cos\omega_0 z/c\ . \tag{33}$$

We can see the occurrence of an amplification factor

$$g = \frac{c^2}{c_0 l \alpha} \sin \omega_0 l / c_0 \ , \tag{34}$$

which now has a more complex structure; it depends on the wave velocity c_0 and frequency ω_0 of the excitation (it is a spectral amplification factor).

This result may throw an interesting light upon the so-called amplification site effect.[10]-[14] It is well known that the ground displacement, velocity and acceleration may exhibit large local variations from site to site. In the light of the above result it is easy to see that a local inhomogeneity surrounded by a different environment may behave as a buried bar, and the normal modes set in this inhomegeneity may exhibit large amplifications factors. The effect is enhanced for a stiff inhomogenity ($c \gg c_0$), but for low attenuation factors α it may appear also for soft inhomogenities. The conditions for its occurence are the resonance and seismic waves with wavelengths shorter than the linear dimension of the inhomogeneity. We note that it is easy to see that the bar-shape of the inhomogeneity is irrelevant; the amplification may occur for inhomogeneities of any shape; the necessary conditions are $c_0/\omega_0 < l$ (excitation wavelength shorter than the dimension of the inhomogeneity) and $\omega_0 = \omega_n = c\alpha_n/l$, where α_n is a numerical coefficient which gives the eigenfrequency ω_n (increasing with incresaing n); these conditions imply $c_0 < c\alpha_n$.

We can see that there exists a discontinuity between the soil displacement u_0 and the displacement u of the bar at the points of the bar. If we allow for a finite extension d of the bar, along, say, the transverse direction x, then, the soil displacement along this direction can be written as $u_0 \cos \omega_0(t - x/c_0)$ and the bar (internal) displacement is $u_0 \cos \omega_0(t - x/c)$. We can see that the displacement is continuous at $x = 0$, while the continuity at $x = d$ is attained only for $\omega_0 d(1/c_0 - 1/c) = (-1)^n 2\pi n$, where n is any integer; this is a vibration condition for the bar thickness d, which, in general is not satisfied (for a given frequency ω_0); equally well, it can be viewed also as a condition for the frequency ω_0. The continuity of the displacement along the bar is not fulfilled (for given soil displacement).

Coupled oscillators. We examine here the necessary conditions for two coupled vibrating bodies be approximated by two coupled dimensionless (point) harmonic oscillators;[15] to this end we use the model of coupled vibrating bars described above. From equations (1) the motion of the n-th normal mode is described by the equation

$$\rho \ddot{u}_n + \mu \kappa_n^2 u_n = \rho \ddot{u}_n + \rho \omega_n^2 u_n = 0 \ ; \tag{35}$$

we can see that the z-dependence becomes irrelevant, and we may view u_n as a global representation of the dispalcement of a point oscillator; this equation is written for the shear modes, but it has the same form, with μ replaced by $\lambda + 2\mu$, for the longitudinal modes. This is the equation of the harmonic oscillators with a set of eigenfrequencies. For frequencies near a certain eigenfrequency ω_1 (*e.g.*, the fundamental frequency) we may limit ourselves to only one harmonic-oscillator equation, written as

$$\ddot{u}_1 + \omega_1^2 u_1 = 0 \ ; \tag{36}$$

we have introduced the label 1 because a similar equation is written for another oscillator, denoted by 2, coupled to the former.

At the joining point $z = 0$ bar 1 acts with a force density (per unit area) $\mu_1 u_1' \mid_{z=0}$ on bar 2, while bar 2 acts with a force $\mu_2 u_2' \mid_{z=0}$ on the former (for shear displacement, equations (25)). The forces which act upon the bars viewed as oscillators are $\mu_{1,2} u_{1,2}' \mid_{z=0} S$, where S is the area of the joining surface. The derivatives of the displacement can be represented as $u_{1,2}' \mid_{z=0} \simeq u_{1,2}/d_{1,2}$, where $d_{1,2}$

are some fictitious distances, introduced for controlling the dimensionality of the equations. It follows that the interaction forces are of the order $\mu_{1,2}u_{1,2}S/d_{1,2}$. In order to conserve energy we must have $\mu_1 S/d_1 = \mu_2 S/d_2$; indeed, $\mu_1 Sd_2 = \mu_2 Sd_1$ is the interaction energy transfered between the two oscillators. This is a necessary condition for the two bars be approximated by oscillators. Therefore, we introduce the elastic interaction constant

$$K = \mu_1 S/d_1 = \mu_2 S/d_2 \tag{37}$$

and write the equations of motion for the two oscillators

$$m_{1,2}\ddot{u}_{1,2} + m_{1,2}\omega_{1,2}^2 u_{1,2} + Ku_{2,1} = 0 \ , \tag{38}$$

where we introduced the mass $m_{1,2}$ for each oscillator. It is worth comparing the interaction constants K with the oscillator constants

$$m_{1,2}\omega_{1,2}^2 \simeq m_{1,2}\frac{c_{1,2}^2}{l_{1,2}^2}\alpha_{1,2n}^2 = (\mu_{1,2}S/l_{1,2})\alpha_{1,2n}^2 \ , \tag{39}$$

where $\alpha_{1,2n}$ are numerical factor from the eigenfrequencies $\omega_{1,2n} = (c_{1,2}/l_{1,2})\alpha_{1,2n}$; these factors increase with increasing n. For lower frequencies and $l_{1,2}$ of the same order of magnitude as $d_{12,}$ all the oscillation constants $m_{1,2}\omega_{1,2}^2$ and K are of the same order of magnitude. This implies a severe restriction upon the coupled-oscillators approximation, since, rigourously speaking, these energies are not equal; it originates in the circumstance that the model of coupled oscillators requires the eigenfrequencies and the coupling constant derive from different forces, while for elastic bars these quantities have the same common orgin - the elastic force. However, if we give up the assumption of a sharp joining surface and consider that the two bars are welded, then there exists a smooth joining and the difference between the two interaction forces and the transferred interaction energies is taken over by the welding; during motion we have a mechanical work dissipated in the welding. In these conditions we may assume $m_1\omega_1^2 \neq m_2\omega_2^2 \neq K$.

The potential energy associated with these two oscillators reads

$$V = \frac{1}{2}m_1\omega_1^2 u_1^2 + \frac{1}{2}m_2\omega_2^2 u_2^2 + Ku_1u_2 \ ; \tag{40}$$

it must have a minimum for $u_{1,2} = 0$; this condition implies

$$K^2 < m_1m_2\omega_1^2\omega_2^2 \ . \tag{41}$$

With the notations introduced above this inequality reads

$$l_1l_2 < d_1d_2\alpha_{1n}^2\alpha_{2n}^2 \ , \tag{42}$$

which can be satisfied, especially for higher eigenfrequencies.

It is convenient to introduce a parameter $0 < \gamma < 1$ through

$$K^2 = m_1m_2\omega_1^2\omega_2^2(1-\gamma) \ ; \tag{43}$$

for $\gamma = 1$ there is no coupling, for $\gamma = 0$ the coupling is maximal. The parameter γ is a dimensionless coupling constant; we are interested in γ close to zero (maximal coupling). Also, we introduce the notations $k_{1,2} = K/m_{1,2}$, such that the system of equations (38) can be written as

$$\ddot{u}_{1,2} + \omega_{1,2}^2 u_{1,2} + k_{1,2}u_{2,1} = 0 \ . \tag{44}$$

The eigenfrequencies of this system of equations are the roots $\Omega_{1,2}$ of the equation

$$\Delta = \Omega^2 - (\omega_1^2 + \omega_2^2)\Omega^2 + \omega_1^2\omega_2^2\gamma = 0 \ ; \tag{45}$$

we get

$$\Omega_1^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 + \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2\gamma}\right] \simeq \omega_1^2 + \omega_2^2 \tag{46}$$

and

$$\Omega_2^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 - \sqrt{(\omega_1^2 + \omega_2^2)^2 - 4\omega_1^2\omega_2^2\gamma}\right] \simeq \frac{\omega_1^2\omega_2^2}{\omega_1^2 + \omega_2^2}\gamma \ . \tag{47}$$

We can see that Ω_1^2 increases from ω_1^2 up to $\omega_1^2 + \omega_2^2$, while Ω_2^2 decreases from ω_2^2 ($\omega_2^2 < \omega_1^2$) down to zero for K^2/m_1m_2 going from zero to its maximum value $\omega_1^2\omega_2^2$ (for γ going from 1 to 0); the coupling lowers the low eigenfrequency and raises the high eigenfrequency.

For a realistic use of the coupled-oscillator model we consider the two oscillators as corresponding to a building (oscillator 2) and its foundation (oscillator 1). For a stiff foundation, such that $\omega_1 > \omega_2$ the eigenfrequencies of the building are reduced to an appreciable extent (down to zero), while the eigenfrequencies of the foundation are increased by the coupling. For a soft foundation ($\omega_1 < \omega_2$) the situation is reversed, the eigenfreqiencies of the building are raised by the coupling and those of the foundation are reduced.

The solutions of the homogeneous system of equations (44) are the real part of

$$u_{1,2}^0 = A_{1,2}e^{i\Omega_1 t} + B_{1,2}e^{i\Omega_2 t} \ , \tag{48}$$

with complex constants $A_{1,2}$, $B_{1,2}$. These constants satisfy the system of equations (44) for $\Omega = \Omega_{1,2}$, respectively:

$$(\omega_1^2 - \Omega_1^2)A_1 + k_1A_2 = 0 \ , \ (\omega_2^2 - \Omega_2^2)B_2 + k_2B_1 = 0 \ ; \tag{49}$$

we get

$$\begin{aligned} A_2 &= \tfrac{\Omega_1^2 - \omega_1^2}{k_1}A_1 \simeq \tfrac{\omega_2^2}{k_1}A_1 \ , \\ B_1 &= \tfrac{\Omega_2^2 - \omega_2^2}{k_2}B_2 \simeq -\tfrac{\omega_2^2}{k_2}B_2 \ ; \end{aligned} \tag{50}$$

the solution is the real part of

$$u_1^0 \simeq A_1e^{i\Omega_1 t} - \tfrac{\omega_2^2}{k_2}B_2e^{i\Omega_2 t} \ , \ u_2^0 \simeq \tfrac{\omega_2^2}{k_1}A_1e^{i\Omega_1 t} + B_2e^{i\Omega_2 t} \ . \tag{51}$$

Let us assume now that the foundation (oscillator 1) is subject to a force $\theta(t)f_0e^{-\alpha t}\cos\omega_0 t$, $\alpha \ll \omega_0$, arising from the ground motion; the equations of motion of the two oscillators

$$\ddot{u}_1 + \omega_1^2u_1 + k_1u_2 = f\theta(t)e^{-\alpha t}\cos\omega_0 t \ , \ \ddot{u}_2 + \omega_2^2u_2 + k_2u_1 = 0 \ , \tag{52}$$

where $f = f_0/m_1$; a particular solution is the real part of

$$u_{1,2} = a_{1,2}e^{i\omega_0 t - \alpha t} \ . \tag{53}$$

The constants $a_{1,2}$ are given by

$$a_1 = f\frac{\omega_2^2 - \widetilde{\omega}_0^2}{\widetilde{\Delta}} \ , \ a_2 = -f\frac{k_2}{\widetilde{\Delta}} \ , \tag{54}$$

where $\widetilde{\Delta} = (\widetilde{\omega}_0^2 - \Omega_1^2)(\widetilde{\omega}_0^2\text{-}\Omega_2^2)$ and $\widetilde{\omega}_0 = \omega_0 + i\alpha$; adding the solution $u_{1,2}^0$ of the homogeneous system of equations (equations (51)) we get the full solution

$$\begin{aligned} u_1 &= A_1 e^{i\Omega_1 t} - \tfrac{\omega_2^2}{k_2} B_2 e^{i\Omega_2 t} + f\tfrac{\omega_2^2 - \widetilde{\omega}_0^2}{\widetilde{\Delta}} e^{i\widetilde{\omega}_0 t} \ , \\ u_2 &= \tfrac{\omega_2^2}{k_1} A_1 e^{i\Omega_1 t} + B_2 e^{i\Omega_2 t} - f\tfrac{k_2}{\widetilde{\Delta}} e^{i\widetilde{\omega}_0 t} \ . \end{aligned} \tag{55}$$

The (complex) constants A_1, B_2 are determined from the initial conditions $u_{1,2}(t=0) = 0$, $\dot{u}_{1,2}(t=0) = 0$.

We focus on the resonance of the building, where $\omega_0 = \Omega_2$ ($\alpha \ll \Omega_2$) and $\widetilde{\Delta} \simeq \alpha\Omega_1^2(\alpha - 2i\Omega_2)$; the initial conditions give $A_1 \simeq 0$ and

$$B_2 \simeq \frac{fk_2}{4\Omega_1^2\Omega_2^2}\left(1 + i\frac{\Omega_2}{\alpha}\right) \ ; \tag{56}$$

the displacements are

$$\begin{aligned} u_1 &= -\tfrac{f\omega_2^2}{4\Omega_1^2\Omega_2^2}\left(\cos\Omega_2 t - \tfrac{\Omega_2}{\alpha}\sin\Omega_2 t\right)(1 - e^{-\alpha t}) + O(\alpha) \ , \\ u_2 &= \tfrac{fk_2}{4\Omega_1^2\Omega_2^2}\left(\cos\Omega_2 t - \tfrac{\Omega_2}{\alpha}\sin\Omega_2 t\right)(1 - e^{-\alpha t}) + O(\alpha) \ . \end{aligned} \tag{57}$$

We can see that the original damped excitation is lost in time and for long time both the building and the foundation oscillate with the resonance frequency Ω_2 of the building; the amplitudes of the oscillations are enhanced by the attenuation factor $1/\alpha$, as expected; the oscillation amplitude of the foundation is controlled by the exciting force, while the amplitude of the building is controlled by the coupling constant. We note that we have considered above oscillations without a damping factor; a damping factor a damping factor affects the contribution of the normal modes and adds to the attenuation factor of the excitation.

Inserting K and $\Omega_{1,2}$ from equations (43), (46) and (47) we get the leading contribution to the oscillation amplitude of the building

$$u_{20} = \frac{fk_2}{4\alpha\Omega_1^2\Omega_2} = \frac{f_0}{4\alpha}\sqrt{\frac{1-\gamma}{\gamma}}\frac{1}{\sqrt{m_1 m_2(\omega_1^2 + \omega_2^2)}} \ ; \tag{58}$$

comparing it with the amplitude $u_{20}^{is} = f_0/m_2\omega_2^2$ of an isolated building (without foundation) subject to the same force and oscillating with the same frequency $\Omega_2 \ll \omega_2$, we get

$$u_{20} = u_{20}^{is}\frac{\omega_2}{4\alpha} \cdot \sqrt{\frac{1-\gamma}{\gamma}} \cdot \frac{\omega_2}{\sqrt{\omega_1^2 + \omega_2^2}} \ ; \tag{59}$$

we can see the enhancement factors ω_2/α arising from resonance and $\sqrt{(1-\gamma)/\gamma}$ arising from the coupling. We note the resemblance of the amplification factor ω_2/α with the amplification factor $\simeq c/l\alpha$ of the embedded bar (ω_2 being of the same order of magnitude as c/l).

It is worth connecting the force f_0 to the amplitude u_0 of the ground displacement u_0, in order to compare the displacement of the building with the ground displacement. According to the discussion above, assumig a good coupling between ground and the foundation and a relatively homogeneous structure groud+foundation+building we may use $f_0 = Ku_0$ (since, for a shear coupling, f_0 is of the order $\mu_g u_0^{'}$, where μ_g is the rigidity modulus of the soil); we get the oscillation amplitude of the building

$$u_{20} \simeq \frac{\omega_1\omega_2}{4\alpha\sqrt{\omega_1^2+\omega_2^2}}\frac{1-\gamma}{\gamma}u_0 \ , \tag{60}$$

where we can see again the occurrence of an amplification factor $\simeq \omega_c/\alpha$, where $\omega_c = \omega_1\omega_2(1-\gamma/4\gamma\sqrt{\omega_1^2+\omega_2^2}$ is a characteristic frequency of the structure.

Finally we give the solution of coupled oscillators subject to a damped force (shock) without oscillations, *i.e.* for $\omega_0 = 0$:

$$\begin{aligned} u_1 &= \tfrac{f\omega_2^2}{\Omega_1^2\Omega_2^2}\left(e^{-\alpha t} - \cos\Omega_2 t + \tfrac{\alpha}{\Omega_2}\sin\Omega_2 t\right) + O(\alpha^2) \ , \\ u_2 &= -\tfrac{fk_2}{\Omega_1^2\Omega_2^2}\left(e^{-\alpha t} - \cos\Omega_2 t + \tfrac{\alpha}{\Omega_2}\sin\Omega_2 t\right) + O(\alpha^2) \ ; \end{aligned} \tag{61}$$

we can see that the shock excites the oscillations with the lowest frequency (Ω_2), similar with an oscillating shock), and there is no enhancement of the oscillations, as expected.

Concluding remarks. The vibrations of an elastic bar extending above the ground surface with one end embedded in the ground are described and the response of the bar to various ground excitations applied to its lower end is calculated. An oscillating shock with a sharp wavefront is considered as the most interesting excitation, since it is deemed that such a shock may correspond to the seismic main shock with its long tail, produced by the so-called surface waves. At resonance it is shown that the bar exhibits amplification factors for displacement, velocity and acceleration, which may attain large values. The amplification factors are given by a combination of the shock duration, the height of the bar above the ground surface and the velocity of the elastic waves in the bar; they arise as a consequence of the excitation of the normal modes in the bar. Two bars coupled along their length are also considered; it is shown that for bars with very different elastic properties the eigenfrequencies of the system are given mainly by the "softer" bar. Such an information may be useful for composite structures, including, for instance, voids. A bar completely buried in the ground may serve as a model for local inhomogeneities in the soil. It is shown that such a bar exhibits spectral amplification factors, which may correspond to the well known site amplification factors documented by the seismic records. Making use of the information gained from the vibrating bars we examine the formulation of the model of two coupled harmonic oscillators, and its application to the structure building-foundation. The coupling lowers the low frequency of the system and raise the upper frequency. The response of two coupled oscillators to an oscillating shock is calculated, and amplification factors similar with the vibrating bars are highlighted.

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